Wireless Communications

Small Scale Fading

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Small-Scale Fading

- Describe the rapid fluctuations of the amplitudes, phases, or multipath delays of a radio signal over a short period of time or travel distance
  - The large-scale path loss effects might be ignored
  - Fading is caused by multipath waves
    - Rapid changes in signal strength
    - Random frequency shift
    - Time dispersion (echoes)
Factors Influencing Small-Scale Fading

- Multipath propagation
  - Multiple signal paths
  - Intersymbol interference
- Speed of the mobile
  - Doppler shift
  - Random frequency modulation
- Speed of surrounding objects
  - Surrounding objects move at a greater rate than the MS
- The transmission bandwidth of the signal
  - Comparing to the bandwidth of the multipath channel
Doppler Shift

- Named after Chris Doppler, an Australian scientist
- The apparent change in frequency and wavelength of a wave due to the moving receiver or the waving transmitter

A mobile moving at a constant velocity \( v \)
Along a path between X and Y with the distance \( d \)
Signals received from a remote source S
The source is very far away, same

The difference in path lengths
\[
\Delta l = d \cos \theta = v \Delta t \cos \theta
\]

The phase change
\[
\Delta \phi = \frac{2 \pi \Delta l}{\lambda} = \frac{2 \pi v \Delta t}{\lambda} \cos \theta
\]
Doppler Shift

- Doppler shift, the apparent change in frequency

\[ f_d = \frac{1}{2\pi} \frac{\Delta \phi}{\Delta t} = \frac{v}{\lambda} \cos \theta \]

- Moving toward the direction of arrival of the wave → the Doppler shift is positive (increasing frequency)
- Moving away from the direction of arrival of the wave → the Doppler shift is negative (decreasing frequency)
Enhance our understanding…

- Consider a transmitter which radiates a sinusoidal carrier frequency of 1850 MHz. For a vehicle moving 60 mph, compute the received carrier frequency if the mobile is moving (a) directly toward the transmitter, (b) directly away from the transmitter, (C) in a direction which is perpendicular to the direction of arrival of the transmitted signal.
Impulse Response Model

• The **impulse response model** can be used for the **small-scale variations** of a mobile radio signal ?!

• So far, we know
  
  • The impulse response is a wideband channel characterization and contains all information
  
  • The mobile radio channel may be modeled as a linear filter with a time varying impulse response

• We need the proof…
Impulse Response Model

- For a fixed position \( d \), the channel between the Tx and the Rx can be modeled as a *linear time invariant system* \( h(t) \).
- Due the multipath fading, the channel transfer function should be a function of the position of the Rx, \( h(d, t) \).

The received signal \( y(d, t) \) is given by:
\[
y(d, t) = x(t) \otimes h(d, t) = \int_{-\infty}^{\infty} x(\tau) h(d, t - \tau) d\tau
\]

For a causal system, \( y(d, t) \) simplifies to:
\[
y(d, t) = \int_{-\infty}^{t} x(\tau) h(d, t - \tau) d\tau
\]

When \( d = vt \), we have:
\[
y(vt, t) = \int_{-\infty}^{t} x(\tau) h(vt, t - \tau) d\tau
\]
Impulse Response Model

- The mobile channel can be modeled as a linear time varying channel, changing with time and distance

\[ y(t) = \int_{-\infty}^{t} x(\tau) h(vt, t - \tau) d\tau = x(t) \otimes h(vt, t) = x(t) \otimes h(d, t) \]

Since \( v \) is constant over a short time or time interval,

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t, \tau) d\tau = x(t) \otimes h(t, \tau) \]

- \( x(t) \): the transmitted bandpass waveform
- \( y(t) \): the received waveform
- \( h(t, \tau) \): the impulse response of the time varying multipath radio channel
- \( \tau \): the channel multipath delay for a fixed value of \( t \)
Impulse Response Model

- Discrete model
  - Equal time delay segments ⇒ excess delay bins
  - $\tau_0$: the first arriving signal
  - $N$: the total number of possible multipath components

\[
h_b(t, \tau) = \sum_{i=0}^{N-1} h(t, \tau_i) \delta(t - t_i)
\]

*delay bin $\Delta \tau = \tau_{i+1} - \tau_i$*
Impulse Response Model

- Discrete model (cont.)
  - The model may be used to analyze transmitted RF signals having bandwidths less than $2/\Delta \tau$
  - Excess delay $\tau_i$: the relative delay of the $i^{th}$ multipath component as compared to the first arriving component
  - Maximum excess delay of the channel: $N\Delta \tau$
Impulse Response Model

- Discrete model (cont.)
  - If the channel impulse response is assumed to be time invariant or at least wide sense stationary
    \[
    h_b(\tau) = \sum_{i=0}^{N-1} a_i \exp[j \theta_i] \delta(\tau - \tau_i)
    \]
  - Power delay profile of the channel over a local area
    \[
    P(\tau) = k |h_b(t, \tau)|^2
    \]
Parameters of Mobile Multipath Channels

- Review: Power delay profile
  - A plot of relative received power as a function of excess delay with respect to a fixed time delay reference
  - Averaging instantaneous power delay profile measurements over a local-area

\[ |r(t_0)|^2 = \sum_{k=0}^{N-1} a_k^2(t_0) \]
Time Dispersion Parameter

- The mean excess delay
  \[ \tau = \frac{\sum_k a_k^2 \tau_k}{\sum_k a_k^2} = \frac{\sum_k P(\tau_k)\tau_k}{\sum_k P(\tau_k)} \]

- The rms delay spread
  \[ \sigma_\tau = \sqrt{\bar{\tau}^2 - (\bar{\tau})^2} \]
  \[ \bar{\tau}^2 = \frac{\sum_k a_k^2 \tau_k^2}{\sum_k a_k^2} = \frac{\sum_k P(\tau_k)\tau_k^2}{\sum_k P(\tau_k)} \]

- The maximum excess delay
  \[ \tau_{\text{max}} = \tau_X - \tau_0 \]
  \( \tau_0 \): the first arrival
  \( \tau_X \): the max. delay at which a multipath component is within X dB
Time Dispersion Parameter

RMS Delay Spread = 46.40 ns

Maximum Excess Delay < 10 dB = 84 ns

Threshold Level = -20 dB

Mean Excess Delay = 45.05 ns
Time Dispersion Parameter

- The rms delay spread and mean excess delay are defined from a single power delay profile, which is averaged over a local area.
- Many measurements are made at many local areas in order to determine a statistical range of multipath channel parameters.
- In practice, such parameters depend on the choice of noise threshold used to process $P(x)$ analysis.
- …Related to bandwidth.
# Example: RMS Delay Spread

<table>
<thead>
<tr>
<th>Environment</th>
<th>Freq. MHz</th>
<th>RMS Delay Spread</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Urban</td>
<td>910</td>
<td>1300 ns avg.</td>
<td>New York City</td>
</tr>
<tr>
<td>Urban</td>
<td>892</td>
<td>10~25 µs</td>
<td>San Francisco</td>
</tr>
<tr>
<td>Suburban</td>
<td>910</td>
<td>200~310 ns</td>
<td>Typical case</td>
</tr>
<tr>
<td>Suburban</td>
<td>910</td>
<td>1960~2110 ns</td>
<td>Extreme case</td>
</tr>
<tr>
<td>Indoor</td>
<td>1500</td>
<td>10~50 ns</td>
<td>Office building</td>
</tr>
<tr>
<td>Indoor</td>
<td>850</td>
<td>270 ns max.</td>
<td>Office building</td>
</tr>
<tr>
<td>Indoor</td>
<td>1900</td>
<td>70~94 ns avg.</td>
<td>Office building</td>
</tr>
</tbody>
</table>
Practice: page 201, ex. 5.4

- Compute the RMS delay spread for the following power delay profile:
- If BPSK modulation is used, what is the maximum bit rate that can be sent through the channel without needing an equalizer?

\[ \frac{\sigma_r}{T_s} \leq 0.1 \]
Coherence Bandwidth

- A defined relation derived from the rms delay spread
- A statistical measure of the range of frequencies over which the channel can be considered “flat”
- The range of frequencies over which two frequency components have a strong potential for amplitude correlation

\[ B_c \approx \frac{1}{50\sigma_\tau} \]  
As the bandwidth over which the frequency correlation function is above 0.9

\[ B_c \approx \frac{1}{5\sigma_\tau} \]  
As the bandwidth over which the frequency correlation function is above 0.5
Doppler Spread and Coherence Time

- Describe the time varying nature of the channel in a small-scale region
- Doppler spread $B_D$
  - A measure of the spectral broadening caused by the time rate of change of mobile radio channel
  - As the range of frequencies over $[f_c-f_d, f_c+f_d]$ which the received Doppler spectrum is essentially non-zero
- Slow fading channel: if the baseband signal bandwidth is much greater than $B_D$, the effect of Doppler spread are negligible at the receiver
Doppler Spread and Coherence Time

- Coherence time \( T_C \)
  - The time domain dual of Doppler spread
  - Characterize the time varying nature of the frequency dispersiveness of the channel

\[
T_C \approx \frac{1}{f_m} \quad \text{The maximum Doppler shift } f_m = v/\lambda
\]

\[
T_C \approx \frac{9}{16\pi f_m} \quad \text{The time over which the time correlation function is above 0.5}
\]

\[
T_C = \sqrt{\frac{9}{16\pi f_m^2}} = \frac{0.423}{f_m} \quad \text{Practical}
\]
Doppler Spread and Coherence Time

- Coherence time is actually a statistical measure of the time duration over which the channel impulse response is essentially invariant.
- Coherence time is the time duration over which two received signals have a strong potential for amplitude correlation.
- Two signals arriving with a time separation greater than $T_C$ are affected differently by the channel.
- If the reciprocal bandwidth of the baseband signal is greater than the coherence time of the channel, then the channel will change during the transmission of the message, thus causing distortion at the receiver.
Practice: page 202, ex. 5.5

- Calculate the mean excess delay, rms delay spread, and the max. excess delay (10dB) for the multipath of the channel. Calculate the coherence bandwidth.
Types of Small-Scale Fading

- Depending on the relation between the signal parameters: bandwidth, symbol period
- Depending on the relation of the channel parameters: delay spread, Doppler spread
- Types
  - Time dispersion fading
  - Frequency selective fading
  - Frequency dispersion fading
  - Time selective fading
Multipath Time Delay Spread

- Flat fading
  - A constant gain and linear phase response over a bandwidth
  - A bandwidth is greater than the BW of the transmitted signal
  - Most common type
  - The spectral characteristics of the transmitted signal are preserved at the receiver, even though the strength of the received signal changes with time
  - *Narrowband* channels or amplitude varying channels

\[
B_s << B_c
\]
\[
T_s >> \sigma_\tau, \quad 10\sigma_\tau
\]
Multipath Time Delay Spread

- Frequency selective fading
  - A constant gain and linear phase response over a bandwidth
  - A bandwidth is smaller than the BW of the transmitted signal
  - More difficult to model
  - The received signal includes multiple versions of the transmitter waveform which are faded and delayed
- Wideband Channel or Intersymbol Interference (ISI)

\[
B_s > B_c \\
T_s < \sigma_{\tau} \quad 10\sigma_{\tau}
\]
Flat Fading
Frequency Selective Fading
Due to Doppler Shift

- Fast fading
  - The channel impulse response changes rapidly within the symbol duration
  - Resulting in frequency dispersion → signal distortion
  - Only dealing with the rate of the change of the channel due to the motion. It does not imply flat/frequency selective fading
  - In practice, fast fading only occurs for very low data rates

\[
T_s > T_c \\
B_s < B_D
\]
Due to Doppler Shift

- Slow fading
  - The channel impulse response changes slower than the transmitted baseband signal
  - The channel may be assumed to be static over one or several reciprocal bandwidth intervals $\Rightarrow$ the Doppler spread of the channel is much less than the bandwidth of the baseband

\[
T_s << T_c \\
B_s >> B_D
\]
Types of Small-scale Fading

Based on multipath time delay spread

- Flat fading
  - BW of signal < BW of channel
  - Delay spread < Symbol period

- Frequency selective fading
  - BW of signal > BW of channel
  - Delay spread > Symbol period

Based on Doppler spread

- Fast fading
  - High Doppler spread
  - Coherent time < Symbol period
  - Channel variation faster than baseband signal variations

- Slow fading
  - Low Doppler spread
  - Coherent time > Symbol period
  - Channel variation slower than baseband signal variations
Types of Small-scale Fading

- **Flat Slow Fading**
- **Flat Fast Fading**
- **Frequency Selective Slow Fading**
- **Frequency Selective Fast Fading**

Symbols:
- $T_S$: Transmitted Symbol Period
- $T_C$: Transmitted Symbol Period of Transmitting symbol
- $B_S$: Transmitted Baseband Signal Bandwidth
- $B_C$: Transmitted Baseband Signal Bandwidth
- $B_d$: Transmitted Baseband Signal Bandwidth
Practice: page 253, 5.30

- For each of the three scenarios below, decide if the received signal is best described as undergoing fast fading, frequency selective fading or flat fading.
  - A binary modulation has a data rate of 500 kbps, $f_c=1$ GHz and a typical urban radio channel is used to provide communications to cars moving on a highway.
  - A binary modulation has a data rate of 5 kbps, $f_c=1$ GHz and a typical urban radio channel is used to provide communications to cars moving on a highway.
  - A binary modulation has data rate of 10 bps, $f_c=1$ GHz and a typical urban radio channel is used to provide communications to cars moving on a highway.
Rayleigh Distribution

- To describe the statistical time varying nature of
  - The received envelope of a flat fading signal
  - The envelope of an individual multipath component
  - The envelope of the sum of two quadrature Gaussian noise signals

- Probability Density Function (PDF)

\[
P(r) = \begin{cases} 
\frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) & 0 \leq r \leq \infty \\
0 & 0 > r 
\end{cases}
\]

\(r\): the rms value of the received voltage signal before envelope detection \(\sqrt{2\sigma}\)

\(\sigma^2\): the time average power of the received signal before envelope detection, variance
Rayleigh Distribution

- Cumulative Density Function (CDF): the probability of the received signal does not exceed a specified value $R$

$$P(R) = \Pr(r \leq R) = \int_0^R P(r)dr = 1 - \exp \left( -\frac{R^2}{2\sigma^2} \right)$$

- The mean value and variance of the Rayleigh distribution

$$r_{\text{mean}} = E[r] = \int_0^\infty rP(r)dr = \sigma \sqrt{\frac{\pi}{2}} = 1.2533\sigma$$

$$\sigma_r^2 = E[r^2] - E^2[r] = \int_0^\infty r^2P(r)dr - \sigma^2 \frac{\pi}{2} = 0.4292\sigma^2$$

- The median value of $r$

$$\frac{1}{2} = \int_0^{r_{\text{median}}} P(r)dr \Rightarrow r_{\text{median}} = 1.177\sigma$$
Rayleigh Distribution

PDF

CDF
Rayleigh Distribution

- Without LOS - there are many objects in the environment that scatter the radio signal before it arrives at the receiver
- The urban environment
  - Build-up city center
- Small-scale fading effect
Ricean (Rice) Distribution

- The small-scale fading envelope distribution
- When there is a dominant stationary (nonfading) signal component present (LOS)
- As the dominant signal becomes weaker → Rayleigh

\[
P(r) = \begin{cases} 
\frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) I_0\left(\frac{Ar}{\sigma^2}\right) & 0 \leq r \leq \infty, A \geq 0 \\
0 & o < r
\end{cases}
\]

A: the peak amplitude of the dominant signal
I(.): Bessel function of the first kind and zero-order
Ricean Distribution

\[ K = \frac{A^2}{2\sigma^2} \]

\[ K(dB) = 10\log \frac{A^2}{2\sigma^2} \]

- Ricean factor \( K \) – the ratio between the deterministic signal power and the variance of the multipath
  - \( A \to 0, \ K \to -\infty \) dB, the dominant path decreases \( \to \) Rayleigh
  - \( K \gg 1 \): less severe fading
  - \( K \ll 1 \): severe fading
Ricean Distribution

![Graph showing Ricean distribution with different values of \( \sigma \) and \( A \)]
Nakagami Distribution

- An empirical model that fits measured data in many mobile environment better than Rayleigh or Ricean model

\[ f_\alpha(x) = \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^m x^{2m-1} \exp\left(-\frac{m}{\Omega} x^2\right) \]

\[ \Omega = E[x^2]; \quad m = \frac{(E[x^2])^2}{\text{Var}[x]} \geq \frac{1}{2} \]

\[ m = \begin{cases} 
\frac{1}{2}, & \text{one sided Gaussian} \\
1, & \text{Rayleigh pdf} \\
> \frac{1}{2}, & \text{approximates Ricean} \\
\to \infty, & \text{no fading} 
\end{cases} \]
Nakagami Distribution

- Power distribution

\[ f_{\alpha^2}(x) = \left( \frac{m}{\Omega} \right)^m x^{m-1} \frac{x^m}{\Gamma(m)} \exp\left( -\frac{m}{\Omega} x \right) \]

\[ F_{\alpha^2}(x) = P[\alpha^2 \leq x] \]

\[ = \int_0^x f_{\alpha^2}(y) \, dy \]

\[ = \frac{1}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^m \int_0^x y^{m-1} \exp\left( -\frac{m}{\Omega} y \right) \, dy \]

\[ = \frac{1}{\Gamma(m)} G\left( m, \frac{m}{\Omega} x \right) \]

\[ G(a,x) = \Gamma(a) - \Gamma(a,x) = \int_0^x y^{a-1} e^{-y} \, dy \]
Practice: page 251, 5.19

- Show that the magnitude (envelope) of the sum of two independent identically distributed complex (quadrature) Gaussian sources is Rayleigh distributed. Assume that the Gaussian sources are zero mean and have unit variance.
Practical Models

- Clark model
- Saleh and Valenzuela indoor statistical model
- Two-ray Rayleigh fading model
- SIRCIM and SMRCIM indoor and outdoor statistical models
- And more…
Level crossing rate by Rice

- Level crossing rate (LCR): the expected rate at which the Rayleigh fading envelope, normalized to the local rms signal level, crosses a specified level in a positive-going direction

\[ N_R = \int_0^\infty r p(R, \dot{r}) \, dr = \sqrt{2\pi} f_m \rho e^{-\rho^2} \]

The time derivative of \( r(t) \)

The joint density function of \( r \) and at \( r=R \)

The value of specified level \( R \), normalized to the local rms amplitude \( R/R_{rms} \)
The Average Duration

The Average Duration: the average period of time for which the received signal is below a specified level $R$.

$$
\bar{\tau} = \frac{1}{N_R} \Pr[r \leq R] \\
\Pr[r \leq R] = \frac{1}{T} \sum_i \tau_i \\
\Pr[r \leq R] = 1 - \exp(-\rho^2) \\
\tau_i : \text{the duration of the fade} \\
T : \text{the observation interval of the fading signal}
$$

$$
\frac{-\tau}{e^{\rho^2} - 1} = \frac{\rho f_m \sqrt{2\pi}}{\rho f_m \sqrt{2\pi}}
$$
For a Rayleigh fading signal, compute the positive-going level crossing rate for $\rho=1$, when the maximum Doppler frequency is 20 Hz. What is the maximum velocity of the mobile for this Doppler frequency if the carrier frequency is 900 MHz?
Practice: page 225, ex. 5.8

- Find the average fade duration for threshold levels $\rho = 0.01, 0.1$ and $1$, when the Doppler frequency is $200$ Hz.
- Find the average fade duration for a threshold levels of $\rho = 0.707$ when the Doppler frequency is $20$ Hz. For a binary digital modulation with bit duration of $50$ bps, is the Rayleigh fading slow or fast? What is the average number of bit errors per second for the given data rate. Assume that a bit error occurs whenever any portion of a bit encounters a fade for which $\rho < 0.1$.

The average duration of a signal fade helps determine the most likely number of signal bits that may be lost during a fade.
Summary: Small Scale Fading

- Parameters of mobile multipath channels
  - Time dispersion parameters and coherence bandwidth
  - Doppler spread and coherence time

\[
\begin{align*}
\bar{\tau} &= \frac{\sum a_k^2 \tau_k}{\sum a_k^2} = \frac{\sum P(\tau_k) \tau_k}{\sum P(\tau_k)} \\
\sigma_\tau &= \sqrt{\bar{\tau}^2 - (\bar{\tau})^2} \\
\bar{\tau}^2 &= \frac{\sum a_k^2 \tau_k^2}{\sum a_k^2} = \frac{\sum P(\tau_k) \tau_k^2}{\sum P(\tau_k)} \\
B_c &\approx \frac{1}{50\sigma_\tau} \\
B_c &\approx \frac{1}{5\sigma_\tau} \\
f_d &= \frac{1}{2\pi} \frac{\Delta \phi}{\Delta t} = \frac{v}{\lambda} \cos \theta \\
T_c &\approx \frac{1}{f_m} \\
T_c &\approx \frac{9}{16\pi f_m} \\
T_c &= \sqrt{\frac{9}{16\pi f_m^2}} = 0.423 \frac{1}{f_m}
\end{align*}
\]
Summary: Small Scale Fading

- Types of small-scale fading
  - Flat fading vs. frequency selective fading
  - Fast fading vs. slow fading

<table>
<thead>
<tr>
<th>Flat fading</th>
<th>Frequency selective fading</th>
</tr>
</thead>
<tbody>
<tr>
<td>BW of signal &lt; BW of channel</td>
<td>BW of signal &gt; BW of channel</td>
</tr>
<tr>
<td>Delay spread &lt; Symbol period</td>
<td>Delay spread &gt; Symbol period</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fast fading</th>
<th>Slow fading</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Doppler spread</td>
<td>Low Doppler spread</td>
</tr>
<tr>
<td>Coherent time &lt; Symbol period</td>
<td>Coherent time &gt; Symbol period</td>
</tr>
<tr>
<td>Channel variation faster than baseband signal variations</td>
<td>Channel variation slower than baseband signal variations</td>
</tr>
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</table>
Summary: Small Scale Fading

- Rayleigh fading distribution

\[ P(r) = \begin{cases} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) & 0 \leq r \leq \infty \\ 0 & 0 > r \end{cases} \]

\[ P(R) = \Pr(r \leq R) = \int_0^R P(r) \, dr = 1 - \exp\left(-\frac{R^2}{2\sigma^2}\right) \]

- Level crossing rate and average duration
Page 250, example 5.12: The fading characteristics of a CW carrier in an urban area are to be measured. The following assumptions are made (1) The mobile receiver uses a *simple vertical monopole*. (2) Large-scale fading due to path loss is ignored. (3) The mobile has no line-of-sight path to the base station. (4) The pdf of the received signal follows a Rayleigh distribution.

- Derive the ratio of the desired signal level to the rms signal level that maximizes the level crossing rate. Express your answer in dB.
- Assuming the maximum velocity of the mobile is 50 km/hr, and the carrier frequency is 900 MHz, determine the maximum number of times the signal *envelope will fade below the level found* in (a) during a 1 minutes test.
- How long on average will each fade in (b) last?
Describe all the physical circumstances that relate to a stationary transmitter and a moving receiver such that the Doppler shift at the receiver is equal to (a) 0 Hz, (b) $f_{d_{\text{max}}}$, (c) $-f_{d_{\text{max}}}$ and (d) $f_{d_{\text{max}}/2}$