A multi-axial, multimechanism based constitutive model for the comprehensive representation of the evolutionary response of SMAs under general thermomechanical loading conditions

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Abstract
We present a fully general, three dimensional, constitutive model for Shape Memory Alloys (SMAs), aimed at describing all of the salient features of SMA evolutionary response under complex thermomechanical loading conditions. In this, we utilize the mathematical formulation we have constructed, along with a single set of the model’s material parameters, to demonstrate the capturing of numerous responses that are experimentally observed in the available SMA literature. This includes uniaxial, multi-axial, proportional, non-proportional, monotonic, cyclic, as well as other complex thermomechanical loading conditions, in conjunction with a wide range of temperature variations. The success of the presented model is mainly attributed to the following two main factors. First, we use multiple inelastic mechanisms to organize the exchange between the energy stored and energy dissipated during the deformation history. Second, we adhere strictly to the well established mathematical and thermodynamical requirements of convexity, associativity, normality, etc. in formulating the evolution equations governing the model behavior, written in terms of the generalized internal stress/strain tensorial variables associated with the individual inelastic mechanisms. This has led to two important advantages: (a) it directly enabled us to obtain the limiting/critical transformation surfaces in the spaces of both stress and strain, as importantly required in capturing SMA behavior; (b) as a byproduct, this also led, naturally, to the exhibition of the apparent deviation from normality, when the transformation strain rate vectors are plotted together with the surfaces in the space of external/global stresses, that has been demonstrated in some recent multi-axial, non-proportional experiments.

1. Introduction

For centuries, metals have played an important role as structural materials. The human ability to understand material behavior (mechanical, thermal, electrical, etc.), which is derived from microstructural attributes, and utilize this information to engineer different material properties for a variety of applications has enabled the development of new alloys and composites. With advancements in science and technology, new classes of multifunctional materials have emerged, among which “ordinary” Shape Memory Alloys (SMAs), and High Temperature SMAs (HTSMA) have a particular significance in advanced engineering applications (Hartl et al., 2010). In almost all engineering fields including civil, mechanical, aerospace and bio-medical engineering, SMAs are known primarily for one fundamental and unique property – the ability to remember...
and recover their original shape after being subjected to large strains. These unique characteristics of SMAs have made them desirable for use in various sensor, actuator, energy absorption, and vibration damping applications.

1.1. Motivation

SMAs have a unique ability to recover imposed deformation merely through the application of heat (even under high applied loads, therefore resulting in high actuation energy densities) making them attractive for many engineering applications. In addition, unlike conventional metals that recover less than 1% strain before plastic deformation, SMAs undergo a diffusionless, thermoelastic martensitic phase transformation as a result of a twinning process that allows complete recovery of strains as large as 8–12% (Grabe and Brühns, 2009; Schwartz, 2002; Wang et al., 2008). The key to this characteristic is transformation between a high symmetry austenite phase, \( A \) (parent), and a lower temperature, low symmetry martensite phase, \( M \) (daughter). This transformation is a first order (abrupt change in lattice parameter), diffusionless (no rearrangement of atoms), solid-to-solid phase transformation (Bhattacharya, 2003), where the lower symmetry \( M \) phase can have different orientations (called variants) that are all related to the same high symmetry parent \( A \) orientation (the so-called correspondence variant). The number of \( M \) variants depends upon the change in symmetry during transformation, which is dependent on the crystal structures of the parent and daughter phases, respectively.

The existence of these two phases leads to the characteristic one way shape memory effect (OWSME), where a daughter phase deformed at low temperature (typically below the martensite finish temperature, \( M_f \)) has significant amounts of remnant strains upon unloading (pseudoplasticity), but returns to its original undeformed shape when heated to an elevated temperature (above the austenite finish temperature, \( A_f \)). Alternatively, when the same material is mechanically stressed at temperatures above the \( A_f \), it recovers its initial shape (with very little or no residual strain) immediately upon unloading. The ability to recover significant amounts of stress-induced strain at temperatures above the \( A_f \) is known as pseudoelastic (or superelastic) behavior. Both of these unique properties can be utilized to great advantage in many engineering applications.

In order to assess the potential of SMAs in advanced engineering applications, numerous experiments have been conducted over the past two decades. Most of these experimental studies have been limited to uniaxial tension and compression tests under isothermal conditions (Adharapurapu et al., 2006; Firstov et al., 2008; Gall et al., 2001, 2002; Gedouin et al., 2010; Lissenden et al., 2007; Patoor et al., 2006; Wada and Liu, 2008a,b; Wang et al., 2003, 2008). In addition to asymmetry in tension and compression (ATC), and OWSME, these experiments revealed the microscopic deformation behavior of SMAs. It was concluded that the complex microscopic and macroscopic behavior is due to the broad variation in the mechanical response with temperature, loading rate, strain range, specimen geometry, thermomechanical history, etc.

Another practical property of SMA is observed when the material is subjected to thermal cycling conditions, usually in the presence of some biasing stress. In this case, the material responds with a significant evolutionary character. As this occurs, a set of biased (preferred) variants are developed depending upon the loading conditions and prior thermomechanical history. This manifests into further exhibited behaviors such as biased OWSME, two-way shape memory effect (TWSME) and multiple-way shape memory effect (MWSME). Such “superthermal” characters are the keys to utilizing SMAs in actuator and active vibration isolation systems under cyclic loading conditions. Experiments have shown that evolution of the microstructure while thermal cycling under an applied stress (also known as training) leads to changes in transformation temperatures, hysteresis width and shape, as well as accumulated strains corresponding to the different underlying microstructures at the lower and upper bounds of the thermal excursion (Gall et al., 2001; Kockar et al., 2008; Lissenden et al., 2007; Patoor et al., 2006; Wada and Liu, 2008a,b; Wang et al., 2003, 2008).

We are motivated by the experimental observations and potential applications of SMAs in modern engineering. Our primary goal here is to develop a general, three dimensional (3D) constitutive material model to describe the two important classes of SMA behaviors; (i) supermechanical (e.g. pseudoelasticity, pseudoplasticity, ATC, isothermal major/minor attraction loops, etc.), and (ii) superthermal (e.g. cyclic thermomechanical training, isobaric major/minor loops, evolution under thermal loading, etc.). However, prior to doing so, we will briefly review the available SMA constitutive modeling approaches.

1.2. Constitutive modeling of SMA: state of art

Over the years, researchers presented many alternative approaches for modeling the complex responses exhibited by SMAs, following one of two approaches; (i) micromechanical, by considering the grain level microstructural changes of the SMA, and (ii) phenomenological, by isolating different energies associated with the phase transformation through internal stress variables.

Micromechanical models are mainly helpful in understanding the micro-scale behavior, such as nucleation, interface motion and twin growth because they utilize the microstructure of the two phases and the crystallographic texture of the material to predict response. However, they are difficult to implement for large scale structural problems. These models use thermodynamics laws to describe the transformation and utilize micromechanics to develop macro-scale state equations and kinetics (Gao et al., 2000; Govindjee and Miehe, 2001; Huang et al., 2000). For polycrystalline SMA models, micromechanics methods utilize single crystal response based on variant formation to describe polycrystalline response based on an assembly (using a set of micro–macro “transformation” rules) of single crystal grains (McDowell and Lim, 2002; Peng et al., 2008; Sittner and Novak, 2000). Most of these models take the transformation strains of the martensitic variants into account, and some, also consider self-accommodation of variants, reorientation of variants, interaction of variants, and other features of martensitic transforma-
tion (Auricchio et al., 2003; Auricchio and Sacco, 1997; Auricchio and Taylor, 1997; Brocca et al., 2002; Peultier et al., 2006a,b; Sadjadpour and Bhattacharya, 2007a,b; Siredey et al., 1999; Sittner and Novak, 2000; Thamburaja, 2005). These models employ numerous sets of constraint equations (e.g. positivity of phase fractions, conservation of mass, grain boundary conditions, grain interaction, etc.) in the minimization problem, which sometimes lead to a lack of convergence, and hence no unique solution (Bhattacharya, 2003) depending upon the formulations. Therefore, several assumptions are employed in order to reduce the intricacy of the problem. Furthermore, different types of averaging schemes have been employed in order to move from single crystal to polycrystalline response, but these attempts have proven to be highly restrictive in nature. Bhattacharya has concluded that the behavior of polycrystalline SMA material can be significantly different from that of a single crystal, rendering it very difficult to study the micromechanics of polycrystals (Bhattacharya, 2003). Models such as these have good potential for increased predictive capabilities, but require a large number of internal variables, extensive experimentation, and complicated parameterization procedures to achieve this objective.

On the other hand, the phenomenological models assume macroscopic energy functions that depend on internal state variables to describe global mechanical response while all the microscopic details are ignored. The evolution equations are often derived in conjunction with the second law of thermodynamics. These models are categorized as phenomenological, since they seek solutions to boundary value problems on the structural level by energy minimization, as in classical plasticity. Most of the early constitutive models (Auricchio and Sacco, 1997; Boyd and Lagoudas, 1994, 1996a,b; Brinson, 1993), and a large majority of more recent SMA modeling attempts (Kan and Kang, 2010) have been based on the use of the martensitic variants’ volume fractions as the primary internal state variables in an attempt to account for the influence of the microstructure. In one case, a rate independent one dimensional (1D) model based on Gibbs free energy was developed by Bo and Lagoudas (1999a,b,c), Lagoudas and Bo (1999). In this, the evolution of internal state variables (martensitic volume fraction, macro-transformation strain, and back and drag stresses) during phase transformation was proposed based on the micromechanical analysis over the representative volume element. The energy balance equation for the heat exchange during phase transformation was derived using the first law of thermodynamics, whereas the evolution equations for transformation strains, back, and drag stresses were described by a methodology similar to classical plasticity models. The model was very good in predicting the experimentally observed SMA behaviors under cyclic loading, and was studied and further extended to various 3D forms by Entchev and Lagoudas (2004), Lagoudas and Entchev (2004), Popov and Lagoudas (2007), Qidwai and Lagoudas (2000a,b). Typically, the treatment of SMA response under cyclic loading has been achieved through the introduction of two separate deformation measures; that is transformation strain and plastic strain (Arghavani et al., 2010; Auricchio et al., 2007; Feng and Sun, 2007; Hartl et al., 2010; Kan and Kang, 2010; Zaki and Moumni, 2007).

The study of SMA behavior under proportional and non-proportional, as well as multi-axial loading conditions has also been an active area of research. Very few multi-axial experiments have been reported in the literature (Grabe and Bruhns, 2009; Lim and McDowell, 1999; McNaney et al., 2003; Tobushi et al., 1991). Hence, most of the models are based on uniaxial test data. Papadopoulos highlighted the variations in experimental observation under multi-axial loading conditions (McNaney et al., 2003). Helm and Haupt (2003) proposed a three dimensional material model to describe the behavior of SMAs in multi-axial loading conditions, assuming small deformation and elastic behavior to be isotropic (Helm and Haupt, 2003). Bouvet et al. (2004) presented a model (limited to pseudoelastic response only) by introducing a novel concept of two different yield surfaces; one for forward phase transformation, and the other one for the reverse phase transformation (Bouvet et al., 2004). These surfaces were defined as a function of the second and third invariants of the stress tensor, in order to take into account the well-known tension–compression asymmetry. In addition, the concept of a scalar equivalent transformation strain quantity was introduced to determine the martensite volume fraction. Recently, an extension of the unified model of Leclercq and Lexcellent (1996) has been proposed by Panico and Brinson (2007), accounting for the effects of multi-axial stress states and non-proportional loading histories. This model is able to account for the evolution of both twinned and detwinned martensite, and martensite reorientation depending on the loading direction, by separating inelastic strain into two contributions, one from the creation of detwinned martensite and the other from the reorientation of previously existing martensite variants (Panico and Brinson, 2007). It is able to capture the main features of SMA behavior under multi-axial loading conditions, but lacks tension–compression asymmetry and response under cyclic loading conditions. An overview of the different SMA models can be found in Birman (1997), Lagoudas et al. (2006), Patoor et al. (2006). Furthermore, a more recent review with comparative assessment for the specific application in 1D cyclic situations for a number of newly proposed SMA models has been also provided by Kan and Kang (2010).

1.3. On the link between classical plasticity concepts and material modeling of SMA behavior

A careful examination of the various existing SMA formulations has revealed that the majority of the SMA constitutive models typically utilize well-known concepts of classical plasticity (Khan and Huang, 1995), as well as corresponding thermodynamic formulations to account for various transformation related phenomena associated with SMA behavior. In general, phase fractions and transformation strains are selected as internal state variables, with the rule of mixtures being used to compensate for the changes in “apparent” elastic moduli between the two phases. A linear relationship between phase fraction and transformation strain is typically employed, which limits the model to proportional loading cases only. The residual strains in pseudoelastic/pseudoplastic responses, and accumulation of residual strain during thermomechanical cycles, are usually attributed to the development of plastic strain, retained martensites, as well as other micromechanical defects. In particular, there is an ongoing discussion in the recent literature on the development of plastic strains during phase
transformation (e.g. see Kan and Kang, 2010; Saint-Sulpice et al., 2009). However for the most part, this has not been verified experimentally. Moreover, from the micromechanical point of view, Bhattacharya has emphasized that SMAs typically show very little plastic deformation and shear slip, particularly with regard to the avoidance of violating the conditions of non-degenerate phase transformation in compliance with the Cauchy–Born hypothesis (see Section 3.5 in Bhattacharya’s monograph (Bhattacharya, 2003)). Furthermore, separate transformation surfaces and flow rules of the non-associative type are often times employed to distinguish between the forward and reverse transformation.

A closer look at the observed behavior associated with plasticity and shape memory characteristics will reveal significant differences between these two properties of the material. For example, plastic flow is governed by the stress state of the material (i.e. stress limited surface), whereas SMAs exhibit transformation under constant stress during thermal loading. Similarly, under the conditions of cyclic ratcheting, the accumulated residual strain grows unboundedly in a plastic material, whereas a saturated finite (bounded) amount of transformation strains are produced in the counterpart situation over an SMA material. Furthermore, there is virtually no difference between yielding under tension and compression in a plastic material system, which is in sharp contrast to the marked differences observed in transformation-induced strains under tension, shear, and compression in SMA alloys. Furthermore, pressure sensitivity is sometimes exploited to account for the tension/compression asymmetry often observed (sometimes this can lead to transformation under a pure hydrostatic pressure state in contradiction with experimental evidences; e.g. see p. 146, Table 9.1 in Bhattacharya (2003)).

However, note that some of the above mentioned approaches constitute a radical deviation from the well established plasticity theories (see Fig. 1). In turn, this may possibly lead to problematic treatments that require careful consideration and warrant further attention.

From the viewpoint of mathematical soundness, classical plasticity theory rests on three fundamental principles (see Fig. 1); (i) normality, (ii) convexity of a single transformation surface defining both forward and reverse conditions, and (iii) associativity of the corresponding flow (Chen and Saleeb, 1994; Khan and Huang, 1995). In our view, these well established concepts are essential in any extension to the comprehensive modeling of SMA behavior. Unless compelling evidence from future experimental results suggest otherwise, we will adhere to all these desirable attributes in formulating the generalized framework presented here. In particular, we will later demonstrate (Section 4) that our framework, without violating any of the above three

Fig. 1. Shape memory effect, plasticity, and underlying mathematical theories.
conditions, will enable the capturing of all observed SMA characteristic responses that have been reported in the literature (Helm and Haupt, 2003; Lexcellent and Blanc, 2004; Lim and McDowell, 1999).

Clearly, there is a need for the development of an efficient, multi-scale, numerically robust computational model that can assess the SMA performance accurately and can be implemented in FE codes for practicing engineers. Hence, the work reported here is primarily motivated by the need to develop a sufficiently general three dimensional material model, accounting for the important effects of the phase transformations (collectively referred to as superthermal and supermechanical effects) in SMA. For this purpose, a multimechanism based, viscoelastoplastic framework with underlying mathematical constraints was adopted to develop a fully general, 3D SMA constitutive model. It is our objective to demonstrate the capabilities of our model without violating the well established mathematical requirements mentioned above (see Fig. 1). To this end, we extend previously developed work by Saleeb et al. (2001). The features targeted in our modeling are inspired by a number of recent experiments reported in the literature. This includes uniaxial and cyclic training tests (Auricchio et al., 2007; Lim and McDowell, 1995), and multi-axial stress states (tension–torsion experiments) with different controls (strain-, stress-, mixed-controls) as reported in Grabe and Bruhns (2009), Helm and Haupt (2003), McNaney et al. (2003), Noebe et al. (2005), Padula II et al. (2008), Panico and Brinson (2007).

1.4. The overall outline of the presentation

An outline of the remainder of the paper is as follows. In Section 2 we present an outline for the new proposed multimechanism based SMA model. This is followed in Section 3 by a detailed mathematical description involving the energy storage and energy dissipation and ensuing evolution equations for the set of conjugate “stress-like” and “strain-like” internal variables governing the behavior of the model.

In particular, there are a number of distinguishing features in these aforementioned Sections 2 and 3 that are emphasized in our formulation, which differ significantly from the recent alternative proposals reported for SMA modeling:

(a) We completely avoided the notion of phase fractions (Auricchio et al., 2009; Kan and Kang, 2010; Panico and Brinson, 2007; Popov and Lagoudas, 2007; Saint-Sulpice et al., 2009; Zaki and Moumni, 2007) as internal variables, and directly utilized a second-order tensor instead to capture all the transformational and orientational effects in deformations within the framework of a single flow rule governing both forward and reverse transformations.

(b) We do not use any of the “so-called” additional plastic strains (Feng and Sun, 2007; Hartl et al., 2010; Kan and Kang, 2010; Zaki and Moumni, 2007) to handle cyclic loads.

(c) We avoided the use of any “artificial” scalar accounting for the number cycles (Bo and Lagoudas, 1999b; Kan and Kang, 2010; Saint-Sulpice et al., 2009; Zaki and Moumni, 2007) in treating the training effects in SMA.

(d) We maintained the view that all transformation-induced deformations are essentially purely deviatoric in SMA, even when they exhibit marked ATC characteristics. This is in sharp contrast to the physically unattractive alternative of introducing the ATC through a purely pressure term (Auricchio et al., 2009; Kan and Kang, 2010; Qidwai and Lagoudas, 2000b) resulting in associated volume change (e.g. significant dilatation).

(e) We completely avoid any decomposition of strains into transformational and reorientational separately, as was used in some recent models (Arghavani et al., 2010; Panico and Brinson, 2007), or into transformational and separate visco-plastic strains in treating time-dependency of HTSMA as in (Hartl et al., 2010).

(f) Throughout all our derivations, we strictly adhere to a complete potential structure with a single associative flow rule, and strict normality relationship in the evolution equations, written in terms of the generalized space of both external stress tensor and internal state, stress-like tensors. This is in contrast to many recent alternative proposals as in (Popov and Lagoudas, 2007; Saint-Sulpice et al., 2009).

In Section 4, we present an extensive number of applications to demonstrate the capabilities of the proposed model in capturing experimentally observed SMA response characteristics. These include a wide range of variations of both stresses, and temperature and include monotonic, cyclic, as well as non-proportional, multi-axial loading conditions. In particular, it is demonstrated that, even with the use of a single set of material parameters, it is possible to capture a host of the observed phenomena (transient as well as stabilized) without resorting to the use of specialized formulations and/or independent parameter sets to capture each particular response regime exhibited by the same SMA material. This is in sharp contrast to the more common approach utilized in the alternative works reported over the years in modeling SMA; e.g. pseudoelasticity vs. pseudoplasticity, proportional vs. non-proportional loading conditions, monotonic vs. cyclic loadings, virgin vs. trained/stabilized materials, isobaric vs. isothermal, major vs. minor loops, and so on (e.g. see Arghavani et al., 2010; Auricchio et al., 2007, 2009; Bo and Lagoudas, 1999a,b,c; Hartl et al., 2010; Lagoudas and Bo, 1999; Levitas and Ozsoy, 2009; Paiva et al., 2005; Panico and Brinson, 2007; Peng et al., 2008).

Finally, a number of important concluding remarks are given in Section 5.

2. Proposed multimechanism based 3D SMA model

For convenience in later discussions, we use in this and in the following sections, a number of simple illustrations to introduce several definitions and terminologies. For instance, Fig. 2 shows an idealized “flag-like” stress–strain curve for pseudoelastic
response of SMA under uniaxial isothermal loading conditions. This is generally observed at a temperature, $T$, much higher than the finish temperature, $A_f$, of the transformation to the parent $A$ phase. The material can be loaded from zero stress (point "0" in Fig. 2) in the $A$ (parent) phase, and will follow essentially a purely elastic response until a critical stress level is reached (point "1" in Fig. 2). At this point, the start of the stress-induced, forward phase transformation from $A$ to $M$ occurs. In the idealized case, regions 1–2 is limited by a nearly constant stress (small hardening). This stress limitation in the present 1D representation corresponds to the limit stress transformation surface (states) in the 3D case. The end of the rapid transformation region is signified by the appearance of a rehardening region "2–3". State "3" corresponds to a fully-transformed material according to the classical viewpoint of the idealized material systems (i.e. a fully martensitic structure). Note that this latter region is bounded by the transformation, strain "magnitude", which is a key material property in SMA modeling (signifying the maximum volume fraction of all produced "M" phases at the end of transformation). This corresponds to a transformation strain bounding/limit surface in the 3D case. The part "3–4–5–6" represents the unloading, where region "4–5" corresponds to a rapid reverse transformation from martensite to austenite. Note that in real experiments, the final state "6" (complete unloading to zero stress) typically indicates the existence of "small" amounts of residual strains, which may be attributed to residual (retained) martensite. Also, note that many existing models in the SMA literature typically discard this feature; i.e., the idealization of purely austenite for state "6". Finally, the differences in critical stress levels for the onset of phase transformations (forward versus reverse), as well as the maximum amount of residual strain, give rise to a "flag" shaped stress–strain curve for these materials.

2.1. Scope and outline of formulation

For the purposes of this paper, we limited ourselves to the range of small deformations. In our mathematical formulation and the associated numerical implementation schemes, we followed the details given in Saleeb et al. (2001). A summary of some key points is given below.

We adopted a complete potential-based framework in terms of strain energy (Gibb's type, $\Phi$) and dissipation ($\Omega$) functions (Arnold and Saleeb, 1994; Saleeb et al., 2001; Saleeb and Wilt, 1993). These are expressed in terms of a number of internal state variables of the tensorial type that allow us to account for the unique features of the SMA response. With reference to the recent literature on SMA modeling, we refer to Luig and Bruhns (2008), where the importance of the use of such internal variables with directional properties is emphasized (in contrast to other "scalar" type quantities such as phase fractions that prevailed in the early developments of SMA models).

In particular, we utilize the following internal state variables (where "a" and "b" are counters denoting specific mechanisms):

1. A decomposed total strain tensor, $e_{ij} = e_{ij}^{\text{re}} + e_{ij}^{\text{I}} + e_{ij}^{\text{th}}$, where $e_{ij}^{\text{re}}$ accounts for the possible rate dependency of the individual phases ($A$ and $M$) within the material; $e_{ij}^{\text{I}}$ stands for the transformation induced deformations; and $e_{ij}^{\text{th}}$ denotes the thermal strain. For simplicity, and also because of their insignificant magnitudes (relative to the major part of transformation-induced strain, $e_{ij}^{\text{I}}$), we will discard thermal strains in all subsequent discussions.

Fig. 2. Schematic of typical stress strain curve for pseudoelastic response of SMA: (0) parent zero stress state; (1) beginning stress of forward phase transformation; (2) end of forward phase transformation; (3) phase in a rehardening(reorientation) region; (4) start of reverse phase transformation; (5) end of reverse phase transformation; (6) residual state. Note that the critical states marked in 1D correspond to limiting surfaces in 3D.
(2) A set of stress-like tensors, \( q_y^{(a)} \), and their associated strain-like conjugates (fluxes), \( p_y^{(a)} \), accounting for the rate dependencies of \( \varepsilon_y^{(a)} \).

(3) A set of nonlinear kinematic hardening stress tensors, \( a_y^{(b)} \), and their conjugates, \( c_y^{(b)} \), accounting for the partitioned energy storage/dissipation in conjunction with \( \varepsilon_y^{(b)} \).

Note that items (2) and (3) above will play a major role in the future treatment of high transformation temperature materials (e.g. NiTiPdX, and other ternary or quaternary SMA alloys). More specifically, regarding item (3) above, we have elected to partition the various SMA mechanisms as follows (see the schematic in Fig. 3):

1. The first \((b = 1)\) accounts for the rapid developments of transformation strains near the critical state (state “1” depicted in Fig. 2) and is used to control the amounts and evolution of residual transformation strains (see part “5–6” in Fig. 2).
2. The second \((b = 2)\) accounts for the possible gradual hardening in the transformation regime (i.e. the “flag” region of pseudoelastic response depicted in Fig. 2). Together with (1) above, this mechanism is also used to control the amount of thermal energy for transformation in superthermal effects.
3. The third \((b = 3)\) and final internal mechanism in the SMA block of Fig. 3 provides for an ever-increasing hardening function (as a novel extension of Saleeb and Arnold (2004), Saleeb et al. (2001)) that naturally leads to a limiting internal force denoting the completion of all phase transformations; see regions “2–3–4” in Fig. 2, and the final part in Fig. 4 (for uniaxial case) and Fig. 11 (for biaxial case in Section 4). This also enables us to bypass any complications in the “traditional” treatment of the condition of “a unit value for the sum of all volume-fractions-of-M-phases”, as utilized by others in the literature (e.g. Bekker and Brinson, 1997, 1998; Popov and Lagoudas, 2007; Qidwai and Lagoudas, 2000a,b).
4. The additional hardening-recovery mechanisms \((b = 4 \text{ to } N\) in Fig. 3) are reserved for the evolutionary character of SMA deformation response under cyclic thermomechanical load effects, both in pseudoelastic and pseudoplastic regions.

It is also important to note that the same evolution (rate) equations for the inelastic strains, \( \varepsilon^I \), are used in both the forward and the reverse transformations (see Eq. (27) in Section 3). This will completely bypass the need for any non-associative flow rules (e.g. see Lagoudas et al., 2006; Patoor et al., 2006; Qidwai and Lagoudas, 2000b). These evolutionary equations were developed through the use of transformation “yield” functions with curved meridians and a distorted (non-circular) shape in the deviatoric plane (along the lines described in Chapters 5 and 7 of Chen and Saleeb, 1994.; see also further elaboration in Fig. 13). This provides limiting conditions for phase transformation (i.e. from austenite to martensite, as well as possible reorientations of martensite variants). This will also enable the modeling of asymmetry in tension/shear/compression, thus often observed experimentally for SMA materials. We avoid the explicit use of phases (and the ensuing cumbersome
computations as in Refs. (Auricchio et al., 2003; Govindjee and Kasper, 1999; Lagoudas et al., 2006; Wang and Dai, 2010)). See specially the recent review on some advantages and some limitations on the use of multiple phase fractions to differentiate between twinned, detwinned, reoriented phase fractions under general loading (Popov and Lagoudas, 2007). Instead, we account, in $\mathbf{e}^I$, for all deformation-producing transformations; i.e. direct $A-M$, detwinning of $M$-variants, reorientations of any $M$-variant under non-proportional loading.

For the numerical integration scheme, we utilize the implicit backward Euler method for its unconditional stability and robustness (Saleeb and Wilt, 1993; Saleeb et al., 1998b, 2000). A concise form is developed by exploiting the mathematical structures of the model equations, leading to a very efficient implementation of the update and its algorithmic (consistent), (material) tangent stiffness. In particular, the closed-form, expressions for the tangent stiffness arrays, whose dimensions are independent of the number of variables employed, are derived (i.e. the stress tensor, tensorial viscoelastic internal parameters, and tensorial viscoplastic state variables). Especially, the dimension of the resulting arrays is only determined by the underlying problem dimensions (e.g. six for three dimensional problems, four for plane-strain, three for plane-stress problems etc.). These expressions and tangent stiffness have been proved effective in implementing the Newton iterative scheme utilized in the integration.

A brief outline of the formulation is given in Section 3. Finally, we refer to Saleeb and Arnold (2004), Saleeb et al. (2000), Saleeb et al. (2002) for many details of the derivations and algorithmic developments alluded to above.

3. Outline of the SMA multimechanism material model

We begin by summarizing the basic equations governing the behavior of a SMA material element. The following discussion is limited to the case of small deformations under isothermal conditions and stress free initial (virgin) state. A Cartesian frame of reference is utilized, along with indicial notation (wherein summation is implied for repeated "subscripts"). We also utilize “superscript” letters placed between parentheses as indices to identify sets of internal state parameters and when needed the summation over these will be indicated explicitly by the summation symbol.

Furthermore, it is noted that we do not consider here any heat transfer aspects (e.g. release/absorption of latent heat, etc.) for a formal thermodynamical treatment of the coupled thermomechanical problem. However, we give below a number of supporting arguments to justify our choice.

Firstly, under different rates of loading, it is often observed that the change in response of an “ordinary” SMA is due to the generation and absorption of latent heat. This causes local change in the temperature, leading to the “apparent” rate dependency in response. These were the main points emphasized in Grabe and Bruhns (2008), Hartl et al. (2010) Lim and McDowell (1999), Peyroux et al. (1998), Shaw and Kyriakides (1995), Tobushi et al. (1998) considering the pseudoelastic response with significant latent heat generation during the $A \rightarrow M$ transformation. On the other hand, there are other reported cases where ordinary transformation of NiTi SMAs exhibit significant relaxation and creep in the pseudoplastic regime, which was attributed to genuine viscoplasticity (Helm and Haupt, 2001, 2003). Note that in these later cases the main strain producing mechanism is detwinning/orientation of $M$ phase product (unlike the $A \rightarrow M$ transformation alluded to above).

Furthermore, significant variations in the SMA response are observed under very high rate of loading (Nemat-Nasser et al., 2005a,b; Nemat-Nasser and Guo, 2006), which cannot be explained solely on the basis of latent heat exchanges. Clearly, there is a debate in the literature on the heat transfer and rate dependency aspects in SMA response that needs to be concluded by systematic and more elaborate experimental programs.

In summary, the important points regarding the consideration of heat transfer and rate effect in our modeling approach are as follows:

1. In the application of HTSMA, viscoplasticity plays an important role in the rate dependency effects at elevated temperatures (Hartl et al., 2010; Kumar and Lagoudas, 2010; Lexcellent et al., 2005; Mukherjee, 1968). Therefore, it must be included in the constitutive modeling approaches targeting to capture the HTSMA response. We reemphasize that our intention is to provide a unified treatment for both ordinary SMAs (the focus in the present work), as well as the HTSMA response (considered in our future papers).
2. Under "strict" isothermal conditions and slow rates of loading, the effect of latent heat exchange leads to changes in specimen’s temperature; hence the testing condition can then be looked at as non-isothermal (or variable thermal–mechanical loading). This can easily be accounted for in our approach by simply providing the varying temperature history, along with the other mechanical controls (stress/strain), as applied “load controls” (i.e., input) during the analysis.

With the special emphasis on isothermal conditions and/or moderate or slow rate of temperature change, heat transfer effects will be minimized in the SMA response. Therefore, all the non-isothermal effects are simply reduced to a mere temperature-dependency of few material coefficients in the developed equations (e.g. see also the last part in Table 2 in Section 4).

Note, however, that this does not affect in any “significant” way the capability of the resulting model (e.g., we refer to Section 4 below with an extensive set of numerical results demonstrating the many desirable phenomena exhibited by the present SMA model). Finally, because of their rather small values (relative to the very significant magnitudes of transformation-induced strains), we here elected to discard any thermal strains in the subsequent discussion.
All equations are written for the general 3D case; but the forms are also directly applicable in subspaces (e.g., two-dimensional, 2D, plane-stress or plane-strain, etc.), see (Saleeb et al., 1998a,b; Saleeb and Wilt, 1993). For conciseness, we define the total number of dissipative viscoelastic state variables (each of the second-order tensor type) as $M$; i.e., the corresponding non-equilibrium stress tensor (Arnold et al., 2001; Saleeb and Arnold, 2001) are $q^{(a)}_{ij}$ ($a = 1, 2, \ldots, M$) and their associated conjugate (strain-like) tensors are $p^{(a)}_{ij}$. In addition, we define the equilibrium stress tensor $(\sigma_{ij})_{eq}$ and strain conjugate $e^{(eq)}_{ij}$. Similarly, for the viscoplastic mechanisms, we use the notation $\gamma^{(b)}_{ij}$ ($b = 1, 2, \ldots, N$) for the back stresses (kinematic hardening) and $\gamma^{(b)}_{ij}$ for their conjugate or dual (strain-like) variables, for a total of $N$ viscoplastic (second-order) tensorial state variables.

3.1. Potential and state equations

The total strain tensor, $\varepsilon_{ij}$, is assumed to be decomposed into two components; i.e. a reversible (i.e., elastic/viscoelastic), $\varepsilon^{e}_{ij}$ and irreversible (i.e. transformation-induced/viscoplastic), $\varepsilon^{i}_{ij}$, components:

$$\varepsilon_{ij} = \varepsilon^{e}_{ij} + \varepsilon^{i}_{ij}. \quad (1)$$

Recall that we utilize the tensor $\varepsilon^{i}_{ij}$ to implicitly account for all transformation-induced deformations; i.e., forward/reverse transformations between $A$ and $M$ phases at higher temperature, the detwinning of $M$-phase variants at lower temperature, as well as reorientations, variant coalescence and other allied effects under non-proportional states of stresses and strains. We also intend to include part of the loading rate- and time-dependency for treating HTSMA materials in the evolution equations of $\varepsilon^{i}_{ij}$ (see Eqs. (9) and (12) below).

The two fundamental energy potentials; that is, the Gibb's, complementary function, $\Phi$, and dissipation function, $\Omega$, are assumed to be decomposed into:

$$\Phi(\sigma_{ij}, \gamma^{(a)}_{ij}; q^{(a)}_{ij}) = \Phi_{R}(\sigma_{ij}, q^{(a)}_{ij}) + \Phi_{IR}(\gamma^{(a)}_{ij}), \quad (2)$$

$$\Omega(\sigma_{ij}, \gamma^{(a)}_{ij}; q^{(a)}_{ij}) = \Omega_{R}(q^{(a)}_{ij}) + \Omega_{IR}(\sigma_{ij}, \gamma^{(a)}_{ij}), \quad (3)$$

where the functional dependencies on the external $(\sigma_{ij})$ and internal $(q^{(a)}_{ij} and \gamma^{(a)}_{ij})$ state variables, with $(a = 1, 2, \ldots, M)$ and $(b = 1, 2, \ldots, N)$, and the conjugate variables $\varepsilon^{i}_{ij}$ (and also $\varepsilon^{e}_{ij}$, $\varepsilon^{(e)}_{ij}$ (also $p_{ij}$), and $\gamma_{ij}$, respectively, are given by the following equation of state:

$$\varepsilon_{ij} - \varepsilon^{i}_{ij} = \frac{\partial \Phi_{R}}{\partial (\sigma_{ij})_{eq}}, \quad (4)$$

$$\varepsilon^{e}_{ij} - p^{(a)}_{ij} = \frac{\partial \Phi_{R}}{\partial q^{(a)}_{ij}}, \quad (5)$$

$$\gamma^{(b)}_{ij} = \frac{\partial \Phi_{IR}}{\partial \gamma^{(b)}_{ij}}. \quad (6)$$

The specific functional form of the potential functions will be described later.

Note that in Eq. (5) only a single viscoelastic strain component is associated with all viscoelastic stress components. Also, for the viscoelastic response, the total stress is decomposed into an equilibrium stress, $(\sigma_{ij})_{eq}$, and a non-equilibrium (dissipative) reversible stress, $q_{ij}$, i.e.,

$$\sigma_{ij} = \sigma_{ij}^{(e)} + q_{ij}, \quad \text{where } q_{ij} = \sum_{a=1}^{M} q^{(a)}_{ij}. \quad (7)$$

Similarly, for the viscoplastic response, the total stress is decomposed again into an equilibrium, $(\sigma_{ij} - \gamma_{ij})$, and non-equilibrium, $\gamma_{ij}$, components, where the total back stress tensor is defined as,

$$\gamma_{ij} = \sum_{b=1}^{N} \gamma^{(b)}_{ij}. \quad (8)$$

From Eqs. (4)-(6), the corresponding rate forms are then,

$$\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{i}_{ij} = \frac{d}{dt} \left( \frac{\partial \Phi_{R}}{\partial (\sigma_{ij})_{eq}} \right) = \frac{\partial^{2} \Phi_{R}}{\partial (\sigma_{ij})_{eq} \partial (\sigma_{kl})_{eq}} \dot{(\sigma_{kl})}_{eq}, \quad (9)$$

$$\dot{\varepsilon}^{e}_{ij} - \dot{p}^{(a)}_{ij} = \frac{d}{dt} \left( \frac{\partial \Phi_{R}}{\partial q^{(a)}_{ij}} \right) = \frac{\partial^{2} \Phi_{R}}{\partial q^{(a)}_{ij} \partial q^{(a)}_{kl}} \dot{q}^{(a)}_{kl}, \quad (10)$$

$$\dot{\gamma}^{(b)}_{ij} = \frac{d}{dt} \left( \frac{\partial \Phi_{IR}}{\partial \gamma^{(b)}_{ij}} \right) = \frac{\partial^{2} \Phi_{IR}}{\partial \gamma^{(b)}_{ij} \partial \gamma^{(b)}_{kl}} \dot{\gamma}^{(b)}_{kl} \quad \text{or } \dot{\gamma}^{(b)}_{ij} = \left[ \frac{\partial^{2} \Phi_{IR}}{\partial \gamma^{(b)}_{ij} \partial \gamma^{(b)}_{kl}} \right]^{-1} \dot{\gamma}^{(b)}_{ij}, \quad (11)$$

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$$\dot{\varepsilon}^{e}_{ij} - \dot{p}^{(a)}_{ij} = \frac{d}{dt} \left( \frac{\partial \Phi_{R}}{\partial q^{(a)}_{ij}} \right) = \frac{\partial^{2} \Phi_{R}}{\partial q^{(a)}_{ij} \partial q^{(a)}_{kl}} \dot{q}^{(a)}_{kl}, \quad (10)$$

$$\dot{\gamma}^{(b)}_{ij} = \frac{d}{dt} \left( \frac{\partial \Phi_{IR}}{\partial \gamma^{(b)}_{ij}} \right) = \frac{\partial^{2} \Phi_{IR}}{\partial \gamma^{(b)}_{ij} \partial \gamma^{(b)}_{kl}} \dot{\gamma}^{(b)}_{kl} \quad \text{or } \dot{\gamma}^{(b)}_{ij} = \left[ \frac{\partial^{2} \Phi_{IR}}{\partial \gamma^{(b)}_{ij} \partial \gamma^{(b)}_{kl}} \right]^{-1} \dot{\gamma}^{(b)}_{ij}, \quad (11)$$
where the over dot represents the time derivatives (i.e., \( \dot{\cdot} \)). Eqs. (10) and (11) above constitute the rate equations for governing all of the (non-equilibrium) state variable for each inelastic mechanism.

Similarly, from the dissipation function (Eqs. (3) and (8)), the corresponding flow and evolution (rate) equations in terms of the “conjugate” internal state variables are obtained as follows:

\[
\dot{\gamma}^{(b)}_{ij} = -\frac{\partial \Omega^R_{ib}}{\partial q^{(b)}_{ij}}, \quad b = 1, 2, \ldots, N. \tag{13}
\]

\[
\dot{p}^{(a)}_{ij} = \frac{\partial \Omega^R_{ia}}{\partial q^{(a)}_{ij}}, \quad a = 1, 2, \ldots, M. \tag{14}
\]

### 3.2. Specific potential functional forms

For completeness, we review here the specified functional forms that have been utilized in the previous work by Arnold et al. (1995, 2001), Saleeb and Arnold (2001, 2004), Saleeb et al. (2001), Saleeb and Wilt (1993) to motivate the several “significant” extensions needed for SMA materials. We have utilized quadratic form for the viscoelastic contribution and the same “power-type” nonlinear forms for the viscoplasticity functions \( H(b) \), \( \Omega_1 \), and \( \Omega_2^{(b)} \) in terms of the invariants of their respective arguments \( (x^{(b)}_a, (\sigma_{ij} - x^{(b)}_a) \) and \( x^{(b)}_a \), respectively). However, a few, carefully-selected, functional forms for the hardening function, \( h(G^{(b)}) \) for the three dedicated mechanisms of energy storage (with no recovery contribution acting in any of them) in the SMA block \( b \in \{1, 2, 3\} \), and \( h(G^{(b)}) \) for \( b \geq 4 \) (with possible thermal recovery acting in each for the HTSMA cases) in the remaining dissipative mechanisms (see Fig. 3) will be investigated.

The specific forms for the reversible and irreversible potentials and driving functions are as follows:

\[
\Phi_R(\sigma_{ij}, x^{(a)}_{ij}) = \frac{1}{2} \sigma_{ij} E_{ijkl}(\sigma_{lk}) + \frac{1}{2} \sum_{a=1}^{M} q^{(a)}_{ij} [\mathbf{M}^{(a)}_{ijkl}]^{-1} q^{(a)}_{kl} + \sum_{a=1}^{M} q^{(a)}_{ij} p^{(a)}_{ij}. \tag{15}
\]

\[
\Phi^R(\sigma_{ij}, x^{(b)}_{ij}) = \sigma_{ij} e_{ij} + \sum_{b=1}^{N} H_{ib}(G^{(b)}), \tag{16}
\]

and

\[
\Omega^R(q^{(a)}_{ij}) = \frac{1}{2} \sum_{a=1}^{M} q^{(a)}_{ij} [H^{(a)}_{ijkl}]^{-1} q^{(a)}_{kl}. \tag{17}
\]

\[
\Omega^R((\sigma_{ij} - x_{ij}), x^{(b)}_{ij}) = \Omega_1(F(\sigma_{ij} - x_{ij})) + \sum_{b=1}^{N} \Omega_2^{(b)}(G^{(b)}(x^{(b)}_{ij})), \tag{18}
\]

where

\[
F(\sigma_{ij} - x_{ij}) = \frac{1}{\kappa^2} \left[ \frac{1}{2\rho^2} (\sigma_{ij} - x_{ij}) \right]_{ijkl} (\sigma_{lk} - x_{lk}) + a_1 (\sigma_{kk} - x_{kk}) + a_2 (\sigma_{kk} - x_{kk})^2 - 1, \tag{19}
\]

\[
G^{(b)}(x^{(b)}_{ij}) = \frac{1}{2\kappa^2(b)} (x^{(b)}_{ij}) \mathbf{M}^{(b)}_{ijkl} x^{(b)}_{kl}. \tag{20a}
\]

\[
g^{(b)}(x^{(b)}_{ij}) = \gamma^{(b)}(Z_m)_{ijkl} x^{(b)}_{kl}. \tag{20b}
\]

\[
\rho = \frac{1 + c\sqrt{d}}{1 + c\sqrt{d} + k}, \tag{20c}
\]

and where the specific Gibb’s functions entering the expressions for dissipation (Eq. (18)) and Gibb’s energies (Eq. (16)) are:

\[
\Omega_1(F) = \int \frac{\kappa^2 F_m}{2\mu} \, dF, \quad \Omega_2^{(b)}(G^{(b)}) = \kappa^2(b) \int \frac{r(G^{(b)})}{h(G^{(b)})} \, dG^{(b)}. \tag{21a}
\]

\[
H_{ib} = \begin{cases} 
\kappa^2(b) \int \frac{1}{h(G^{(b)})} \, dG^{(b)}, & \text{for } b = 1, 2, 3 \\
\kappa^2(b) \int \frac{1}{h(G^{(b)})} \, dG^{(b)}, & \text{for } b \geq 4.
\end{cases} \tag{21b}
\]
Note that in the above, \(E_{ijkl}\) and \(M_{ijkl}\) are fourth order tensors of viscoelastic stiffness moduli (with the corresponding compliance tensors \([E_{ijkl}]^{-1}\) and \([M_{ijkl}]^{-1}\), respectively); \(\eta_{\alpha}^{ij}\) (\(\alpha = 1, 2, \ldots, M\)) are the fourth order tensorial viscosity coefficients associated with the \(\alpha\)th dissipative viscoelastic mechanisms (e.g. for the use in applications with HTSMA). All the latter tensors are taken to correspond to isotropic behavior in the present applications reported in Section 4 (with corresponding material constants \(E_\alpha\) and \(\eta_{\alpha}^{ij}\) in \(E_{ijkl}\) and \(M_{ijkl}\), respectively). Typically, in applications, one takes \(\eta_{\alpha}^{ij} = \rho_{\alpha}^{ij} M_{ijkl}^{\alpha}\), where \(\rho_{\alpha}^{ij}\) is the characteristic relaxation time for each of the \(\alpha\)th viscoelastic mechanism. As alluded to earlier no considerations are made here for HTSMA, and the “viscoelasticity” will simply be reduced to an “elastic” part (see Table 2 in Section 4).

The \(\Omega_i\) and \(\Omega_x^{(b)}\) are the inelastic dissipations due to transformation-induced strain and associated hardening, and static (thermal) recovery (in HTSMA materials), respectively. Each of the functions \(H_{ij}\) is taken to be nonlinear in terms of the internal state tensor \(\chi_{ij}^{(b)}\), and/or its conjugate internal strain \(\gamma_{ij}^{(b)}\) for each of the hardening mechanisms. To emphasize the function dependency in the potentials, e.g. \(\Phi_R, \Phi_I\) and \(H_{ij}\), the corresponding arguments are shown in parentheses.

Furthermore, in compliance with the experimental fact that inelastic deformation of most SMAs are “essentially” independent of hydrostatic stress (incompressible), we selected the fourth order tensor, \(M_{ijkl}\), to be purely deviatoric (see Saleeb and Wilt, 1993; Saleeb et al., 2002), and Eq. (33) below). The limited amount of pressure sensitivity and associated volumetric dilatancies (if any) will be treated separately in our future work through the terms containing \(M_{ijkl}\).

The remaining functions that are needed to be assumed are those defining the hardening and recovery processes within the SMA material. For the thermal recovery, we take here a “power type” function (as before), that is, for \(b \geq 4\):

\[
r(C^{(b)}) = R_{ij}^{(b)}[G^{(b)}]^{m_{ij}},
\]

Note that Eq. (22) is shown here only for completeness since all applications in Section 4 were restricted to “ordinary” SMA material systems (not the HTSMA class operating at high temperatures). We have therefore, suppressed the recovery terms of all mechanisms here.

For the hardening functions, we separately treat in the sequel the individual energy-storage mechanisms (for \(b = 1, 2, 3\)) in the SMA block of Fig. 3 compared to the remaining mechanisms (i.e. \(b \geq 4\)) in the counterpart dissipative block.

3.2.1. SMA Energy storage block

The driving forces for the evolution of hardening in every one of these dedicated energy storage mechanisms are given by function “h” whose arguments are internal strains \(\gamma_{ij}^{(b)}\):

\[
h(g^{(b)}) = \rho_{ij}^{(b)} H_{ij} \left(\sqrt{g^{(b)}}\right)^{\frac{\beta_{ij}^{(b)} - 1}{\beta_{ij}^{(b)}}} \quad \text{for } b = 1, 2,
\]

\[
h(g^{(b)}) = \rho_{ij}^{(b)} H_{ij} \left[1 + \left(\frac{\sqrt{g^{(b)}}}{k_{ij}^{(b)}}\right)^{\beta_{ij}^{(b)}}\right] \quad \text{for } b = 3,
\]

where

\[
\rho_{ij}^{(b)} = \frac{1 + c_{ij} \sqrt{d_{ij}}}{1 + c_{ij} \sqrt{d_{ij}} + k_3^{(b)}},
\]

\[
k_3^{(b)} = \cos 3\theta^{(b)}.
\]

with Lode’s angle \(\theta^{(b)}\) being evaluated from the invariants of internal state tensor \(\chi_{ij}^{(b)}\) as per the well-known relations (e.g. Chen and Saleeb, 1994).

3.2.2. SMA energy dissipation block

On the other hand, for each of the dissipative mechanisms, we use functions with internal stress arguments that explicitly exhibit limit/bounding/saturation states (see Saleeb and Arnold, 2004). To this end, we make use of the Heaviside function \(h(L)\) with the loading index \(L = \chi_{ij}^{(b)} I_{ij}\), where \(I_{ij}\) is defined in Eq. (30) below, to account for the effects of cyclic and non-proportional loadings:

\[
h(C^{(b)}) = H_{ij} \left[1 - \left(\frac{\sqrt{G^{(b)}}}{\rho_{ij}^{(b)}}\right)^{\frac{\beta_{ij}^{(b)}}{\beta_{ij}^{(b)}}} h(L)\right], \quad \text{for } b \geq 4.
\]

Note that \(\rho_{ij}^{(b)}\) is defined as before. Note also that \(k_3\) in Fig. 3 and every \(k_3^{(b)}\) in Eq. (23d) above is always bounded in the range \(-1\) to \(+1\); \(c, \) and \(c_{ij}\) are positive material constants; and each of the material constants \(d\) and \(d_{ij}\) in Eqs. (20c) and (23c)
satisfy the convexity conditions of being positive and greater that one in magnitude (see Table 1 for a sampling of the “convexity” constraints connecting \(d\) and \(c\), or any of the pairs \(d_{(b)}\) and \(c_{(b)}\) for \(b = 1, 2, \ldots, N\)). It is also remarked here that all forms in Eqs. (23a), (23b), and (24) contain the same number of material constants.

### 3.3. Important remarks and implications

Several important remarks are in order here regarding the above selected forms:

1. There are several convex functions and a number of generalized normality rules employed here; i.e., “\(F\)”, “\(G\)”, and “\(g^{(b)}\)” as given in Eqs. (19) and (20); and Eqs. (4)–(6) and (12)–(14), respectively. In particular, note that with Eqs. (12), (19), (27), and (30) the normality of \(\bar{\sigma}^i_q\) is now in terms of generalized/effective stresses \((\sigma_g - \sigma_{lq})\), and not only the stress \(\sigma_g\). This latter property will prove essential in capturing some of the important observations in recent SMA experiments on non-proportional multi-axial loading (see also Section 4.A.6).

2. The dual nature of the arguments in the “\(h\)” functions will enable us to provide critical/limit states simultaneously for both internal stresses and transformation strains. While the function “\(h\)” for \(b = 1, 2\) tends to reduce rapidly and become bounded when internal “transformation” strains, \(\gamma_{(b)}^{(b)}\), tend to grow unboundedly, the corresponding state variables for \(b = 3\) (under the same conditions) will increase gradually to become unbounded. On the other hand, for every mechanism \(b \geq 4\), any tendency for the internal stresses to increase will eventually render all corresponding “\(h\)” bounded. Indeed, it is this “orchestrated” partitioning of evolutions in the tensorial hardening mechanisms under stress- and transformation strain controlled conditions that enabled the SMA model to produce the “favorable” test cases of Figs. 14 and 15 (see also LD = Limits on the Duals in Fig. 1).

3. As a corollary to (1) above, the appropriate use of “\(h\)” functions will automatically bypass any need for such alternative “specialized” treatment (often with further complications) as the Lagrange multipliers (Lagoudas et al., 2006; Patoor et al., 2006; Siredey et al., 1999), “polyconvex” constraints on the phase fractions (Govindjee and Kasper, 1999; Govindjee and Miehe, 2001), or use of indicator (singular) functions (Auricchio et al., 2007; Lim and McDowell, 1999) in order to introduce some of the “critical” states mentioned above.

4. As a result of the specific generalized normality rule (12)–(14), the thermodynamic admissibility condition in the form of dissipation inequality (2nd law of thermodynamics) is now guaranteed by the non-negativity of the dissipation rate functions \((\Omega, \Omega_b, \Omega_g)\), which is trivially satisfied by the convexity condition alluded to the item 1 above; i.e. a convex dissipation function \(F\) in Eq. (18) and \(\Omega_g\), and \(\Omega_g\) (see Eqs. (3) and (17)–(20)). For further details on these mathematical aspects of the presented model, we refer to Saleeb and Arnold (2001), Saleeb and Wilt (1993).

### 3.4. Material constants and temperature effects

Finally, in the above, \(\kappa, \mu, n, c, d\) represent inelastic flow material constants, whereas the \(H_{(b)}, \beta_{(b)}, c_{(b)}, d_{(b)}\) are hardening material constants. The \(R_{(b)}\) and \(m_{(b)}\) are recovery material constants; and the constants \(\kappa_{(b)}\) are hardening threshold stresses for the individual viscoplastic mechanisms \((b)\), where \(b = 1, 2, \ldots, N\), depending upon the form of the hardening function assumed in Eqs. (23) and (24).

From the theoretical standpoint, it is possible for any of these material parameters to be made dependent on temperature; however, if done “arbitrarily”, this will often require further complications in model characterization from experiments. Consequently, and also to indicate the great potential of the presented SMA model in the simplest setting, we have elected to restrict temperature-dependency to only a few inelastic hardening mechanisms; i.e., \(b = 1, 2,\) and 4. Furthermore, among the many available parameters governing these three mechanisms, we have chosen one single parameter; i.e., the threshold \(\kappa_{(b)}\) for this purpose (see also the last part of Table 2 in Section 4).

### 3.5. Resulting multi-axial flow and evolution equations

The following expressions, along with the particular functional forms above, constitute the governing associated flow and evolution equations in the present viscoelastoplastic model:

<table>
<thead>
<tr>
<th>Degree of ATC</th>
<th>Values of (c_{(b)}) in (c) (or (c_{(b)}) ≤ (c_{(b)}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increasing</td>
<td>0.565 (extreme case)</td>
</tr>
</tbody>
</table>

**Table 1**

Convexity constraints for different degrees of ATC.
Table 2
Set of material parameters used for simulated test cases.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Units</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Viscoelastic mechanisms</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>“Deflated” elastic stiffness modulus, $E$</td>
<td>MPa</td>
<td>35,000</td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu$</td>
<td>–</td>
<td>0.3</td>
</tr>
<tr>
<td>$E^{(1)}$</td>
<td>MPa</td>
<td>1.0</td>
</tr>
<tr>
<td>$\rho^{(1)}$</td>
<td>s</td>
<td>$10^{23}$</td>
</tr>
<tr>
<td><strong>Viscoplastic mechanisms</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of viscoplastic mechanisms</td>
<td>–</td>
<td>6</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>MPa</td>
<td>55</td>
</tr>
<tr>
<td>$n$</td>
<td>–</td>
<td>5</td>
</tr>
<tr>
<td>$\mu$</td>
<td>MPa s</td>
<td>$10^3$</td>
</tr>
<tr>
<td>$\kappa_{ij} b = 3, 5, 6$</td>
<td>MPa</td>
<td>8, 30, 350</td>
</tr>
<tr>
<td>$m_{ij} b = 1, 2, \ldots, 6$</td>
<td>–</td>
<td>2, 1, 1, 1, 1, 1, 1</td>
</tr>
<tr>
<td>$\rho_{ij} b = 1, 2, \ldots, 6$</td>
<td>–</td>
<td>14, 10, 1, 1, 1</td>
</tr>
<tr>
<td>$R_{ij} b = 1, 2, \ldots, 6$</td>
<td>$s^{-1}$</td>
<td>0.0, 0, 0, 0, 0, 0</td>
</tr>
<tr>
<td>$\Delta_{ij} b = 1, 2, \ldots, 6$</td>
<td>MPa</td>
<td>$50 \times 10^3, 30 \times 10^3, 100, 10 \times 10^3, 15 \times 10^3, 1150$</td>
</tr>
<tr>
<td><strong>Distortion Constants (used only for tension compression asymmetry, and yielding surface/stress transformation)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>–</td>
<td>0.56</td>
</tr>
<tr>
<td>$d$</td>
<td>–</td>
<td>1.01</td>
</tr>
<tr>
<td>$c_{ij} b = 1, 2, \ldots, 6$</td>
<td>–</td>
<td>0.56, 0.56, 0.56, 0.56, 0.56, 0.56</td>
</tr>
<tr>
<td>$d_{ij} b = 1, 2, \ldots, 6$</td>
<td>–</td>
<td>1.01, 1.01, 1.01, 1.01, 1.01, 1.01</td>
</tr>
<tr>
<td><strong>Temperature variation of strength parameters $\kappa_{ij}$ (interpolated linearly between temperatures, $T_1$ and $T_2$)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mechanisms</td>
<td>$T_1$</td>
<td>$T_2$</td>
</tr>
<tr>
<td>$b = 1, 2$</td>
<td>65 °C</td>
<td>120 °C</td>
</tr>
<tr>
<td>$b = 4$</td>
<td>30 °C</td>
<td>78 °C</td>
</tr>
<tr>
<td>$a$ In accordance with the available experimental evidences on “ordinary” SMA materials, we assumed that all inelastic (transformation-induced) deformations are deviatoric in nature, with zero values for both $a_1$ and $a_2$ in Eq. (19). Furthermore, we restricted all applications in Section 4 to the initially-isotropic material case, with $\xi$ and $\zeta$ being zero in Eqs. (33).</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\tilde{\sigma}_{ij} = E_{ijkl} (\dot{x}_{ij} - \dot{\varepsilon}_{ij}) + \sum_{a=1}^{M} q_{ijkl}^{(a)} ,
\]
\[
\dot{q}_{ijkl}^{(a)} = M_{ijkl}(\dot{x}_{ij} - \dot{\varepsilon}_{ij}) - M_{ijkl} h_{ij}^{-1} q_{ijkl}^{(a)} ,
\]
\[
\dot{\varepsilon}_{ij} = \begin{cases} f(F) \Gamma_{ij} & \text{if } F \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad \dot{f}(F) = \frac{F^n}{2\mu}
\]
\[
\pi_{ijkl}^{(b)} = Q_{ijkl} \left[ \dot{\varepsilon}_{ijkl} - \frac{\dot{\gamma}(G^{(b)}) \dot{h}(G^{(b)})}{h(G^{(b)})} \right] ,
\]
where the nonlinear fourth order $Q_{ijkl}$-operator for each hardening mechanisms can be straightforwardly derived from Eqs. (20)–(24) above. For example, we show below the case corresponding to $b = 4$:
\[
Q_{ijkl}^{(b)} = \left[ \frac{\partial^2 H_{ijkl}^{(b)}}{\partial \varepsilon_{ijkl}^{(b)}} \right]^{-1} = h(G^{(b)}) \left( Z_{ijkl} + \frac{h'(G^{(b)})}{h(G^{(b))}} \left[ 1 - \frac{h'(G^{(b)})}{h(G^{(b))}} (2\kappa_{ijkl}^{(b)} G^{(b)}) \right] \pi_{ijkl}^{(b)},
\]
\[
\Gamma_{ij} = \frac{\partial F}{\partial (\tilde{\sigma}_{ij} - \Delta_{ij})} ,
\]
\[
q_{ijkl} = \sum_{a=1}^{M} q_{ijkl}^{(a)} , \quad \pi_{ijkl} = \sum_{b=1}^{N} \pi_{ijkl}^{(b)} .
\]
In the above, we have made use of the following (recalling the earlier form of function $\dot{h}(L)$ in the part preceding Eq. (24) above):
\[
\dot{h}'(L) = \frac{\dot{h}(L) \partial h(L)}{\kappa_{ijkl}^{(b)} \partial C^{(b)}}, \quad \text{for } b = 4, 5, \ldots, N.
\]
Note $(Z_{ijkl})_{ijkl}$ is the “generalized” inverse of $\mathcal{M}_{ijkl}$ (see Ref. [Saleeb and Wilt, 1993] for further elaboration on this). Details regarding intermediate steps, extension to cases involving different degrees of initial anisotropy, which is important in
considering “texture” effects (e.g. see Ref. (Khan and Huang, 1995)) in SMA materials, and the numerical solution of these equations have been previously discussed (see Saleeb et al., 2001, 2003; Saleeb and Wilt, 1993). In particular, for a simpler case of anisotropy; i.e. with transverse isotropy (as in thin-walled cylinders) we have:

\[ \mathcal{H}_{ijkl} = \hat{P}_{ijkl} - \zeta A_{ijkl} - \frac{1}{2} \zeta S_{ijkl}, \]  

where

\[ \hat{P}_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{3} \delta_{ij} \delta_{kl}, \] 

\[ A_{ijkl} = \frac{1}{2} (D_{ik} \delta_{jl} + D_{il} \delta_{jk} + D_{jl} \delta_{ik} + D_{jk} \delta_{il}) - 2D_{ij} D_{kl}, \] 

\[ S_{ijkl} = 3D_{ij} D_{kl} - (D_{ij} \delta_{kl} + D_{kl} \delta_{ij}) + \frac{1}{3} \delta_{ij} \delta_{kl}, \] 

\[ \tilde{D}_{ij} = \tilde{v}_i \tilde{v}_j. \] 

In the above equations, \(0 \leq \zeta \leq 1\) and \(0 \leq \zeta \leq 1\) are the non-dimensional material strength ratios for anisotropy. In addition, \(\delta_{ij}\) is the Kronecker delta, and \(\tilde{v}_i\) represents the unit vector along the material fiber direction. Note that, when \(\zeta = \zeta = 0\) (i.e. isotropic case), \(\mathcal{H}_{ijkl}\) simply reduces to \(\hat{P}_{ijkl}\), leading to the classical von Mises-type forms (e.g. in some terms of function \(G^{(b)}\)). Note also that \(\zeta\) and \(\zeta\) will require experimental measurements of the transformation onset critical stresses with loading on the specimen aligned in different directions relative to the anisotropy direction \(\tilde{v}_i\) of the SMA material (i.e. four measurements involving two shear and two tension/compression components in the transverse plane normal to \(\tilde{v}_i\) as well as longitudinally along \(\tilde{v}_i\); see Saleeb and Wilt (1993) for further details). Data from experiments of this latter type are presently lacking in the literature. Therefore, all applications in Section 4 are restricted to the initially isotropic case.

3.6. Some useful guidelines for “estimating” the various material parameters

We are listing below some “simple” guidelines in order to help appreciating the role played by various material parameters for our SMA Material model presented above. This will prove quite useful in the actual “exercise” in characterizing an actual SMA material with specific composition, heat treatment, etc. (see also the further elaboration in Section 4).

1. \(E, v\) are taken of the order of SMA elastic modulus and Poisson’s ratio.
2. \(\kappa, n, \mu\) are inelastic, rate dependent response governing parameters. Currently, we are focusing on “nearly” rate-independent responses in SMA. Typically, \(\kappa\) signifies the critical transformation (“yield”) stress at the lowest temperature of interest (below \(M_f\)). The \(n\) (positive constant) shows the severity of rate effect, and \(\mu\) (ranging from \(10^5\) to \(10^{11}\)) for “large” values of \(n\) will give nearly rate independent behavior for the targeted range of slow and moderate loading rates (e.g. strain rate of \(10^{-2}\) s\(^{-1}\)–\(10^{-5}\) s\(^{-1}\)) we are focusing on.
3. The ratio of \(K_{(b)}/H_{(b)}\) (where \(b = 1 \rightarrow 6\)) determines the transformation strain ranges after which each hardening mechanism will reach its limiting values governed by \(K_{(b)}\). For example, \(K_{(3)}/H_{(3)}\) gives a good estimate for the beginning of rehardening strain (i.e. completion of forward transformation. See regions 2–3 in Fig. 2).
4. \(\beta_{(b)}\) (where \(b = 1 \rightarrow 6\)) dictates how fast each mechanism will show its maximum effect (i.e. degree of roundness in stress-strain or strain–temperature curves).
5. Mechanisms 1, 2, and 3 are used as energy storage mechanisms. The mechanisms 1 and 2 are mainly to account for temperature-dependency under variable thermal loading. These two mechanisms lead to transitional responses ranging from pseudoelastic to pseudoplastic responses as temperature decreases. Mechanism 3 is responsible for the termination of phase transformation.
6. Mechanisms 4, 5, and 6 are dissipative mechanisms, leading to hysteretic behavior in SMA responses. Mechanisms 4 and 5 will sense the hysteresis width and depth. Mechanism 6 works as an “ever evolving” mechanism to provide evolution under sustained cycles (e.g. see Fig. 19(a) in Section 4 below), and in determining the “long term” slopes of stress-transformation strain curves well into the highly developed phase transformation regime (see 2–8% strain transformation region in the pseudoplastic case reported in Fig. 4(b) below).
7. The \(c\) and \(d\) for flow and hardening mechanisms determines the ATC. The limiting values of these parameters are given in the Table 1. Due to the lack of proper experimental results and for simplicity, we used same value for every mechanism leading to the extreme ATC. However, these parameters will have different values for an actual SMA.
4. Results for simulation of tests demonstrating the modeling capabilities

A number of important modeling aspects have been described in the previous sections. We will present the results from several numerical simulations to confirm that the aforementioned concepts can be easily utilized to obtain aspects of the SMA response using our fully general 3D framework.

To this end, we consider the qualitative aspects of our SMA model under uniaxial, as well as multi-axial (both strain- and stress controlled) loading conditions. More specifically, we are presenting numerical simulations of the following cases:

A. Supermechanical aspects

1. Asymmetry in pseudoelastic and pseudoplastic response in tension and compression (strain controlled simulation).
2. Responses in strain controlled, proportional and non-proportional, multi-axial deformation path tests.
3. Responses in stress controlled, non-proportional, multi-axial deformation path tests.
4. Major/minor attraction loop character.
5. Projection on deviatoric stress (and strain) plane for the limiting transformation stress (and strain) surfaces.
6. Apparent deviation from normality under non-proportional loading path.

B. Superthermal aspects

7. One way shape memory behavior (OWSME).
8. OWSME and allied effects under non-proportional multi-axial loading conditions.
9. Transient response and typical behavior under isobaric tests (and the counterpart “isostrain”/constrained “stress-recovery”) conditions.
10. Evolutionary character under thermomechanical cycling.

The simulations are performed using a UMAT (user material) routine developed for the use with the commercial FE program ABAQUS® (ABACUS, 2008). The material parameters utilized in tests are listed in Table 2. We refer readers to Section 3 for a detailed definition of the various material constants.

The selection of parameters was based upon data taken from experimental observations reported for various SMA alloys having different compositions. To the best of our knowledge, there is presently a definite lack of complete/comprehensive sets of experimental results (proportional/non-proportional loading, isobaric cycles, constrained recovery, shear vs. tensile vs. compressive loading modes, sustained cycles to attraction/saturation states, extended thermomechanical cycles, etc.) describing all the features of one particular SMA system (that is a given composition with specific heat treatment). This renders it completely infeasible to attempt a direct characterization of our presented general model with one source (or one SMA) of data. Instead we have opted to emphasize qualitative aspects, based on what we perceive as “averaged” SMA behavior for typical ranges of transformation strains, start and finish temperature ranges, and/or stress magnitudes of test cases in the contemporary literature. However, a more systematic and automated characterization procedure will be called for to keep up with the extensive data required to completely determine all the material coefficients in the present general model with its targeted comprehensive scope of SMA evolutionary response (under sustained cycles with general loads). This will be part of our future work; guided by some of our previous experiences reported in Saleeb et al. (1998a, 2002).

With the above background in mind and facilitated by the “simple” guidelines we provided earlier in Section 3.6, we utilized the information from various reported experimental observations to construct a single parameter set for use with the model. The experimental results that the model predictions were compared to were obtained for different compositions of NiTi based SMA, along with some other non-NiTi based SMAs. Note also that, the distortion constants are used only for the cases of ATC (Fig. 4), and for the purpose of plotting the limiting transformation surfaces and transformation strain rate vector in Figs. 14 and 15, respectively. Furthermore, a strain rate of 10^{-4} per seconds (s^{-1}), stress rate of 0.125 MPa/s, and temperature rate of 0.012 °C/s were maintained in all simulations. While performing numerical tests, “as received” (virgin) material state had been assumed. In all subsequent plots, stresses, strains, and temperature are reported in MPa, percentage (%), and degree Celsius (°C), respectively. Furthermore, normal stress, shear stress, normal strain, shear strain, and temperature will be denoted by σ, τ, ε, γ, and T, respectively, in all the figures.

4.A.1. Asymmetry in tension and compression in pseudoelastic and pseudoplastic regimes

Asymmetry in tension and compression is often observed experimentally for many SMAs. To demonstrate this aspect, even in the case of inelastic incompressibility, we utilized material parameters as in Table 2. The two extremes of response under a 10% strain controlled test were obtained by changing the working temperature; i.e. 150 °C (T > Af) for pseudoelasticity, and 30 °C (T < Mf) for pseudoplasticity.

The model’s ability to capture the different degrees of asymmetry under strain controlled loading conditions is demonstrated in Fig. 4 for the two cases; i.e. pseudoelastic and pseudoplastic. Note that the ratios of maximum absolute stress in compression and in tension for the pseudoelastic and pseudoplastic responses are 1.417 and 1.218, respectively, and is in agreement with the experimentally observed behavior of NiTi (Adharapurapu et al., 2006; Lim and McDowell, 1999;
Liu et al., 1998). Finally, although the rate dependency aspect is not emphasized here, note that the present model is intrinsically rate dependent; so although this aspect is not emphasized here, we can still observe some small differences in the predicted response with an order of magnitude change in the applied strain rates, but all other pseudoelastic and pseudoplastic aspects remained intact [stresses in MPa, strains in %].

4.A.2. Responses in strain controlled, proportional and non-proportional, multi-axial deformation path tests

To demonstrate the complete 3D aspect of the model, several multi-axial deformation path tests have been considered. First, a radial deformation path (the normal and shear strains were proportional to each other; i.e. \( \gamma = \sqrt{3} \tan \phi \cdot \varepsilon \)) was considered, where each loading path belonged to one loading and unloading cycle. Values of the angle \( \phi \) equal to 0° (pure tension), 22.5°, 45°, 67.5° and 90° (pure shear) were used in the analysis (Fig. 5(a)) and the magnitude of the strain in each of the loading paths was 4%. The response in stress space is shown in Fig. 5(b). Qualitatively, the model response is similar to that observed experimentally (see Fig. 3(b), Helm and Haupt, 2003). Also, stress–strain plots are shown in Fig. 5(c) and (d). Note the significant opening in the hysteresis stress–strain loops plotted individually in Fig. 5(c) and (d), as opposed to the apparent narrower loop in the stress space in Fig. 5(b).

Next, we considered a non-proportional loading path in the form of a square (Fig. 6(a)) in strain space, where the first tensile loading corresponds to incomplete transformation followed by subsequent shear loading at constant normal strain. The model prediction of the conjugate stress path in Fig. 6(b) is similar to that observed experimentally (Fig. 6(c)).

The equivalent stress vs. equivalent strain plot for the square path test is shown in Fig. 7. The model is able to predict the sharp change in equivalent stress due to the formation of new \( M \) variants during shear loading. At the end of tension unload-
ing, there is an increase in the equivalent stress with decreasing applied equivalent strain, as observed experimentally (see Fig. 7(b)). The micromechanical ground for this behavior is not yet clear, however, it has been suggested that such behavior is due to the memory effect in SMA (McNaney et al., 2003).

When the model is subjected to a similar square-loading path, but the first tension loading reaches a level of strain corresponding to complete phase transformation, the material reorientation effect in the subsequent shear loading (Fig. 8) is observed. Unlike the previous case, the equivalent stress reduces monotonically with reducing equivalent strain. This further confirms the material reorientation effect during shear loading.

Next, a non-proportional deformation path test in the form of a butterfly in strain space was considered (Fig. 9(a)). The resulting stress path (Fig. 9(b)) is very similar to that observed experimentally in Fig. 9(c) (see also Fig. 8 in Grabe and Bruhns (2009)). Note that in all the path tests, the material was able to recover almost completely (i.e. very small residual stress). Qualitatively, the model is able to demonstrate all the characteristics of proportional and non-proportional, strain controlled loading path tests using the same parameterization for all cases.

4.A.3. Responses in stress controlled, non-proportional multi-axial deformation path tests showing continued transformation and reorientation effects

Two slightly different multi-axial stress controlled simulations were conducted to investigate the correctness of the unified theory. First, the material was loaded in tension to a stress of 300 MPa (lower than the critical stress for forward phase transformation), followed by an applied shear stress of magnitude 250 MPa while maintaining the normal stress constant.

Fig. 5. Radial deformation path test (proportional loading) at T = 150 °C: (a) loading path in strain space, (b) response path in stress space, (c) normal stress vs. normal strain, and (d) equivalent shear stress vs. equivalent shear strain [stresses in MPa, strains in %].
After this, the material was unloaded in tension and subsequently in shear. Note that the magnitudes of the applied stresses, individually (either tension or shear alone), do not produce any phase transformation, whereas the combined loading lead to transformation. At the end of the initial tension loading (path AB), the developed normal strain is 1.17% (Fig. 10(b)). The magnitude of normal strain further increases during the subsequent shear loading (2.31% normal strain and 5.84% shear strain in Fig. 10(b)). The transformation happens in the direction of loading, since both normal and shear components are present in path BC. Note also that the experimental result presented in Fig. 10(c) are pertinent to Cu–Al–Zn–Mn SMA (Sittner et al., 1995), whereas the same unified material parameter set was utilized here to obtain the results for this complex multi-axial loading path test. We emphasize again that the material constants for our “averaged” SMA material properties were determined from experimental data of many SMA compositions; hence it does not represent the specific Cu–Al–Zn–Mn SMA tested in Fig. 10(c), leading to the significant quantitative deviations in the predicted response of Fig. 10(c). However, it is important to be reminded that, qualitatively, all the essential measured response characters in Fig. 10(c); for example, note the similarity between Fig. 10(b) and (c) in showing the significant reorientation during path CD, and subsequent reverse transformation, recovering the initial state, at A’ following the initial forward transformation in shear along path BC.

A second stress controlled multi-axial stress test was performed such that the resulting path would appear rectangular (Fig. 11(a)) in stress space. Here, the material was initially loaded in tension up to complete phase transformation. Qualitatively, the loading path “ABC” was similar in the two cases, but the level of applied initial tension stress corresponds predominantly to the elastic regime in Fig. 10(a), whereas it corresponds to a completed phase transformation regime after tension loading in Fig. 11(a). Therefore, during the shear loading, a decrease in the normal strain is observed along path BC.
Note that in both test cases, our model was able to exhibit a significant amount of pseudoelasticity by recovering almost all of the inelastic strain upon unloading. Furthermore, since distortion constants are deactivated in this case, equal amounts of strain are observed in tension and compression. It is important to recall that the same single unified material parameters (Table 2) set was utilized here also. The parameterization was biased towards NiTi based SMA, whereas the experiments were performed on Cu–Al–Zn–Mn based SMA (see Figs. 10(c) and 11(c)). Nevertheless, our model was able to demonstrate the behavior observed experimentally for these non-proportional biaxial tests.

4.A.4. Evolutionary characteristics under isothermal conditions (major/minor loops)

In order to capture the existence of evolutionary character during sustained cycling, uniaxial, stress controlled, loading cycles were simulated; i.e. a number of partial loading and unloading between stresses corresponding to 525 MPa (upper stress level) and 350 MPa (lower stress level) were performed. Fig. 12(a) shows the behavior when the material was subjected to partial loading–unloading cycles in the initial segment of forward phase transformation (minor loops from left). In this case, the mean strain (average of strains at 350 MPa and 525 MPa) increased from 3.644% in the first cycle to the nearly saturated at 5.350% (as shown by the dotted line) in 35 cycles. Alternatively, when the material was initially loaded to nearly a level that completed the forward transformation, and then unloaded and reloaded repeatedly over the same stress range, the mean strain evolved from a starting value of 7.088% and approached 5.738% near saturation (minor loops from right as shown by the arrow in Fig. 12(b)). The loops from each case are attracted towards each other. Note the very good qualitative similarity between our model prediction (to reach an inner loop attraction state) and the experimental counterpart reported in Lim and McDowell (1995). In this latter reference, the minor loop cycles were performed after a number of major loop training cycles on a NiTi SMA alloy. It was observed that four minor loop cycles were needed to reach "saturation" under stress control between 150 MPa and 280 MPa (see Fig. 12(d), with cycles approaching from the left), as opposed to only one cycle to reach the stabilized inner loop cycle (approached from the right Fig. 12(d)) under the same range of stress cycling.

4.A.5. Determination of limiting stress surfaces under strain controlled tests

The determination of the limiting stress surface for initiation of the phase transformation (forward, $A \rightarrow M$ or reverse $M \rightarrow A$) under multi-axial proportional loadings is an important issue in the constitutive modeling of SMAs. This evidently needs multi-axial mechanical tests. In the literature, various experimental as well as theoretical procedures have been outlined in order to predict the transformation surface (Bouvet et al., 2004; Huang, 1999; Lexcellent and Blanc, 2004; Lexcellent et al., 2002, 2006; Lim and McDowell, 1999; Taillard et al., 2006).

The general shape of the transformation surface can be best described by its cross-sectional shapes in the deviatoric plane and its meridians on the meridian planes. The cross sections (Fig. 13) are the intersection curves between the transformation surface and a deviatoric plane (which is perpendicular to the hydrostatic axis). The meridians of the transformation surface are the intersection curves between the transformation surface and the plane containing the hydrostatic axis with constant angle of similarity, $\theta$ (also known as Lode's angle) (Chen and Saleeb, 1994). For an isotropic material, the cross sectional shape of transformation surface contains six-fold symmetry; therefore, the whole trace can be obtained by the symmetry of one sector $0^\circ \leq \theta \leq 60^\circ$ (see part ABC in Fig. 13). Note that the meridian planes corresponding to $\theta = 0^\circ$, $30^\circ$, and $60^\circ$ are called tensile (T), shear (S), and compressive (C) meridians, respectively.
Since phase transformation is essentially governed by shear, it would be more appropriate to plot the trace of the transformation surface in a deviatoric plane that passes through the origin (this plane is called $\pi$-plane), where every stress point represents a state of pure shear with no hydrostatic stress (or volumetric strain) component. Let $r_i$ be the projection of the $\sigma_i$ axis ($i = 1, 2, 3$ for three principal stress/strain axes as in Fig. 13) on the deviatoric plane. Then, the projection of the deviatoric stress (strain) vector on the $\sigma_1$ axis can be written as,

$$ q \cos \theta = \frac{1}{\sqrt{6}} (2s_1 - s_2 - s_3), $$

where $\rho$ is the length of the deviatoric stress (strain) vector and is defined as $\sqrt{2J_2}$, and $s_i$ (for $i = 1, 2, 3$) are the principal deviatoric stresses (strains). We refer readers to the Section 2.11 in Chen and Saleeb (1994) for further details. Furthermore, it can be shown that

$$ q \sin \theta = \frac{1}{\sqrt{2}} (s_2 - s_3). $$

**Fig. 8.** Square deformation path test (non-proportional loading) at $T = 150^\circ$C showing material reorientation effect: (a) loading path in strain space, (b) response path in stress space, and (c) equivalent stress vs. equivalent strain [stresses in MPa, strains in %].
To facilitate the 2D graphical representation, we plot Eq. (35) as abscissa (the $x$-axis) and Eq. (34) as ordinates (y-axis) for different stress states corresponding to different values of $h$ in the range between $0/C_{176}$ and $60/C_{176}$ to obtain one sector of the trace in the $p$-plane. Note that for stress controlled tests we can replace deviatoric stress with deviatoric strain in Eqs. (34) and (35) to obtain the trace of transformation surface in strain space.

To plot the trace of the transformation surface, we considered strain controlled, triaxial tests, with proportional normal, principal, strains (yielding constant magnitude for $\rho$) along each of the three axes. If $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ represent the applied strains along each axis (see Fig. 13), then their values conveniently parameterized by the variable $B$ as defined below:

\[
\begin{align*}
\frac{\varepsilon_2 - \varepsilon_3}{\varepsilon_1 - \varepsilon_3} &= B, \\
\varepsilon_1 + \varepsilon_2 + \varepsilon_3 &= 0, \\
\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 &= \rho^2.
\end{align*}
\]

(36)

Note that $\rho$ is the magnitude of the strain vector. Furthermore, it can be shown that the direction of this strain vector is given by,

\[
B = \frac{2\tan \theta}{\sqrt{3 + \tan^2 \theta}}.
\]

(37)
It is clear from Eq. (37) that the factor, $B$, will increase, monotonically, from 0, to 0.5, to 1 as $h$ varies from $C_1$ (tension path “T”), to $30^\circ$ (shear path “S”), to $60^\circ$ (compression path “C”), respectively.

The trace of six-fold symmetric projection (ABC in Fig. 13) of the transformation surface on the deviatoric plane is plotted following the procedure described below, in which we make use of the relationships given in Eqs. (34)–(37) above.

According to Eqs. (36) and (37), the experimental setup is guided by the two key parameters; the magnitude ($q$) of the control (stress or strain), and directionality ($B$). For demonstration purposes, we used a controlled strain magnitude ($q = 3.5\%$), at temperature, $T = 30^\circ C (T < M_f)$, in preparing the results plotted in Fig. 14(b). Although the results were prepared at the lower temperature, similar results would be obtained at any other temperature. When a strain-control condition was used for the projection of transformation surface, the magnitude of the transformation stress vector monotonically increases from a minimum in mode 1 (“T” mode) to a maximum in mode 3 (“C” mode) with an intermediate value being obtained in mode 2 (“S” mode).

On the other hand, when subjected to a stress controlled condition (Fig. 14(c)) that allows for the complete conversion to oriented martensite ($q = 800$ MPa), a reversal in the nature of the response is observed (compare Fig. 14(d) to (b)). In particular, the limiting value of the resulting transformation strain magnitude now becomes a maximum in the “T” mode, with intermediate in the “S” mode, and a minimum in the “C” mode. Furthermore, the “S” mode response is here shifted more towards the “T” mode, which is opposite to what is observed earlier in the strain controlled condition in Fig. 14(b). This logical, complete, and neat duality between the stress and strain spaces (LD, see Fig. 1) results naturally from the energy partitioning and the associated mathematical constructs that form the basis of the present model as presented in Section 3. Note that in both cases, the linear controls paths lead to nonlinearity in the predicted response paths.

Fig. 10. Stress controlled square path test at $T = 150^\circ C$; (a) applied loading path in stress subspace, (b) response in strain subspace, and (c) experimental observation on Cu-Al-Zn-Mn SMA (Sittner et al., 1995) [stresses in MPa, strains in %].
For the purpose of comparison with experimental observations, we are presenting the projection of the surface in Fig. 14(b) on the plane-stress plane along with the transformation strain rate vector (i.e. $\sigma_3 = 0$) in Fig. 14(e). The transformation strain rate vector can be obtained by two alternatives; (i) numerical differentiation of transformation strain histories obtained experimentally, and (ii) analytically by manipulation of the data stored for global and internal stress state variables. We, here, employed the “five point stencil” method to obtain the strain rate vectors.

It can be seen that the obtained surface in plane-stress subspace is very close to that obtained from a series of biaxial experiments conducted on CuAlBe SMA thin-walled tube specimens (Lexcellent et al., 2002), as shown in Fig. 14(f). The normality of transformation strain rate vector in plane-stress subspace has been well captured by our model.

4.A.6. On the proper treatment and interpretation of apparent deviation from normality

In one of the important fundamental studies dealing with deformational aspects of SMA responses, Lim and McDowell (1999) have presented the results of a comprehensive biaxial loading (tension–torsion) together with the mapping of the limiting stress transformation surfaces and associated strain rate vectors. The main objective in this study was to investigate the validity of the assumptions in classical plasticity theories when extended for the treatment of SMA materials. In particular, the issue of normality of the transformation strain rate vector to the limit surface in the stress space, and the related concepts of the associative flow rules were examined. From this work, three main conclusions were reached; (i) there is a significant deviation between the strain rate vector direction and the normal to the limit transformation surface plotted in the space of applied (global) stress components ($\sigma$ and $\tau$), (ii) simple classical plastic models purely formulated in terms of the “global” stress tensor components (e.g. $J_2-J_3$ type), and associative flow rules are not applicable for the observed SMA
Fig. 12. Minor attractor loops for more sustained loading cycles at $T = 150 \degree C$: (a) minor loops from left, (b) minor loops from right, (c) approach from left (red) and approach from right (green), and (d) experimentally observed behavior for a NiTi based SMA (Lim and McDowell, 1995) [stresses in MPa, strains in %]. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 13. General character of the trace of transformation surface in the deviatoric plane.
Fig. 14. Trace of transformation surface on π-plane at T = 30 °C; (a) controlling strain surface, (b) resulting limiting stress surface, (c) controlling stress surface, (d) resulting limiting strain surface, (e) projection of the critical stress transformation surface in (b) on the plane-stress plane (σ₁ = 0), and (f) experimental result for the critical transformation surface in the plane-stress subspace for CuAlBe SMA (Lexcellent et al., 2002). Note that, stresses are normalized w.r.t. maximum tensile stress in parts (e) and (f) [stresses in MPa, strains in %].
response, and (iii) there is a phase-angle shift between the vectors defining the stress path compared to the strain-path counterpart vectors (see also angles $\phi_i$ in Fig. 15 below). Since that time, these conclusions have been repeatedly misinterpreted by several researchers in the field (Auricchio et al., 2003, 2007; Helm and Haupt, 2003). We view such issues as delicate and requiring further attention. On the other hand, it is our intention in this brief section to show that, when interpreted differently in a more generalized sense involving internal state parameters (to account for the microstructural changes in the SMA material systems) one is indeed directly able to capture both of the above observed phenomena without violating normality nor associativity conditions.

In the work presented by Lim and McDowell (1999), the apparent deviation from the normality condition has been interpreted by some researchers as requiring non-associative flow rules (Auricchio et al., 2003, 2007; Boyd and Lagoudas, 1994; Entchev and Lagoudas, 2004; Helm and Haupt, 2003; Lagoudas and Entchev, 2004). We, therefore, demonstrated such an “apparent” deviation from normality in a strain controlled circular path test (Fig. 15(a)). The resulting vectors are shown in Fig. 15(b) along with the response of the material in stress space. Note that Lim and McDowell performed numerical differentiation to obtain the transformation strain rate vector. We refer to Lim and McDowell (1999) for the details of derivation of transformation strain rate vectors.

The similarity between our analytical results in Fig. 15(b) and the experimental counterpart presented (Fig. 15(c)) can clearly be seen. Note that, despite the similarity in obtained response in stress subspace (see Fig. 5(b) in Grabe and Bruhns (2009)), and the smooth and convex transformation surfaces in our 3D SMA model (Fig. 14), apparent deviation from normality is observed, which is the same as exhibited in the actual results reported by Lim and McDowell (1999).

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**Fig. 15.** Transformation strain rate vector on resulting stress path in (b) from applied circular strain path test in (a) at $T = 150^\circ C$, along with (c) experimental observation on NiTi SMA (Lim and McDowell, 1999); and (d) transformation strain rate vector resulting from a strain path similar to (a) but with radius corresponding to the beginning of forward transformation. Note that stress path is shifted by phase-angle $\phi_i$ ($i = 1 \rightarrow 4$) = $\{11.75^\circ, 6.31^\circ, 4.81^\circ, 11.27^\circ\}$ from respective strain path. Note also that the transformation strain vectors are nearly normal to the transformation surface at the onset of transformation in (d) [stress in MPa, strain in %].
On the other hand, when the material was subjected to similar non-proportional circular loading path with radius 0.175% strain, the resulting transformation strain rate vectors showed the normality in stress subspace (Fig. 15(d)). Note that 0.175% normal strain corresponds to the beginning/onset of the forward transformation in uniaxial tension test. This indicates that, during evolution, the transformation strain rate vectors will continuously change the direction moving from the extreme condition of nearly normal to the other extreme condition of nearly tangential relative to the trace of the transformation surface in the concerned stress subspace, which is also confirmed by Fig. 15(b) and (d). We will leave the discussion of detailed aspects, and the theoretical modeling significance, of changes in these transformation strain rate orientations to a separate future work publication.

The normality conditions in our model are satisfied by Eqs. (9)–(14), given in Section 3. More specifically, these normality conditions are defined in terms of generalized “effective” stresses involving the internal variables not merely in terms of the applied (“global” or “overall”) stresses. That is, individual internal mechanism components exhibit normality with respect to their respective surfaces, whereas the total (global) components, which are byproducts (resultant) of the internal mechanisms, may or may not respect the constraint conditions.

In the following subsections, we will demonstrate the capability of our multimechanism based constitutive model to present the superthermal effects using the same unified material parameters listed in Table 2. We remind the reader again that these parameters were obtained as average parameters reported for binary NiTi. Note that, the distortion constants are disabled in the following subsections.

4.B.1. The one way shape memory effect (OWSME)

The most elementary superthermal effect that has been successfully utilized in the advanced engineering industry, known as OWSME, is considered here. In this, the material was loaded under controlled stress to 300 MPa and then unloaded to zero stress at 30 °C (fully martensite). This was followed by heating to 150 °C (>\text{\(A_f\)}) and cooling to 30 °C (<\text{\(M_f\)}) at zero stress. The obtained result is presented in Fig. 16 in 3D strain–stress–temperature space.

During the isothermal loading–unloading, the material follows the path ABC and accumulated a residual strain of 3.25% strain at 30 °C. This residual strain was recovered during the heating process to 0.111% strain at point ‘D’, followed by the final residual strain at base operation temperature (30 °C) of 0.152%. This slight increase is due to the fact that the OWSME test was performed on the virgin specimen, and not the trained one. This transient behavior will be explored in the next sections. As can be seen in Fig. 16, the OWSME behavior predicted is very similar to that observed experimentally, although the magnitudes of the counterparts in Fig. 16(b) are different since the data was for a high temperature NiTiPd alloy (Noebe et al., 2005).

4.B.2. Generalized OWSME and allied effects under non-proportional multi-axial loading conditions

Grabe and Bruhns (2009) presented a series of novel biaxial (tension/torsion) tests under variable temperature conditions. The main objective in these experiments was to study the effects of loading sequence, and interaction between temperature and stress-state on the SMA material response. The experimental protocol involved a square shaped normal/shear stress path (see Fig. 17(a)), in which (i) the material was unloaded in the \text{\(M\)} phase, and (ii) material was unloaded in the \text{\(A\)} phase. Note that the loading in tension and shear for all the cases were done in the \text{\(M\)} phase. It was concluded that the strain recovery (related to OWSME) in the shear and normal directions were independent of each other, and that the shift in the \text{\(M\)} var-

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**Fig. 16.** The one way shape memory effect (OWSME). (a) Model prediction and (b) experimental observation on NiPtTi alloys (Noebe et al., 2005) [stress in MPa, strain in %, temperature in °C].
iant specific transformation temperature was dependent on the stress tensor (or stress state) and prior thermomechanical loading history.

Here, we tried to simulate two of such cases by following the experimental protocol described in Grabe and Bruhns (2009). In case (i), a stress controlled, square-loading path is considered, where the whole loading and unloading sequence is being performed in the M phase \( (T = 30\, \text{°C}) \), followed by heating and subsequent cooling between \( 150\, \text{°C} \) and \( 20\, \text{°C} \) (Fig. 17(a), path ABCDAME). The loading path ABC in case (ii) coincides with the first case along path ABC and until point C (i.e. tension followed by shear loading). Thereafter, the material is heated to \( 150\, \text{°C} \), while maintaining the existing stress state. The unloading in tension and shear is done at high temperature, followed by cooling to the same \( 20\, \text{°C} \) at zero stress (path ABCKLME in Fig. 17(a)).

The obtained results are shown in Fig. 17(b). As observed experimentally, both the tensile and shear strains increase during the initial tension unload, followed by increase in shear strain and decrease in normal strain. At the end of shear unloading, 0.47% and 1.08% residual normal and shear strains, respectively, are obtained, which diminishes to 0.06% and 0.1% during the strain recovery process (i.e. heating followed by cooling).

During the heating process, case (ii) (where unloading occurs in the A phase) shows a slight increase in strains during the initial tension unload, followed by increase in shear strain and decrease in normal strain. At the end of shear unloading, 0.47% and 1.08% residual normal and shear strains, respectively, are obtained, which diminishes to 0.06% and 0.1% during the strain recovery process (i.e. heating followed by cooling).

The apparent differences in measured response segments between AB\(_1\)C\(_1\) & AB\(_2\)C\(_2\) of part (d) in this figure, see further comments in the text of Section 4.B.2 [stress in MPa, strain in %, temperature in °C].

4.B.3. Transient response under isobaric tests (constant stress) and counterpart isostrain test (constant strain with stress recovery) conditions

To demonstrate the transient response of an untrained SMA, the material was first loaded to 250 MPa at 30 °C in the fully martensite state. This was followed by heating to 150 °C (where the material was presumed to be in the fully austenitic

![Fig. 17. One way shape memory effect under biaxial loading conditions. (a) Stress controlled loading paths: (i) path ABCDAME and (ii) path ABCKLME; (b) response in stress–strain–temperature space; (c) 2D projection of response on strain plane; and (d) experimental observation on NiTi SMA (from Tables A.1, A.2 in Grabe and Bruhns (2009)). For the apparent differences in measured response segments between AB\(_1\)C\(_1\) & AB\(_2\)C\(_2\) of part (d) in this figure, see further comments in the text of Section 4.B.2 [stress in MPa, strain in %, temperature in °C].]
state) and subsequent cooling to the original 30 °C, while maintaining the applied stress constant. The obtained results are shown in Fig. 18(a).

Here, the normal strain increases from 2.243% at the end of isothermal loading to 3.218% at 65 °C, and then remains almost constant until 80.7 °C. After that, the strain gradually decreases to a final value of 2.561% at 117.3 °C, followed by a relatively sharper drop to 2.352% at 122.4 °C, and finally 2.180% at 150 °C. Note that this strain is higher than the strain level if the material was loaded at 150 °C (see Fig. 12). This transient behavior (slight increase followed by reduction in strain, see also Fig. 19(c)) is observed only when the initial loading is done at low (lower than $M_f$) temperature (Padula II et al., 2008). This is yet another example of the framework’s ability to capture experimentally observable, transient behavior.

To investigate further into the transient response, the same virgin material was now subjected to 2.25% of normal strain (conjugate to nearly 250 MPa normal stress) at 30 °C, followed by heating to 150 °C at constant strain. This time, the response of the material again showed the complementary behavior of SMA characteristics in stress–strain–temperature space (Fig. 18(b)). Unlike the previous case, the conjugate stress remains constant at 257 MPa until 64.8 °C. It then drops sharply to another constant value of 237 MPa (between 67.2 °C and 81.6 °C), and then increases to 338 MPa. Note that this behavior is opposite to that for the isobaric (stress controlled) test condition, in that the conjugate stress decreased and saturated quickly, whereas the conjugate strain (in the stress-controlled test) increased and showed no signs of saturation.

Finally, we observe, under both test conditions in Fig. 18, the rather significant changes in response during the first cooling phase (following the initial heating from “virgin state”). In particular, note the distinct shapes of these initial “transient” loops in Fig. 18, in comparison to their counterpart “evolved” loops that are attained with training of the material in Fig. 19 that is depicted below.

### 4.8.4. Evolution under thermal cycling at constant stress/strain

The test cases in Fig. 18 were further subjected to five thermal cycles between 30 °C and 150 °C. The evolutionary character of SMA is depicted in Fig. 19. The isobaric case shows consistent evolution of increasing strain at low as well as high temperature (Fig. 19(a)), whereas the constrained recovery case shows a decrease in stress with cycles (Fig. 19(b)). Note that the isobaric case does not show any sign of saturation, which is in good agreement with the results presented by Padula II et al. (2008) (see Fig. 19(c)). The constrained recovery case, however, saturated in almost three cycles (i.e. no significant change in stress at either temperature levels), which is similar to experimental observation on Ti$_{50}$Ni$_{45}$Cu$_{5}$ SMA (Šittner et al., 2000) in Fig. 19(d). Furthermore, we also observe a difference in the average thermal loop width (temperature difference between the average of heating curve and cooling curve during transformation), which is narrower when strain control is used. These unique characters of SMA are very important as they are intimately linked to the conditions that will produce an optimal training condition for the material.

Note that, with very few exceptions, the coefficient of thermal expansion has been discarded in most of the numerical results presented. Typically, the coefficient of thermal expansion of Ni–Ti alloys are of order of 10$^{-5}$/°C. In all our simulations, the highest temperature difference reached was of order of 100 °C. This will lead to a strain difference of order of 0.1% in non-isothermal (or superthermal) cases under stress control, which has little effect on the character of the solution. However, the effect of thermal expansion may be relatively more prominent in the constrained recovery (constant controlled strain and varying temperature cycles) case.
For quantitative comparison, we are also providing the results for the two cases above; i.e., isobaric and constrained recovery tests but now with the effect of “linear” thermal expansion included. The corresponding results are compared in Fig. 19(a) and (b) for the specific value of the coefficient of thermal expansion of $10^{-5}/\degree C$. As mentioned earlier, the thermal strain effect is “hardly visible” compared to the overall transformations strains reported in Fig. 19(a) for the isobaric case. On the other hand, the effect of thermal expansion is relatively “more visible” in the constrained recovery case in Fig. 19(b). However, the main characteristic of the SMA evolutionary response is still maintained (in particular, note the attainment of “quick” saturation of cycles in the present constrained recovery case for both with or without thermal expansion effects).

5. Conclusion

Several important remarks are in order regarding the mathematical developments and numerical test cases presented in the previous sections. Although there is presently no general consensus in the SMA materials’ community on the details of all of the physical mechanisms responsible for the observed behaviors in these materials, it is generally agreed that the deformation response is strongly linked to the microstructural evolution that occurs in response to changes in both the stress and temperature conditions imposed. In the mathematical formulation presented, we have therefore focused on accounting directly for this evolutionary character of the SMA material response through a careful partitioning of the stored and dissipated mechanical energies by utilizing the notion of multiplicity of inelastic mechanisms. The use of individual mechanisms with
dual sets of stress/strain tensorial variables, with markedly different evolution characteristics (rapidity, intensity, etc.), was shown to be effective in capturing many of supermechanical and superthermal phenomena that are important in modeling SMA materials.

The results of an extensive number of test cases were documented to demonstrate the formulation’s modeling capabilities. In this regard, we emphasized two main points:

1. When interpreted in the generalized space of “global” and “internal” state tensorial variables, our strict adherence to the well established mathematical requirements of normality, associativity, convexity, within the context of a single flow rule for both forward and reverse transformation did not detract from our ability to capture the “apparent” deviation from normality (when the transformation strain rate vectors are plotted together with the surfaces in the space of external/global stresses) that has been reported in some recent multi-axial, non-proportional experiments (see Section 4.6).

2. Even with the use of a single set of material parameters, it is possible to capture a host of the observed phenomena (transient as well as stabilized) without resorting to the use of specialized formulations and/or independent parameter sets to capture each particular response regime exhibited by the same SMA material; e.g. pseudoelasticity vs. pseudoplasticity, proportional vs. non-proportional loading conditions, monotonic vs. cyclic loadings, virgin vs. trained/stabilized materials, isobaric vs. isothermal, major vs. minor loops, etc.

In light of these results, a more comprehensive experimental investigation is warranted to guide the further development of the model. Such an approach should be intimately tied to the development of standard practices for the testing and evaluation of SMA materials, as deviations in the testing protocols can lead to unforeseen variability in aspects of the model parameterization. Hence, the proper characterization protocol for SMA/HTSMA materials must be used to address the following three aspects; i.e. the specifics associated with the evolutionary character of internal parameters, quantification of the appropriate material constants, and determination of their associated temperature-dependency ranges. Each of these aspects is critical to obtaining quantitatively accurate predictions across the entire range of observed responses. This will prove essential in future work dealing with the quantitative validation of the model presented in applications involving specific SMA alloys under widely varying conditions of stress and temperatures.

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