Automated Finite Element Analysis of Complex Dynamics of Primary System Traversed by Oscillatory Subsystem

Atef F. Saleeb & Abhimanyu Kumar

The University of Akron, Akron, OH, USA

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The extensive literature on a particular dynamics problem, the bridge-vehicle-interaction, has provided insights and understanding of the many complex issues involved, but the mathematical and numerical work reported remained too highly specialized for practicing engineers. In particular, the use of standard FEA packages is virtually nonexistent. To this end, we propose an alternative approach by considering the primary and secondary systems as deformable solids, undergoing large, overall relative sliding motion. This will directly account for important convective accelerations, and the possible contact/separation. The versatility of the proposed strategy has been demonstrated by the solution of an extensive number of benchmark problems.

Keywords Bridge vehicle interaction, Moving loads and masses, Parametric resonance, Contact interactions, Finite element analysis, Benchmark problems

1. INTRODUCTION

1.1. Motivation

An intriguing class of structural dynamics problems, which always attracted researchers and engineers, involves the dynamic response of a distributed elastic system (continuous structure) induced by a moving elastic subsystem. For example, highways or railway bridges and cranes involve a number of translating elastic subsystems [1]. The dynamic response quantities, such as deflections, stresses, etc., are always higher than those in static cases, and thus require accurate analyses of forces and stresses for a reliable design and service life prediction.

This class of problems can be categorize in two subgroups: (i) The so-called bridge—vehicle—interaction (BVI) problems in which the primary (“bridge”) structure is traversed by one or more secondary systems of oscillators (“vehicles”); and (ii) the “dual” class of axially moving media (AMM), in which a highly flexible primary structure is now interacting with an axially moving secondary system of support/load constraints (e.g. copy and printing machines, magnetic tape drives, hot and cold rolling process, deployment and retrieval in space exploration, fluid transmission system, etc.), thus leading to a case of variable—mass (variable—domain) problem. Unlike standard dynamic problems, where the forces (e.g. applied load, inertia, and elastic interaction force) are fixed in their spatial locations, these non-standard dynamic problems involve forces that are changing their spatial location with time, leading to singular terms in their governing differential equations. The main difficulty stems from the fact that a direct mathematical treatment within the classical Galerkin—assumed mode approach—will eventually lead to many non-standard forms of the inertia (mass), stiffness, as well as the damping operators, as a result of convective velocity and acceleration contributions [2]; e.g., see [3–6].

Over the years, these problems have generated an extensive body of literature because of their importance in many engineering applications. Although the works reported in the literature on this subject bring out many interesting aspects of the problem, the solution approaches reported to date remained highly specialized in nature (both mathematical treatments and the associated details of the algorithmic implementations); i.e., beyond the “ordinary” background acquired in typical engineering curriculum. For overall background see [1, 7] (some other pertinent references will be also mentioned later). Also, despite the availability of several large-scale, commercial general purpose, finite element (FE) packages, there is a lack of guidelines on integration of the specialized schemes for BVI problems to large-scale FE software packages for the benefit of practicing engineers.

Our main motivation here is to fill such a gap for the practitioners by providing a general procedure to solve these problems in present-day FE software packages. This alternative viewpoint offers two distinct advantages. Firstly, all complications due to the need for convective contributions (as in classical treatments) are bypassed. Secondly, the versatility and power of large-scale, commercial general purpose, FEA computations that can now directly be utilized will allow a much wider scope of problems; e.g., ranging from the simple beam/cable type to the more complex frames, plates, shells, two- and three-dimensional (2D/3D)
continua, with straightforward inclusions of such important effects as separation–reattachment, irregularities in road surface, and realistic damping/dissipation materials, etc.

In this paper, we will focus on the issues related to the BVI problems. In the following sections and subsections we will review the work reported in the literature for BVI problems, including selected formulations, followed by a proposed methodology and its numerical implementation. For demonstration purposes, we used ABAQUS [8] as a solver engine but the same procedure is applicable to other large-scale FE packages such as ADINA [9], ANSYS [10], LS-DYNA [11], etc.

1.2. Bridge Vehicle Interaction (BVI) Formulations

The dynamic response of distributed system carrying one or more traveling elastic subsystems has no steady state solution and stability is determined by the transient response solution. The complexity of problem is increased by a set of coupled partial differential equations, which are needed to be solved in order to get the response of the system. In general, three types of problems have been considered in the literature: the moving mass problem, where the coupling stiffness between the distributed structure and the elastic subsystem is assumed infinity; the moving load problem, where the inertia effect of the elastic subsystem is completely neglected; and the more realistic moving oscillator (sprung mass) problem, where significant inertia effect is present and the coupling stiffness is finite.

A number of studies have been conducted to find the dynamic response of a structure subjected to moving loads [5, 12–20]. Yang et al. showed that the response of the system was very different from that of moving load [21]. Pesterev and Bergman showed that the complexity of the problem involving an elastic beam carrying accelerating oscillator was the same as that for an oscillator moving with constant speed [22, 23]. Nikkhoo et al. proposed an approximate solution of these problems by limiting the inertial effect of the moving mass to the vertical acceleration and concluded that the effects of the centripetal and Coriolis accelerations of the moving mass were negligible for velocities below a critical velocity (defined in terms of beams fundamental frequency and span length) [24]. Furthermore, a number of models have been proposed to study the dynamic response of beams resting on an elastic foundation, subjected to moving mass/load [25–27]. A comprehensive literature survey and guidelines for the mathematical formulation of the dynamic behavior of solids under moving loads and moving masses have been provided by Fryba and Tzou [1, 7].

Very few researchers extended the BVI problem from 2D beams to 3D plates and shells [28–30]. Wu developed a new formulation for plate elements to study the response of the system for different configurations of the plate and the paths of moving load/mass [31–34]. To our knowledge, only the works reported by de Faria involve the analysis of some cylindrical and spherical shells under moving load/mass [35, 36]. To this end, it is clear that there is a need for unification of the theories and/or methodology to deal with this class of problems.

1.3. Representative Formulations for BVI

For convenience in later discussions, we will follow the general procedure for the mathematical formulation of the BVI problem in this section. In practice, the BVI system is treated as a very complex system due to the dependency of interaction between vehicle and bridge on many parameters; e.g., vehicle speed, material properties of bridge, and surface roughness. To reduce the mathematical complexity to an extent, typically the “bridge” is modeled as a 2D beam and “vehicles” as a spring–mass–damper system. The limiting cases of BVI analysis (moving mass or moving load) are achieved by changing the spring stiffness. In general, the governing equation of motion for a beam of length \( L \), subjected to \( N \) vehicles, can be represented as,

\[
\rho A \frac{\partial^2 w(x, t)}{\partial t^2} + C \frac{\partial w(x, t)}{\partial t} + EI \frac{\partial^4 w(x, t)}{\partial x^4} = \sum_{i=1}^{N} P_i(t) \delta(x - \xi_i(t)),
\]

where, \( w \) is the transverse deflection of beam at point \( x \) and time \( t \). The \( \rho, E, \) and \( C \) are mass density, linear elastic modulus, and viscous damping coefficient of the beam, respectively. The \( A \) and \( I \) are cross-sectional area and moment of inertia of the beam, respectively. The \( P_i(t) \) is the interaction forcing function due to the motion and location of \( i \)th vehicle. The \( \delta(x - \xi_i(t)) \) is Dirac delta function, where \( \xi_i(t) \) is the instantaneous position of \( i \)th vehicle. Eq. (1) is very similar to the equation of motion of forced vibration of beam, except here the forces are varying with time and instantaneous location of the vehicles.

Since the interaction forces are functions of time and instantaneous location of vehicle, the singular term on the right hand expression of Eq. (1) will eventually lead to convective terms (in terms of velocity and acceleration of vehicles) in the resulting system of equations (see Eqs. (7) and (9) below). As highlighted by Bathe, the convective terms lead to the non-symmetry in the coefficient matrices [2]. Furthermore, additional difficulty in the numerical treatment arises in problems with large convective terms, in which the standard Galerkin method cannot be applied to obtain the solution in these cases. This necessitates further modifications utilizing Galerkin–Petrov, upwind stabilization, etc. [2, 37, 38]. Surprisingly, despite its longstanding history in the area of fluid dynamics, there are very few serious studies on how to appropriately handle the effects of strong convection in the structural-dynamics field of BVI [3, 39]. This is important in view of the fact that many reported formulations dealt with “apparent” stiffness, inertia, and damping operators with convective contributions to handle the interaction dynamics of BVI problems (e.g. [5, 6, 23, 40]). More elaborate discussions of such issues can be found in section 7.4.3 in [2], and sections 3.3 and 22.8 in [38].

In general, the solution procedure adopted in the literature involves modal superposition followed by simplification of Eq. (1) according to the classical Galerkin method [41]. For demonstration purposes, we will consider the simplified case of...
single moving mass, \( M \), moving with initial velocity, \( v_0 \), and a constant acceleration, \( a \) (Fig. 1). Furthermore, continuous contact assumption and only two (\( n \)th and \( m \)th) vibration modes will be utilized for modal superposition. The governing differential equation of motion (Eq. (1)), for this simplified case, will reduce to,

\[
\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + C \frac{\partial w(x,t)}{\partial t} + E I \frac{\partial^4 w(x,t)}{\partial x^4} = M \left( g - \frac{d^2 u(t)}{dt^2} \right) \delta (x - \xi (t)),
\]

where,

\[
\delta (x - \xi (t)) = v_o t + \frac{1}{2} a t^2.
\]  

The function \( u(t) \) is the displacement of mass at time \( t \), and \( g \) is the acceleration due to gravity. Compatibility of the system will lead to,

\[
u(t) = w(\xi(t), t).
\]  

For the sake of representation, we will symbolize \( w(x,t) \) as \( w, u(t) \) as \( u \) and \( \xi(t) \) as \( \xi \). Therefore, upon simplification, the equation of motion of beam can be written as,

\[
\rho A \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} + E I \frac{\partial^4 w}{\partial x^4} = M \left\{ g - \frac{\partial^2 w}{\partial t^2} - 2(v_0 + at) \frac{\partial^2 w}{\partial x \partial t} - (v_0 + at)^2 \frac{\partial^2 w}{\partial x^2} - a \frac{\partial w}{\partial x} \right\} \delta (x - \xi).
\]

For a simply supported beam, the displacement function, \( w(x,t) \), from the first two modes of vibration can be represented as,

\[
w = Q \Phi,
\]

where, \( q_i \) (\( i = n, m \)) are functions of time, \( t \), and the dot symbol indicates a dot (scalar) product between two vectors \( Q \) and \( \Phi \). The substitution from Eq. (6) into Eq. (5), and followed by multiplication of both sides by \( \Phi \), and integrating between 0 and \( L \), we get,

\[
M \dot{Q} + C Q + K Q = M g \Phi |_{x=\xi},
\]

where, we have defined the tow 2×2 arrays \( M_e, C_e \), and \( K_e \) as

\[
M_e = \begin{bmatrix}
\frac{\rho AL}{2} \left\{ 1 + \frac{2M}{\rho AL} \sin^2 \left( \frac{n \pi \xi}{L} \right) \right\} \\
M \left( \sin \left( \frac{n \pi \xi}{L} \right) \sin \left( \frac{m \pi \xi}{L} \right) \right) \\
M \left( \sin \left( \frac{n \pi \xi}{L} \right) \sin \left( \frac{m \pi \xi}{L} \right) \right) \\
\frac{\rho AL}{2} \left\{ 1 + \frac{2M}{\rho AL} \sin^2 \left( \frac{m \pi \xi}{L} \right) \right\}
\end{bmatrix},
\]

\[
C_e = \begin{bmatrix}
\frac{CL}{2} \left\{ 1 + n \frac{2M \pi (v_0 + at)}{CL^2} \sin \left( \frac{2n \pi \xi}{L} \right) \right\} \\
\frac{2M \pi (v_0 + at)}{L} \sin \left( \frac{m \pi \xi}{L} \right) \cos \left( \frac{m \pi \xi}{L} \right) \\
\frac{2M n \pi (v_0 + at)}{L} \cos \left( \frac{n \pi \xi}{L} \right) \sin \left( \frac{m \pi \xi}{L} \right) \\
\frac{CL}{2} \left\{ 1 + n \frac{2M \pi (v_0 + at)}{CL^2} \sin \left( \frac{2m \pi \xi}{L} \right) \right\}
\end{bmatrix},
\]

\[
K_e = \begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix}.
\]

The coefficient matrices in Eq. (7), \( M_e, C_e \), and \( K_e \), are usually termed as apparent mass, damping, and stiffness,
respectively. Note that the purpose of this exercise here is to show the delicacy of the underlying method that has been repeatedly reported in the literature. However, we do not advocate such a method due to its specialized nature and limited scope.

Remarks

1. For the multi-degree-of-freedom (MDOF) case using $N$ terms in modal superposition, each coefficient will be an $N \times N$ matrix; all with time-dependent entries by conveniently specializing the counters $m, n = 1, 2, \ldots, N$.

2. Note the appearance of convective terms in the bracketed terms in the coefficients of Eq. (8). In particular, consider $C_e$ and $K_e$, which, depending on the system parameters ($\rho, A, L, m, v, a$, etc.), can either become positive, negative, or definite/semi-definite during the time history. This will then trigger the significant changes in the character of the governing MDOF system of equations; i.e., from hyperbolic to elliptic or parabolic. Furthermore, the symmetry of stiffness matrix in Eq. (8c) is lost, which has a significant importance in FE procedure.

3. Considering the simplified case of constant vehicle speed and using only the first mode in Eq. (6) for modal superposition, for which the apparent stiffness will then reduce to a scalar,

$$K_e = \frac{\pi^4 EI}{2L^3} \left\{ 1 - \frac{2}{\pi^2} \left( \frac{m}{\rho AL} \right) \left( \frac{\omega^2}{c_f^2} \right) \sin^2 \left( \frac{\pi \xi}{L} \right) \right\}, \quad (9)$$

where, $c_f$ is flexure wave speed of beam ($c_f = (EI/\rho AL^2)^{1/2}$). The critical condition of the second term in the curly brackets of Eq. (9) being numerically equal to 1 will provide a demarcation between subsonic- to supersonic-like response regimes for wave propagation in the beam due to the moving source (the right-hand side of Eq. (7)). This constitutes the basis of the study of wave propagation in weak and/or supporting media (e.g. soil) [42–44]. This will be the direction of our future work; i.e., response of structure under vehicles moving with the speed near to wave speed in the material of the structure.

4. In the terminology of continuum mechanics, the use of two different reference frames is implied in Eqs. (1)–(5). In particular, the $w(x, t)$ and $w(\xi, t)$ are material (Lagrangian) and spatial (Eulerian) fields for the description of transverse vibration of the beam (“bridge”) and moving oscillator (“vehicles”), respectively. This is not the standard format utilized in the classical field of structural dynamics, where a purely Lagrangian formulation is typically used.

Typically, two major approaches are adopted by researchers to solve problems such as Eq. (7); i.e., numerical (discretized) and differential (continuous) forms. Either approach requires special numerical schemes for time integration and its implementation. For a sampling of the many proposals in the literature, we refer the reader to [4–6, 14, 17, 21, 28, 30, 33, 34, 36, 40, 45].

The majority of works employed such simplification as continued/persistent contact, or only the at–rest conditions for the primary and secondary systems irrespective of the number of traversing secondary oscillators in the BVI problems. Also, there has been a notable absence of any applications involving large-scale FEA codes. In fact, these were sometimes dismissed at the outset as inapplicable to the complex situations in the BVI problems [33, 40, 46]. To our knowledge, the only exception here is the one case in which a specialized beam element for a BVI analysis was utilized very recently in ANSYS [10] FEA code by Bowe and Mullarkey [47].

On the other hand, we strongly believe that, if the physics of BVI problems are carefully studied and understood, it is possible to analyze these problems directly, and efficiently using large-scale FE codes for both the transient as well as the steady state solution (if any). Along with the basic kinematical quantities such as displacement and velocity, these FE packages can also report other critical design parameters such as stresses, strains, and interaction forces. Another feature is the robust contact formulations that allow contacting points to separate and subsequently reattach during interaction, leading to a more realistic situation. In the numerical example section, and in order to support our viewpoint, we will present comparison results for different benchmark BVI problems, along with some new, more challenging problems for which there are no previous solutions reported in the literature.

2. Modeling of BVI Problems in Large-Scale FE Software

Our proposed methodology is outlined in the sequel. It is mainly intended to aid the practicing engineers to carry out the analysis (and design) for the complex dynamics of BVI and AMM problems.

The key to our proposed methodology lies in the modeling of the actual physics of the problem within the scope of nonlinear continuum mechanics [2, 48]. This is facilitated through the use of contact interaction capabilities of large-scale FE computations. In this, the large overall sliding (relative) motion between the primary and secondary systems is the main source of nonlinearities. In our modeling, the system and subsystems are two independent sets of objects and the contact/interaction provides a mode of communication for coupling between different response quantities (Fig. 2). This enables complete bypassing of the complications arising from the treatment of singular or convective terms (as alluded to in sec. 1.3 above). Instead, we carry out the physical convection of the material by introducing specified relative motion between the two objects “A” and “B” (the primary and the secondary systems, respectively); i.e., by updating the nodal locations in both of the meshed parts “A” and “B” at all times, using a purely Lagrangian approach. Note that the present use of traditional Lagrangian format is in sharp contrast to the “awkward” alternative of combined Eulerian/Lagrangian
that has been employed almost exclusively in the existing literature on BVI problems (see item (4) under Remarks in section 1.3 above).

An added advantage of using large-scale FE codes is their inbuilt efficient and robust numerical algorithms for integration of equations of motion (both explicit and implicit schemes). In the BVI model, the primary “bridge” configuration can be modeled as beam, frame, plate, shell, or general 2D/3D continua. On the other hand, the modeling of moving secondary system “vehicles” may vary from the simplest lumped load-mass-spring-dashpot type to a general discretized continuum. For both the primary and secondary systems, one may use different elastic or inelastic (dissipative) materials.

In summary, the whole modeling of BVI problems in any large-scale FEA software would require the following three steps:

1. Modeling of the primary system (e.g., beam, plate, shell, and three-dimensional solid).
2. Modeling of the secondary moving subsystems (e.g., point load, point mass, spring-mass-damper system, moving 2D/3D continua, moving supports, etc.)
3. Selection of an appropriate contact formulation between primary system and moving secondary subsystem, with large sliding.

We selected ABAQUS [8] as a vehicle to model and simulate BVI problems, but our modeling approach is applicable to any of the available FEA software codes with (a) extensive material (built-in or user-defined) and element library, (b) nonlinear kinematic capabilities, and (c) contact with large sliding capabilities.

In our treatments, the use of large (finite) sliding is obviously essential, and all the problems are thus treated nonlinearly, even if the deformations are assumed small in the “bridge” and “vehicles.” Also, the selection of appropriate contact formulation is very important due to its direct effect on the convergence and efficiency of the solution. For the benefit of practicing engineers, a brief description of contact interaction capabilities and associated guidelines are given in Appendix 1.

3. NUMERICAL EXAMPLES

To illustrate our methodology, five numerical simulation examples are presented here. Their selections are based on their engineering applicability and the complexity involved in their mathematical treatments as presented in the literature. The problems can be categorized as: (a) the cases involving 2D beams (examples 1–3), and (b) the cases involving 3D plates/shells (examples 4–5). In addition to the benchmarks, we also present a set of new results for the case of a framed “overpass” bridge traversed by a stream of several oscillating vehicles. To our knowledge, there are no prior solutions reported on such complex frame systems in the available literature. In particular, we consider the interactions of axial-shear-flexural deformations in the dynamics, and provide studies for the parametric resonances in the bridge due to vehicle motion and effect of “road” irregularities on the response of bridge/vehicle systems. All FE simulations are modeled and performed in ABAQUS [8], treating each problem as a non-linear dynamics problem (hard pressure overclosure relationship to enforce contact, unless specifically mentioned). The implicit integration scheme (Hilber-Hughes-Taylor method with numerical damping control parameter, $\alpha = -0.05$) of ABAQUS [8], with automatic time increments and default solution convergence control, has been employed in all of the presented examples. Furthermore, only final converged results (after mesh convergence and solution convergence analysis) are reported in the sequel.
3.1. Bridge Vehicle Interaction

The most common BVI problem involves a simply supported beam and a spring mass system moving with constant velocity. Such a numerical example (Fig. 3) from [49] has been considered here.

The problem was solved using three alternative approaches: moving load, moving mass, and moving oscillator (sprung mass). The “Node to Surface Contact” feature of ABAQUS [8] was used to simulate the sliding motion on the frictionless surface of the “bridge.”

The “bridge” was modeled as a beam using 100 “Timoshenko Beam Element” (B21 [8]), and the “vehicle” as a single spring mass system. The “vehicle” model was moved with constant velocity by defining a prescribed time history of its displacement along the beam axis. In the implicit time integration scheme, the time increment size was allowed to vary between the minimum and maximum step size of \(9 \times 10^{-6} \text{ s}\) and 0.002 s, respectively.

The time history traces of the mid-span deflection and acceleration in the beam, and sprung mass displacement and acceleration, are shown in Figs. 4 and 5. Results obtained using the present FE model are in good agreement with those obtained in the reference (Figs. 4–7 in [49]). A “very small” deviation can perhaps be attributed to the fact that all results obtained in the literature were for the case assuming continuous contact conditions, whereas our solution does allow for the separation and reattachment of the “vehicle.” Also, the convergence of the solution for four different mesh sizes is shown in Fig. 6.

Note that, with reference to the simple case of Eq. (9), the coefficient of the sine term in Eq. (9) is approximately 0.0026. This obviously has negligible effect on the apparent stiffness of the system, hence explaining the good resemblance among the different solutions reported before, whether fully considering the effects of convection (e.g. Ref. [47]), or completely discarding them (e.g. Ref. [50]). Our solution in that case also coincided with these reported results.

However, as per our earlier discussion in section 1.3, if the convective terms are not accounted for carefully in the case of large convection, the obtained response components may not be correct. Bowe & Mullarkey [47] highlighted the importance of this for an example involving a cantilever beam where the coefficient of sine term in Eq. (9) is approximately 1.11. We solved this same problem (see details of data in [47]) and we show results in Fig. 7a for the history of the displacement at the cantilever’s tip, and in Fig. 7b for the normalized (with respect to the weight of sprung mass) contact force. As in [47], we remark that the approximate solution by neglecting the convective
FIG. 5. Time histories of (a) vertical deflection; (b) vertical acceleration of the sprung mass. (Color figure available online).

FIG. 6. Mesh convergence study for the mid-span vertical acceleration of the beam. (a) Time history from four mesh sizes, and (b) acceleration at $T = 0.3\,s, 0.6\,s$, and $0.9\,s$, vs. mesh. (Color figure available online).

FIG. 7. Time histories of (a) tip vertical deflection; (b) normalized contact force between sprung mass and the cantilever beam. (Color figure available online).
acceleration in Akin & Mofid [50] is four times higher (i.e., in error) than that from our approach (Fig. 6a). Furthermore, we note a more realistic behavior of the model concerning the periods of separation and reestablishment of the contact between sprung mass and cantilever beam in Fig. 7b leads to the shallow segment near 0.1 sec in the time history of tip vertical deflection of beam (Fig. 7a). Finally, note that the results obtained here (Fig. 7) are in good agreement with the solution presented by Bowe & Mullarkey (Figs. 6b and 7 [47]).

3.2. Dynamic Analysis of Suspended Cable Carrying Two Oscillating Subsystems with Varying Speed

A cable is defined as a long, very flexible and lightweight structural element. In the literature, the dynamic analysis of continuum cable, carrying a moving subsystem, has been considered a particularly difficult problem within the BVI class, due to the non-linear (geometry) kinematics. Therefore, we considered a cable-type problem from the literature [51] as our next numerical example. The schematic of the problem is shown in Fig. 8. Note that a circular cross-section for the cable was chosen with the same cross-sectional area as given in the reference, but with a negligible moment of inertia to satisfy cable properties; i.e., it is very flexible and can carry only tension.

The cable was modeled with 50 Euler–Bernoulli beams (B23), whereas the oscillators were modeled as spring–point mass–dashpot. The analysis was performed in two steps, accounting for fully non-linear geometry and deformations: (a) a static step to get the initial catenary shape (prestressing state); followed by (b) a dynamic step to analyze the response of the cable due to the motion of two moving oscillators. In the static step, a uniform line load of magnitude corresponding to weight per unit length of the cable was applied, whereas the two oscillators were held stationary at left support and forces equal to their respective weights were applied to them. The mid-span vertical deflection of the cable obtained at the end of the static step was 11.7275 m. In the dynamic analysis step (with 10⁻⁶ s as the minimum and 0.1 s as maximum increment size), the spring mass subsystems were moved with constant velocity, \( v = 5 \text{ m/s} \), for the first 90 seconds and then a constant deceleration, \( a = -0.5 \text{ m/s}^2 \), such that the subsystems were brought to rest at \( t = 100 \text{ sec} \). The “vehicle—2” was moved 40 sec. after “vehicle—1” in order to simulate the spacing, \( d \). The time–histories of the cable’s mid-span vertical displacement, the vertical displacement of the “changing” contact point of the second oscillator with the cable, and the “slope” at mid-span of the cable are shown in Fig. 9. Overall, the results are in good agreement with those presented in the reference (Fig. 7 in [51]). Note, however, that Sofi and Muscolino [51] used the “unrealistic” assumption of continuous contact. On the other hand, in Fig. 9, the more realistic treatment involving remittent contact/separation is accounted for (hence, the additional superimposed “small amplitude” oscillations as shown in Fig. 9).

The time histories of spring force developed in oscillators are shown in Fig. 10. As expected, when the second oscillator starts moving on the “already oscillating” cable, thus resulting in instantaneous impact force exerted on the cable, there is a sudden jump in the spring force, leading to a set of “high frequency–high magnitude” forces in the oscillators. The time histories of the spring forces can help in designing an efficient and comfortable vehicle suspension system.

3.2. Parametric Resonance and Effect of Road Surface Irregularity on the Dynamics of System

We considered an “overpass bridge” subjected to a train of 20 moving vehicles (Fig. 11) for the study on parametric
resonance and the effect of surface irregularity. To the best of our knowledge, there is no analytical solution available for this particular type of framed structure. The complexity in the problem arises from the dynamic coupling between the axial and bending deformations modes.

In general, the vehicle or train is represented as arrangement of a spring–mass–damper system, with number degrees of freedom. The complexity in FEA modeling of vehicles in BVI problems depends upon two factors:

1. The amount of available vehicle data.
2. The degree of sophistication and details of the desired vehicle response.

The simplification to vehicle models has been employed in the literature, varying from a single spring–mass–damper system [17, 52] to MDOF systems [18, 19, 52–54]). The simplified FEA representation of the truck model is shown in Fig. 11. The problem data adopted for this problem are as follows:

**Bridge Data**
- Modulus of Elasticity of Bridge, $E = 25$ GPa
- Possions Ratio, $\nu = 0.3$
- Density of bridge material, $\rho = 2600$ kg·m$^{-3}$
- Cross-sectional area of bridge frame, $A = 0.2$ m$^2$
- Moment of inertia of bridge frame, $I = 0.015$ m$^4$
- Distance between two vehicles, $l_c = 8$ m
- Speed of vehicle, $v = 25$ m·s$^{-1}$

**Vehicle Data**
- Mass on rear axle, $M_r = 12233.4$ kg
- Mass on front axle, $M_f = 3058.1$ kg
Mass of rear wheel, \( m_r = 0 \) kg
Mass of front wheel, \( m_f = 0 \) kg
Total load on rear wheel, \( W_r = 120 \) kN
Total load on front wheel, \( W_f = 30 \) kN
Spring stiffness of rear suspension, \( k_r = 100 \) kN.m\(^{-1}\)
Spring stiffness of front suspension, \( k_f = 100 \) kN.m\(^{-1}\)
Distance between axles, \( l_w = 4 \) m

For the extreme case of moving load only, assuming small beam deflections, the condition for resonance in terms of lag time (the time difference between the entrances of two consecutive vehicles) can be given as [55]:

\[
\Delta T = \frac{2\pi n}{\omega},
\]

(10)

where, \( \omega \) and \( n \) are defined as the natural frequency of vibration of the simple beam and the integer multiplier, respectively.

An eigenvalue frequency analysis was performed to obtain the natural frequencies and modes of vibration for the isolated “bridge” structure. The first natural time period reported by ABAQUS/Standard\( ^\text{R} \) was 0.3204 sec. To analyze resonance, we utilized 0.32 sec as the lag time between the entrances of two consecutive vehicles.

The following irregularity function for the vertical profile of the “bridge” deck is assumed (where \( \eta = \) amplitude, \( l^* = \) half wavelength),

\[
\gamma_0 = \frac{\eta}{2} \left[ 1 - \cos \left( \frac{\pi x}{l^*} \right) \right].
\]

(11)

The problem was analyzed for two values of \( \eta \); 0 m for smooth surface, and 0.1 m for rough surface. The whole “bridge” model was modeled with 128 beam (B21) elements (96 elements on deck, and eight elements on each of the four inclined members). Each “vehicle” was modeled as a pair of sprung mass systems (one for front axle and the other for rear axle). We utilized a linear pressure overclosure relationship (with 10\(^{12}\) as contact stiffness to enforce contact constraint). Prior to a full dynamic analysis step, concentrated loads equaling the weight of each “vehicle” were applied on each sprung mass system in a static step of duration 1 sec. The vehicles were moved with prescribed displacement in the dynamic analysis. In this example, we utilized \( 6 \times 10^{-6} \) s and 0.005 s for the limits (i.e., minimum and maximum allowed increment size) of automatic time increment schemes.

For the case of smooth surface (\( \eta = 0 \) m), time histories of the vertical displacement at point “H,” bending moment at point “B,” and axial force at point “G” (midpoint of member BD) are shown in Fig. 12. The time history plots show resonance characteristics in all the response quantities for the moving load case, whereas the resonance characteristics completely disappeared in the case of moving mass approach. As we mentioned before,
the condition for resonance in Eq. (10) was derived for moving load case. Unfortunately, the same condition is being used in the literature to demonstrate resonance phenomena for the case of a moving sprung mass system (e.g. [54]). In the current problem, the ratio of one vehicle mass to total mass of the bridge deck is significant (of value 0.306), and hence the effect of mass on the overall response of system is pronounced here. Furthermore, the effect of spring stiffness on the character of the solution is demonstrated by the beating phenomenon observed in all the response quantities.

We next considered a case with $\eta = 0.1$ m to study the effect of bridge deck surface irregularity on the response of the system. We selected $l^*$ to be equal to 1 m, and analyzed the same problem with moving mass approach only, since the effect of inertia is maximum in the case of moving mass (the extreme case of a moving force is of course not affected by irregularities). The irregularities in bridge deck surface profile were introduced as initial nodal coordinates. The results obtained are shown in Fig. 13, along with the results from smooth surface case. Contrary to some researchers’ observations (e.g. [19, 54]), there are significant amplifications in all of the response quantities due to the “road” roughness simulated here. This is in accordance with the observations made by some researchers [20, 56]. Conclusively, the results in Figs. 12 and 13 clearly indicate that the response of the system is highly dependent on the analysis assumptions and the system parameters.

To validate the accuracy of the dynamic solution, relative error in energy conservation (balance) for different cases of loading and road surface conditions are shown in Fig. 14. For
In this problem, only four (out of a possible total of 16) energy contributions/outputs for the whole model from ABAQUS/Standard® are applicable to this problem; i.e., kinetic energy (ALLKE), internal energy (ALLIE), external work (ALLWK), total energy (ETOTAL). In terms of the applicable energies, ETOTAL for the whole model can be written as [8],

\[
ETOTAL = ALLIE + ALLKE - ALLWK. \tag{12}
\]

Equation (12) must be satisfied in order to satisfy the law of conservation of energy. Note that the energy balance condition is necessary but not sufficient for the accuracy of FEA. Therefore, we define the relative error in energy conservation as,

\[
\text{Relative Error} = \frac{ETOTAL - (ALLIE + ALLKE - ALLWK)}{|ALLIE| + |ALLKE| + |ALLWK|}. \tag{13}
\]

The relative error in energy conservation (Fig. 14) is of order of $10^{-7}$ in all the cases we presented, which is well within the machine precision. It clearly indicates the good quality of the solution of this dynamic problem. We hope that the detailed results documented here will encourage other researchers in the future to use them for comparisons as a new benchmark for any of the other analytical (or semi-analytical) or numerical approaches for BVI problems.
3.4. Dynamic Response of Plate under Moving Distributed Mass

A rectangular plate, simply supported at two edges, subjected to a moving distributed mass (Fig. 15) was analyzed next. The generality of FEA software packages can be clearly seen in this problem, where we solved the present “plated” BVI problem with exactly the same ingredients as in the earlier beam/framed BVI cases.

The plate was modeled with 60 shell elements (S4), similar to the mesh used in [33] (i.e., 10 elements along the length and 6 elements along width); however, the results from two other refined meshes have also been presented. The distributed mass (on finite area) was modeled with 64 shell elements of 0.025×0.025 element size. The thickness of distributed mass was taken as 0.001 m and elasticity modulus was taken as 10 times higher than that of the plate material. Density of material of distributed mass was defined such that the total mass of the distributed mass was 6.5 kg. In [33], a modal damping ratio of 0.005 was applied to all the modes during numerical simulation. ABAQUS allows us to use mass- and stiffness- (α and β, respectively) proportional Rayleigh damping. The relationship between modal damping ratio and Rayleigh damping coefficients can be given as:

$$\alpha = \frac{2\omega_i\omega_j (\omega_i \xi_j - \omega_j \xi_i)}{\omega_i^2 - \omega_j^2}, \quad \beta = \frac{2 (\omega_i \xi_j - \omega_j \xi_i)}{\omega_i^2 - \omega_j^2}. \quad (14)$$

In the above equation, $\omega_i$ (and $\omega_j$) and $\xi_i$ (and $\xi_j$) represent the $i$th (and $j$th) natural frequency of vibration and associated modal damping ratio, respectively.

The Rayleigh damping coefficients were calculated using the first two natural frequencies of vibration of the “isolated” plate (here 150.20 rad/sec, and 369.32 rad/sec), and equal modal damping ratio of 0.005 for each of these two modes. This led to the Rayleigh’s coefficients 1.06775 sec$^{-1}$ and 1.92485×10$^{-5}$ sec. for the mass-proportional ($\alpha$) and stiffness-proportional ($\beta$) damping parts, respectively.

To reduce the number of equations to solve, all the nodes on the left support were restrained to move along the $x$–axis as well as along the $z$–axis, whereas the nodes of distributed mass were allowed to move only along the $x$–axis and $y$–axis.
An equivalent instantaneous pressure load was defined over distributed mass surface to account for the weight of distributed mass. The center of distributed mass was moved, linearly, from \( x = 0.1 \) to \( x = 0.9 \) in 0.4 sec along the center line of the plate, giving a constant speed of 2 m·s\(^{-1}\). The analysis for this example was executed using a minimum time increment size of \( 4 \times 10^{-8} \) s and a maximum allowed time increment size of 0.002 s for the automatic time increment scheme of ABAQUS.

The time history of midpoint deflection of the plate is shown in Fig. 16 (a). The results obtained from the analysis are similar to the second case reported in the reference (Fig. 5 for the case of \( d_x = 0.1 \text{ m}, d_y = 0.4 \text{ m} \) in [33]), thus validating the scope of our methodology. The difference in amplitude of oscillations after 0.2 sec can be attributed to the fact that, unlike the reference solution [33], separation during contact was allowed in our present simulations. To demonstrate contact, separation, and subsequent reattachment, time history of the normalized contact pressure (normalized with respect to applied pressure load on distributed mass) at the midpoint of distributed mass is shown in Fig. 16 (b), where zero pressure represents the duration of separation. Furthermore, there is continuous decrease with cycles in the amplitude of oscillation due to damping, whereas the result from [33] shows an increase (unreasonable) in amplitude of oscillations!

### 3.5. Dynamic Analysis of Curved Panel Subjected to Point Mass

The most general case needed to be analyzed has to be motion of mass in 3D space. We could find only one problem involving
The problem involves dynamic response of curved cylindrical panel subjected to a mass moving in circumferential direction. The schematic of the problem is shown in Fig. 17 (all data are given in global coordinate system; dark edges represent fixed boundary conditions; i.e., edges parallel to the z-axis are fixed, whereas circumferential edges are free).

The curved panel was modeled with shell elements (30 elements along axial direction, and 60 elements along circumferential direction). The moving mass was modeled as a point mass with predefined displacement along the circumferential center line of panel. A constant concentrated load of magnitude equaling the weight of point mass was applied to the point mass in the global y-direction. The x–translation of point mass, $u_x$, with respect to initial position, $\varphi_0 (-\pi/12)$, at any instance of time, $t$, (in terms of radius of the panel, $R$, and global angular velocity, $\Omega(-\pi/24)$) in the global coordinates system can be expressed as follows:

$$u_x = R \{\sin (\varphi_0 + \Omega t) - \sin \varphi_0\} , \quad (15)$$

The motion of mass was controlled by an amplitude function, defined in global co-ordinates, and was obtained by transforming the motion of mass from the local coordinate system to the global coordinate system. Only x–translation of mass was defined, since the problem would become ill posed if the force and displacement in the same direction at a node (i.e., applied weight of the mass and its y–translation) are specified simultaneously. In other words, the y–translation of the point mass was treated as a solution variable. Similar to other examples in this paper, separation between point mass and panel surface was allowed. Except for large relative sliding, we used the “small” panel’s deformation assumption to enable comparison with the reference solution [35]. The time stepping minimum and maximum limits were set to $10^{-6}$ s and 0.01 s, respectively.

The time history response of midpoint of the panel is shown in Fig. 18. The displacements are plotted as a function of angle, calculated at respective time, in order to compare with reference solution.

The result from the analysis shows consistency with the reference solution (Fig. 6 in [35]). Note that our approach was able to capture the small amplitude vibration of the panel’s mid-point due to impact between moving concentrated mass and the panel. The counters of maximum in-plane principal stress (units N·m$^{-2}$) at three different locations of moving mass are shown in Fig. 19. Stress is concentrated at the instantaneous location of the point mass when it was at $-7.5^\circ$ and $+7.5^\circ$, whereas when the point mass was at 0°, the stress was almost uniformly distributed over the plate. This can be attributed to the fact that the point mass was not in contact with panel; therefore no load (either its weight or contact force) was exerted by the point mass. This behavior can only be observed if the separation between the secondary sub-system and primary system is allowed; i.e., a more realistic situation. Also, if the weight of the point mass is significant, local yielding of the panel may be observed. It implies that the proposed formulation in [35], which is limited to linear isotropic elastic material, will require revision in order to handle such situations. However, there will not be any change in the modeling of the problem in large-scale FE packages, except for the change in material data from the linear elastic to inelastic case in the scenario referred to above. This example clearly shows the robustness and generality of the proposed methodology.
4. CONCLUSIONS

The extensive literature presently available on the subject of BVI has provided significant insights and understanding of the many issues involved in this class of problems. However, many mathematical formulations and computational schemes presented in these works remain very highly specialized in nature. This renders them inaccessible by ordinary engineers with typical backgrounds (as covered in regular engineering training). Our main focus here has been on an alternative point of view, which enabled the direct incorporation into large-scale, standard FE computations. The key to the success of the presented methodology for a general BVI problem is in the use of nonlinear mechanics and a purely Lagrangian formulation *ab initio*. This alternative viewpoint offers two distinct advantages:
(i) all complications due to the need for convective contributions in stiffness, inertia, etc. (as in existing BVI treatments) are bypassed; (ii) the versatility and power of large-scale FEA computations allow for a much wider scope of the problems; i.e., by realistically accounting for all interaction forces (be it of any type such as gravitational, inertia, or elastic in BVI) using the large sliding and intermittent impulse/impact capabilities in large-scale FEA packages. Additionally, with the help of robust algorithms implemented in FEA codes, it is feasible to obtain a great amount of useful information for design and parametric studies.

We have presented the solutions of a rather extensive set of the BVI benchmark problems to clearly validate and demonstrate the power of the developed strategy as used in the large-scale FEA code. Consequently, we have been able to consider a rather impressive scope of BVI, ranging from simpler beam/cable type to plates and shells, accounting for the effects of a stream of many vehicles, and associated parametric resonance phenomena. Furthermore, we have presented solutions to new and more challenging problems, for which there are no solution in present literature.

The proposed methodology is independent of the type of problem and solver, thus it can be implemented in any of the available large-scale finite element software (which has the needed ingredients discussed in section 2). This will certainly help practicing engineers to efficiently analyze the many classes of BVI problems.

REFERENCES


APPENDIX 1

Contact/Impact with Large Overall Sliding

ABAQUS gives flexibility of contact interaction modeling by either surface-to-surface or surface-to-node. For the purpose of the BVI problems, surface-based contact interaction is preferred due to its generality; however, node-based contact interaction is equally suitable for the problems involving a single sprung mass system (i.e., single contact point on secondary system). Surface-based contact can be defined for contact between two surfaces with at least one surface of the contacting pair defined by elements [8].

In ABAQUS/Standard, contact conditions are based on a strict master–slave algorithm, where only master is considered as surface. The geometry and orientation of master surface is used to decide the surface normal that passes through slave surface nodes [8]. From the slave surface, only the location of nodes and associated surface area with it are used for contact enforcement, and its normal is irrelevant in the contact algorithm. For successful contact, the slave surface should always be on the same side of the master surface’s outward normal. The master surface can penetrate through the slave surface and, therefore, selection of master-slave surface definition becomes very important for contact analysis [8]. Refined discretization of the slave surface is always helpful in reducing the penetration of the master surface nodes into the slave surface.

The BVI problems involve large sliding of vehicle (slave surface) over the bridge (master surface); therefore finite sliding, which allows arbitrary motion of contacting surfaces, is the most suitable between the two available tracking approaches for the relative motion of contacting surface pairs in ABAQUS/Standard. It also allows for arbitrary separation and rotation of surfaces.

The default contact property is “hard” for pressure–overclosure relationship, frictionless for tangential behavior, and no contact damping to oppose relative motion between interacting surfaces. A hard pressure–overclosure relationship assumes that the contact pressure is transmitted only when the slave surface nodes contact the master surface, and it can be infinity [8]. It thus minimizes the penetration of the slave nodes into the master surface and does not transfer tensile stress across the interfaces. The contact pressure, \( P \), between two surfaces at a point is defined in terms of the “overclosure” or the interpenetration of the surfaces, \( h \), of the two surfaces. In the case of hard contact, the contact pressure, \( P \), is defined as [8],

\[
P = 0 \quad \text{for} \quad h < 0 \quad (\text{open}),
\]

\[
h = 0 \quad \text{for} \quad P < 0 \quad (\text{closed}).
\]

(A.1)

Apart from strictly hard pressure-overclosure relationship, ABAQUS/Standard allows linear, exponential, or tabular piecewise linear function of clearance between surfaces to define pressure-overclosure relationship. The contact pressure can be represented as follows for exponential pressure overclosure relationship,

\[
P = 0 \quad \text{for} \quad h \leq -c,
\]

\[
P = \frac{P_0}{(e - 1)} \left[ \left( \frac{h}{c} + 1 \right) \left( e^{(h/c) - 1} - 1 \right) \right] \quad \text{for} \quad h > -c.
\]

(A.1)

where \( h \) is overclosure and \( c \) is clearance between master and slave surfaces.

At high penetration (\( h > 6c \)), linearized pressure overclosure relationship with a constant slope is used to avoid numerical difficulties.
ABAQUS/Standard® has three methods to enforce contact constraints: (i) direct method; (ii) Penalty method; and (iii) Augmented Lagrange method [8]. With the exceptions of finite sliding, surface-to-surface, hard contact, and 3D, node-to-surface, self-contact with hard pressure overclosure relationship, the direct method is used as the default contact constraint enforcement method. However, the Penalty or Augmented Lagrange method can also be used, but sometimes it leads to some loss of accuracy.

In summary, the following points should be considered during contact modeling for BVI problems:

1. Sufficiently refined meshes should be used for master and slave surfaces definition.
2. The primary surface definition is extended far enough to account for all the expected motions of the subsystems.
3. The use of hard pressure overclosure with the direct contact constraint enforcement method is encouraged.
4. Sufficiently refined time increment is recommended for easy convergence.

Further details can be found in the ABAQUS analysis user’s manual and ABAQUS theory manual [8].