ABSTRACT: The ability for structural alloys to exhibit recovery of state, i.e., return to a softer state following periods of hardening, under varying stress and temperature is known to strongly influence structural response under some important thermomechanical loadings. For example, those involving thermal ratchetting and creep crack growth. Here the influence of dynamic and thermal recovery on the axisymmetric creep buckling response of a circular cylindrical shell under variable uniform axial loading is investigated. The real shell is idealized as a two-membrane sandwich shell while the constitutive model, unlike the commonly employed Odqvist creep law, incorporates a representation of both dynamic and thermal (state) recovery. The material parameters of the constitutive model are chosen to characterize Narloy-Z, a representative copper alloy used in thrust nozzle liners of reusable rocket engines. Variable loading histories investigated include rapid cyclic unloading/reloading sequences and intermittent reductions of load for extended periods of time. The calculated results show that failure to account for state recovery in the constitutive relations can lead to nonconservative predictions of the critical creep buckling time.

INTRODUCTION

The influence of dynamic and thermal recovery on the high-temperature behavior of structural alloys is well recognized (Pugh and Robinson 1976; Robinson and Pugh 1976) (see Figs. 1 and 2). The inclusion of mechanisms in constitutive relations accounting for recovery effects has been shown to play a major role in one of the central structural problems relating to liquid metal breeder reactor design, that of thermal ratchetting (Sartory 1980). Also, recovery effects are believed to play a primary role in creep crack growth under creep/fatigue (variable stress) conditions (Kubo 1983; Robinson 1979).

Recently, Robinson and Arnold (1986) showed that the inclusion of dynamic and thermal recovery mechanisms leads to significant reductions in predictions of critical time in problems of high-temperature creep buckling under variable loading.

The acceleration of creep buckling is shown in Robinson and Arnold (1986) to result from an increase in creep strain rates following inelastic state recovery (Figs. 1 and 2). The previous work (Robinson and Arnold 1986) examined the creep buckling of a column under variable loading. The objective of the present study is to investigate the influence of dynamic and thermal recovery mechanisms on the creep buckling of a circular cylindrical shell under variable axial loading. The shell investigated is an idealized two-membrane “sandwich” shell much in the same spirit as the “sandwich” column investigated earlier (Robinson and Arnold 1986). The sandwich ideal-
FIG. 1. Schematic Representation of Dynamic Recovery. In Absence of Stress Reversals, Stress Relaxations over a Fixed Time, \( ab, cd, \) and \( ef \) Show Successive Hardening. After Stress Reversal \( fgh, \) Stress Relaxation \( hj \) Shows Evidence of (Creep) Softening, i.e., Evidence of Recovery of State [See Pugh and Robinson (1976)]

ization has been employed by Hoff (1954, 1955) for columns and Samuelson (1964, 1967) for cylindrical shells. Most cylindrical shell creep buckling studies utilize this idealization in conjunction with a constitutive model of the Norton type (Hoff 1968; Honikman and Hoff 1971; Samuelson 1964, 1967) in which the steady state creep rate is taken as a function of stress and temperature alone. Some investigations have been conducted using hardening theories which include a primary creep phase (Obrecht 1977; Pan 1971; Sundstrom 1957). These constitutive theories are adequate under constant loading. However, under variable loading they may significantly overpredict the time to buckling, as they do not allow for inelastic state recovery and consequently for rejuvenation of primary creep.

The constitutive model employed in the present study is a nonisothermal unified viscoplastic model developed principally by Robinson, at Oak Ridge National Laboratory and NASA Lewis Research Center (Robinson and Pugh 1976; Robinson and Swindeman 1982). This model embodies a representation of dynamic and thermal recovery in the context of the widely accepted Bailey–Orowan theory (Mitra and McLean 1961). The material parameters employed in the constitutive equations are chosen to represent a copper alloy, Narloy-Z, which is typical of materials used in reuseable rocket engine thrust nozzle liners.
The paper begins by first stating the constitutive model. Secondly, the geometry, basic assumptions, and solution method employed in the sandwich shell idealization are presented. Finally, creep buckling behavior is investigated under constant loading, cyclic loading composed of relatively rapid load reductions and reversals (dynamic recovery), and intermittent reductions of load for extended periods of time (thermal recovery) at elevated temperature. Calculations using a roughly equivalent Odqvist (Norton type) creep model are also made for comparison. Conclusions from the study are then stated.

**Inelastic Constitutive Theory**

A complete nonisothermal multiaxial statement of the constitutive model is reported in Robinson and Pugh (1976) and Robinson and Swindeman (1982) and is not repeated here. For the problem at hand it is assumed that conditions of plane stress exist for both the applied stress \(\sigma\) and internal (or back stress) \(\alpha\). Under this assumption the multiaxial formulation reduces to the following, including the specification of material functions and parameters for the copper alloy Narloy-Z.

Flow laws:
\[ \varepsilon' = \frac{AF^n}{3\sqrt{J_2}} [2(\sigma_x - \alpha_x) - (\sigma_h - \alpha_h)]; \quad F > 0 \text{ and } D > 0 \]

\[ \varepsilon'_h = 0; \quad F \leq 0 \text{ or } F > 0 \text{ and } D \leq 0 \]

\[ \varepsilon' = \frac{AF^n}{3\sqrt{J_2}} [2(\sigma_h - \alpha_h) - (\sigma_x - \alpha_x)]; \quad F > 0 \text{ and } D > 0 \]

\[ \varepsilon'_h = 0; \quad F \leq 0 \text{ or } F > 0 \text{ and } D \leq 0 \]

**Evolutionary laws:**

\[ \dot{\alpha}_s = \frac{H}{G^b} (2\varepsilon'_s + \varepsilon'_h) - \frac{RG^{m-\beta}}{\sqrt{I_2}} \alpha_s; \quad G > G_0 \text{ and } \Xi > 0 \]

\[ \dot{\alpha}_s = \frac{H}{G^b} (2\varepsilon'_s + \varepsilon'_h) - \frac{RG^{m-\beta}}{\sqrt{I_2}} \alpha_s; \quad G \leq G_0 \text{ or } G > G_0 \text{ and } \Xi \leq 0 \]

\[ \dot{\alpha}_h = \frac{H}{G^b} (2\varepsilon'_h + \varepsilon'_h) - \frac{RG^{m-\beta}}{\sqrt{I_2}} \alpha_h; \quad G > G_0 \text{ and } \Xi > 0 \]

\[ \dot{\alpha}_h = \frac{H}{G^b} (2\varepsilon'_h + \varepsilon'_h) - \frac{RG^{m-\beta}}{\sqrt{I_2}} \alpha_h; \quad G \leq G_0 \text{ or } G > G_0 \text{ and } \Xi \leq 0 \]

in which

\[ J_2 = \frac{1}{3} \{(\sigma_x - \alpha_x)^2 + (\sigma_h - \alpha_h)^2 - (\sigma_x - \alpha_x)(\sigma_h - \alpha_h)\} \]

\[ I_2 = \frac{1}{3} \{\alpha_x^2 + \alpha_h^2 - \alpha_x\alpha_h\} \]

\[ S_x \Sigma_x = \frac{1}{9} \{(4\sigma_x - 2\sigma_h)(\sigma_x - \alpha_x) + (\sigma_h - 2\sigma_x)(\sigma_h - \alpha_h)\} \]

\[ S_h \Sigma_h = \frac{1}{9} \{(4\sigma_h - 2\sigma_x)(\sigma_h - \alpha_h) + (\sigma_x - 2\sigma_h)(\sigma_x - \alpha_x)\} \]

\[ S_x a_x = \frac{1}{9} \{(2\sigma_x - \sigma_h)(2\alpha_x - \alpha_h)\} \]

\[ S_h a_h = \frac{1}{9} \{(2\sigma_h - \sigma_x)(2\alpha_h - \alpha_x)\} \]

\[ F = \frac{J_2}{K^2} - 1 \]

\[ G = \frac{J_2}{K_0^2} \]

\[ D = S_x \Sigma_x + S_h \Sigma_h \]
FIG. 3. Geometry of Circular Cylindrical Shell of Radius \( a \), Thickness \( H \), and Subjected to Distributed Axial Load \( (P/2 \pi a) \)

\[ D = S_x a_x + S_h a_h \]  \hspace{1cm} \text{(3j)}

and \( n = 4; \) \( m = 4.365; \) \( \beta = 5.33 \times 10^{-7} (T^2) + 0.8; \) \( K^2 = 69.88 - 6.65 \times 10^{-3} (T); \) \( K_0^2 = K^2(T_0); \) \( A = 1.385 \times 10^{-8}; \) \( H = 1.67 \times 10^4/(3K_0^2)^{(6.895)^{28+1}}; \) \( R = 2.194 \times 10^{-3} (3K_0)^{m-\beta} e^{(Q_0/(T_0-1/T))}; \) \( G_0 = 0.04; \) \( Q_0 = 40,000; \) and \( T_0 = 811 \text{ K}. \)

The values of the material parameters are consistent with stress in MPa, time in h, strain rate in \( h^{-1} \), and temperature \( T \) in degrees Kelvin. The elastic response is characterized in terms of the Young’s modulus \( E \) (MPa) given by

\[ E = 1.47 \times 10^5 - 70.5 (T) \]  \hspace{1cm} \text{(4)}

Eqs. 1–3 incorporate both dynamic and thermal recovery and therefore predict the type of behavior depicted by Figs. 1 and 2. The material parameters were determined from uniaxial tensile and stress relaxation test data as discussed earlier in Robinson and Arnold (1986). Note that each membrane of the sandwich shell will have an associated flow and evolutionary law.
**Sandwich Shell Model**

In the calculations to follow a solid circular cylindrical shell of radius $a$, subjected to a uniformly distributed axial load (see Fig. 3), is imagined to be replaced by an equivalent cylindrical sandwich shell of radius $a$, consisting of two membranes. Each membrane has a thickness $\delta$ and is separated a distance $d$ apart by an incompressible core (see Fig. 4). The core is assumed rigid in shear and yet unable to carry normal stresses.

A comparison of results obtained using the sandwich structural element and those of the solid shell is viable if the parameters and $d$ are chosen such that the area and moment of inertia of the solid shell are preserved (Hoff 1959; Samuelson 1967):

$$\delta = \frac{H}{2} \quad \text{.................................................. (5)}$$

$$d = \frac{H}{\sqrt{3}} \quad \text{.................................................. (6)}$$

in which $H$ = the thickness of the actual shell (see Fig. 3).

The following general assumptions are made when developing the governing equations:

1. The circular cylindrical shell is long enough that end effects are negligible.
2. Radial deflections of the cylindrical shell are axisymmetric.
3. Deflections are small prior to buckling. This is justifiable for relatively thin cylinders.
4. Plane sections remain plane.

![FIG. 4. Sandwich Shell Idealization](image)
5. Deformations due to shear are neglected.
6. Both elastic and inelastic strains are considered to cause deformations.
7. Buckling is induced by a sinusoidal initial imperfection of the cylindrical shell.
8. The cylindrical shell is loaded by a time-dependent uniformly applied axial load \( P(t) \) shown positive in Fig. 3.

**EQUILIBRIUM**

The condition of axial equilibrium, (see Figs. 3 and 4), is

\[
\sigma_{xx} + \sigma_{xx} = \frac{P(t)}{2\pi a} \tag{7}
\]

The condition of moment equilibrium may be expressed as

\[
V' = \frac{1}{2} (\sigma_{xx}' - \sigma_{xx}') \delta d \tag{8}
\]

in which \( V \) = the shear stress resultant per unit length of circumference; and the prime denotes differentiation with respect to \( x \), the axial direction.

Equilibrium of forces in the radial direction gives

\[
-w''_r \delta (\sigma_{xx} + \sigma_{xx}) + \frac{\delta}{a} (\sigma_{hxx} + \sigma_{hxx}) - V' = 0 \tag{9}
\]

in which \( w''_r \) = the curvature of the meridian caused by the total radial deformations \( w''_r \), i.e., \( w''_r = d^2 w_r/dx^2 \). Substituting Eq. 8 into Eq. 9 results in the following equilibrium condition:

\[
-(\sigma_{xx} + \sigma_{xx}) w''_r + (\sigma_{hxx} + \sigma_{hxx}) \frac{1}{a} \frac{d}{d} (\sigma''_{xx} - \sigma''_{xx}) = 0 \tag{10}
\]

Assuming the radius \( a \) to be constant, the rate forms of Eqs. 7 and 10 are written as

\[
\dot{\sigma}_{xx} + \dot{\sigma}_{xx} = \dot{\sigma}_r \tag{11}
\]

\[
(\dot{\sigma}_{xx} + \dot{\sigma}_r) K_T + (\sigma_{xx} + \sigma_{x}) \dot{K}_T + (\dot{\sigma}_{hxx} + \dot{\sigma}_h) \frac{1}{a} \frac{d}{d} (\dot{\sigma}''_{xx} - \dot{\sigma}''_{xx}) = 0 \tag{12}
\]

respectively, where \(( \cdot )\) indicates differentiation with respect to time.

**COMPATIBILITY**

Three compatibility relations result from the preceding assumptions and geometric considerations. The first (valid for \( d/a \ll 1 \) and small displacements), shown in rate form, is

\[
\dot{e}_h = \frac{\dot{w}_e}{a} \tag{13}
\]

Here \( e_h \) is the total hoop strain and \( w_e \) is the effective radial displacement \((w_e = w_r - w_{initial})\).
A second compatibility relationship accounts for incompressibility of the core, which demands that the radial displacements of the inner and outer membranes remain equal (i.e., \( w_i = w_e \)). The corresponding rate form is

\[ \dot{e}_h = \dot{e}_e. \quad (14) \]

The final compatibility condition obtained is

\[ \dot{K}_e = \frac{1}{d} \left( \dot{e}_x - \dot{e}_e \right). \quad (15) \]

in which

\[ \dot{K}_e = \frac{-d^2 \dot{w}_e}{d\gamma^2}. \quad (16) \]

**CONSTITUTIVE RELATIONS**

Here, under plane stress conditions, the total strain rate tensor \( e_{ij} \) may be written in terms of both the inner and outer membranes as

\[ \dot{e}_{xx} = \frac{1}{E} \left( \dot{\sigma}_{xx} - \nu \dot{\sigma}_{hh} \right) + \dot{e}_{x} \quad (17) \]

\[ \dot{e}_{x} = \frac{1}{E} \left( \dot{\sigma}_{x} - \nu \dot{\sigma}_{hh} \right) + \dot{e}_{x} \quad (18) \]

\[ \dot{e}_{hh} = \frac{1}{E} \left( \dot{\sigma}_{hh} - \nu \dot{\sigma}_{xx} \right) + \dot{e}_{h} \quad (19) \]

\[ \dot{e}_{h} = \frac{1}{E} \left( \dot{\sigma}_{h} - \nu \dot{\sigma}_{x} \right) + \dot{e}_{h} \quad (20) \]

in which \( E \) and \( \nu \) are the Young’s modulus and Poisson ratio, respectively, and are assumed to be identical for both inner and outer membranes. The inelastic strains utilized in Eqs. 17–20 correspond to those described previously (see Eqs. 1–3).

**SOLUTION METHOD**

As formulated earlier, the nine unknowns \( \sigma_{xx}, \sigma_{x}, \sigma_{hh}, \sigma_{h}, e_{xx}, e_{x}, e_{h}, e_{h}, w_T \) are solved for utilizing nine equations; two equilibrium (Eqs. 11 and 12); three compatibility (Eqs. 13–15); and four constitutive (Eqs. 17–20). Owing to the complexity of the material model a numerical solution to the aforementioned equations will be sought. However, as these equations involve second derivatives of such primary variables as stress and radial displacement, advantage of previous analytical work (Hoff 1968; Honikman and Hoff 1971) will be taken to simplify the problem and reduce numerical inaccuracies. That is, the following Fourier series representations are assumed:

\[ w_T = f_0 + \sum_{j=1}^{\infty} f_j \cos (\lambda_j x). \quad (21) \]
in which \( \lambda_j = \pi j/L \) is the half-wavelength in the axial direction; and \( g_j \) are the known initial imperfection amplitudes of the shell.

The rate forms are identical to the aforementioned, whereby all time dependency is placed upon the coefficients. Note that the \( g_t \) coefficients are prescribed and are not a function of time.

The literature supports such forms (Hoff 1968; Honikman and Hoff 1971) and has shown that a single cosine term suffices to describe the buckled shape. It is realized that inclusion of additional terms would add to the accuracy of such a solution. However, the additional complexity involved in obtaining the correct higher mode wavelengths appears unmerited. Therefore only one cosine term will be considered in the present study.

The problem is now reduced to the determination of the coefficients \( f_0, f_1, A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1 \) of the Fourier series. This is accomplished by substituting the aforementioned definitions, Eqs. 21–26, into the nine governing equations.

Utilizing Eqs. 11, 23, and 24 and equating like terms, an expression relating \( A_j \) and \( B_j \) is obtained:

\[
\dot{A}_0 = \dot{\sigma}_v - \dot{B}_0 \quad \text{(27)}
\]

\[
\dot{A}_1 = -\dot{B}_1 \quad \text{(28)}
\]

Combining Eqs. 11 and 12 and taking into account the appropriate definitions results in the following:

\[
(\dot{\sigma}_v, f_1 + \dot{\sigma}_v, f_1)\lambda_1^2 \cos (\lambda_1 x) + \frac{1}{a} (\dot{D}_0 + \dot{C}_0) + (\dot{D}_1 + \dot{C}_1) \cos (\lambda_1 x)
\]

\[
- \frac{d}{2} (\dot{A}_1 - \dot{B}_1)\lambda_1^2 \cos (\lambda_1 X) = 0 \quad \text{(29)}
\]

Then substituting Eq. 28 into Eq. 29, an expression relating \( D_j \) to \( C_j, B_j \), and \( f_j \) is obtained:

\[
\dot{D}_0 = -\dot{C}_0 \quad \text{(30a)}
\]

\[
\dot{D}_1 = - (\dot{\sigma}_v, f_1 + \dot{\sigma}_v, f_1) a\lambda_1^2 - ad\lambda_1 \dot{B}_1 - \dot{C}_1 \quad \text{(30b)}
\]
Next, three additional expressions are obtained by combining the compatibility relations (Eqs. 13-15) with the equilibrium (Eq. 11) and constitutive equations (17-20). These are

\[ w = a \left[ \frac{1}{E} (\dot{\sigma}_h - \nu \dot{\sigma}_x) + \dot{\epsilon}_h^f \right] \] .......................... (31)

\[ \dot{\epsilon}_h^f - \dot{\epsilon}_h = \frac{1}{E} \left[ \dot{\sigma}_h - \dot{\sigma}_h + 2
\nu \dot{\sigma}_x - \nu \dot{\sigma}_y \right] \] .......................... (32)

\[ \dot{K} = \frac{1}{Ed} \left[ 2 \dot{\sigma}_x + \nu (\dot{\sigma}_h - \dot{\sigma}_h) - \dot{\sigma}_y \right] + \frac{1}{d} (\dot{\epsilon}_x^f - \dot{\epsilon}_x) \] .......................... (33)

Substituting Eqs. 21-26 along with 30 into Eqs. 31-33 and expressing the result in matrix form yields

\[
\begin{bmatrix}
-1 \quad \frac{E}{a} - \nu - \cos (\lambda, x) \\
1 \quad \frac{E}{a} + \nu + \cos (\lambda, x) \\
2 \nu \cos (\lambda, x) \\
\end{bmatrix}
\begin{bmatrix}
\frac{\dot{\epsilon}_h}{\dot{\epsilon}_h^f} \\
\frac{\dot{\epsilon}_h - \dot{\epsilon}_h^f}{\dot{\epsilon}_x} \\
-2 \nu \cos (\lambda, x) \\
\end{bmatrix}
\begin{bmatrix}
\dot{C}_0 \\
\dot{B}_0 \\
\dot{C}_1 \\
\dot{B}_1 \\
\end{bmatrix}
\]

= \begin{bmatrix}
E \dot{\epsilon}_h - \nu \dot{\sigma}_v \\
E \dot{\epsilon}_h^f - aK_f \dot{\sigma}_v \\
E (\dot{\epsilon}_h^f - \dot{\epsilon}_h) - (1 - \nu aK_f) \dot{\sigma}_v \\
\end{bmatrix} \] .......................... (34)

Examining Eq. 34, the problem appears to be indeterminate as the number of unknowns exceeds the number of available equations. This difficulty, however, is surmounted by employing the collection method (Reddy 1984) where each of the three equations is satisfied at two distinct stations along the axial direction. Both the left- and right-hand sides of the system of equations are dependent on the \( x \) location. Using this method a \( 6 \times 6 \) system of equations is obtained. Gaussian elimination with full pivoting is then employed to solve for the six rate coefficients. Once \( \dot{C}_j, \dot{f}_j, \) and \( \dot{B}_j \) are known, the \( A_j \) and \( D_j \) unknowns are obtainable from Eqs. 27, 28, and 30. The non-rate coefficients are then obtained by numerically integrating the known rate coefficients.

This solution procedure was verified by comparing solutions for the classical elastic symmetrical buckling of a cylindrical shell, as given by Timoshenko and Gere (1961), under uniform axial compression. Similarly, Hoff's rigorous inelastic analytical solution, utilizing a Norton type material model for creep exponents of three and five (Hoff 1968; Honikman and Hoff 1971), was compared. Details regarding this verification may be found in Arnold (1987).

**RESULTS**

All calculated results presented are isothermal with the temperature taken to be 811° K (1,000° F). Although chosen somewhat arbitrarily, the geometric properties \( a, H, f^0 \), and the applied stress \( (\sigma_v) \) were chosen so that the ratios \( a/H, f^0/H, \) and \( \sigma_v/\sigma_F \) conform to the conclusions drawn by Ob-
Schöle (1977) and Samuelson (1964, 1967) as to restrictions on these ratios in order to ensure axisymmetric buckling. The radius \( a \) was taken as 10.0 cm, the actual shell thickness \( H \) as 0.1, and initial imperfection \( f^0 \) as 0.001. The Euler critical stress \( \sigma_E \) is then easily found, using

\[
\sigma_E = -\frac{2Ed}{a\sqrt{1 - \nu^2}}
\]

(35)

to be 1,085 MPa. The loading/unloading ramp rate \( |\dot{\sigma}_E| \) is taken to be 1,276 MPa/s.

The system of equations (34) was integrated using a multistep self-adapting Adams Bashforth predictor-corrector method with a fourth order Runge-Kutta method as a starter. The calculations were performed in double precision on a Prime 850 with an upper error bound of \( 5 \times 10^{-5} \). In all cases, the criterion defining the critical time to buckling \( (t_{cr}) \) is defined as

\[
\eta = \frac{w_T}{d} = 1.0
\]

(36)

which is equivalent, under constant load, to the criterion

\[
\sigma_{\sigma_0} = 0.0
\]

(37)

i.e., the axial stress in the outer membrane becoming zero.

**BEHAVIOR UNDER CONSTANT LOAD**

In all calculations the compressive load commences from a state of zero loading, where the deforming membrane elements are considered in a virgin state (i.e., \( \alpha_{\xi_i} = \alpha_{\xi_o} = \alpha_{h_i} = \alpha_{h_o} \approx 0.0 \)). The load is then increased to a nominal value of the applied stress \( \sigma_E = P/2 \pi a \delta = -103 \) MPa. Thus, with the Euler stress \( \sigma_E \) equal to 1,085 MPa (as indicated earlier) the nominal value of the stress ratio \( \rho \) is

\[
\rho = \frac{\sigma_E}{\sigma_E} = \frac{103}{1,085} = 0.095
\]

(38)

This is within the stress range suggested by Obrecht (1977) as being consistent with axisymmetric cylindrical buckling.

Here, the calculated results for a constant load, with \( \rho \) held constant at the aforementioned value, are presented. Fig. 5 shows the response of \( \eta(\tau) \), the nondimensional radial displacement versus normalized time, for five different values of initial (nondimensional) displacements

\[
\zeta = \frac{f^0}{H}
\]

(39)

that is, \( \zeta = 0.01, 0.02, 0.04, 0.06, \) and 0.1. The nondimensional time (\( \tau = t/t_0 \)) in Fig. 5 is normalized with respect to the critical time for the initial imperfection \( \zeta = 0.01 \), i.e., \( t_0 = 0.8 \) h.

Fig. 5 indicates that the critical time to buckling is reduced by more than a factor of 25 with an order of magnitude increase in the initial imperfection. Thus, as expected, the shell problem appears to be quite sensitive to the

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magnitude of initial imperfection. Similar sensitivity is observed when the stress ratio $p$ is increased. In particular, as the limiting case when $p = 1$ is approached, instantaneous elastic buckling occurs.

The validity of the selected buckling criteria ($\eta = 1.0$) is apparent from Fig. 5, where $\eta$ is allowed to progress to a value of 3 prior to termination of the solution process. Clearly, little additional information is gained once $\eta$ exceeds 1.0.

**Behavior under Variable Load**

For the sake of comparison, all subsequent calculations are taken to have an initial imperfection $\zeta = 0.01$ and are presented in terms of the nondimensional time $\tau = t/t_0$.

**Dynamic Recovery**

The effect of rapid load reductions and reversals superimposed on the constant load $\sigma_v = -103$ MPa ($p = 0.095$) are now considered. As shown in the inserts of Fig. 6, load interruptions occur at time intervals of $\tau_0 = 0.15$ with varying amplitudes. These include reductions (in the tensile direction) of $\sigma$, to $-51.5$ MPa and to 0 MPa (insert a), to $+51.5$ MPa (insert b), and a complete reversal to $+103$ MPa (insert c). The calculated effects of these loading histories on the creep buckling response $\eta(\tau)$ are shown in the respective curves a, b, and c of Fig. 6.

Curve a corresponds to the loading histories indicated in insert a, i.e.,
with reductions to $-51.5$ MPa and $0$ MPa. The response curve for each is identical to that of Fig. 5 for the same initial imperfection $\xi = 0.01$, indicating that no change in the buckling response has occurred over that under constant load. Response curve $b$ corresponds to insert $b$ in which there is a partial load reversal to $\sigma_v = +51.5$ MPa. Here a measurable change in the calculated creep buckling response is observed, the critical time having diminished from the constant load case by about 30%. Finally, in curve $c$, which corresponds to the loading history of insert $c$ involving a complete load reversal, a reduction of more than 35% in the time to buckling is noted. This reduction in both curves $b$ and $c$ is attributed to the presence of dynamic recovery (Fig. 1) where creep (or relaxation) is observed to be accelerated with stress reversals—even in the absence of significant reversed inelastic strain. Although, to the knowledge of the writers, these effects have not been observed directly in creep buckling phenomena, it is expected that such effects can occur, on the basis of the experimental observations illustrated schematically in Figs. 1 and 2. The influence of dynamic recovery is best understood by considering the associated state space $(\sigma_{ij}, \alpha_{ij})$. However, as this is a four-dimensional space, for the present problem, a comparable but reduced invariant space is desired for easy visualization. The representation selected is that of $\Sigma$ versus $\nabla$ as shown in Fig. 7, where

$$\Sigma = \sqrt{j_2} \text{sgn} (\sigma_v)$$

(40)

FIG. 6. Nondimensional Radial Displacement versus Time for Variable Loading Histories which Induce Dynamic Recovery, i.e., Those Depicted in Inserts a, b, and c
FIG. 7. Invariant State Space for Inner Membrane Showing State Path for Constant Load (Curve a, Fig. 6)

\[ V = \sqrt{I_2 \text{sgn}(\alpha_x)} \]  
\[ J'_2 = \frac{1}{2} S_{ij} S_{ij} = \frac{1}{3} (\sigma_x^2 + \sigma_h^2 - \sigma_x \sigma_h)_{i,o} \]  
\[ I_2 = \frac{1}{2} \alpha_y \alpha_{ij} = \frac{1}{3} (\sigma_x^2 + \sigma_h^2 - \alpha_x \alpha_h)_{i,o} \]

This representation of the state space is analogous to that employed in Robinson and Arnold (1986) for the one-dimensional column buckling problem. In Fig. 7 and all subsequent representations of the state space, compressive \( \sigma \) and \( \alpha \) are shown as positive and are plotted upward and to the right, respectively. Hereafter, the relevant quadrants of the state space will be referred to as the first \( (\Sigma V > 0) \) and the fourth \( (\Sigma V < 0) \).

In Fig. 7 the trajectory of the state point \( (\Sigma, V) \) for the inner membrane is shown for the constant load, responsive curve a in Fig. 6. The segment 0A corresponds to the path of the state point during initial load-up to \( \sigma_x = -103 \text{ MPa} \). Inelasticity is indicated over path 0A by an increase in \( V \). With \( \sigma_x \) held constant, the inner membrane then creeps under nearly constant axial stress, and the state point moves toward B. As the geometric nonlinearity becomes prevalent, the axial stress in the inner membrane increases rapidly (as the outer decreases) while the inner hoop stress decreases (as that in the outer increases) and the state point moves toward C. Point C corresponds to the buckled condition \( \eta = 1 \) in Fig. 6 (curve a). The effect of abrupt load...
changes (insert a, Fig. 6) are vertical (elastic) trajectories in the state space, resulting in no overall change in the state path 0ABC from that just considered for a constant load. This, of course, results in the same creep buckling response observed earlier for the constant load.

The state path of Fig. 8 relates to the loading history of insert b in Fig. 6, which now produces partial stress reversals in the inner (and outer) membrane elements. Here evidence of state recovery is clearly visible, as the stress point trajectories now follow curved paths as the state point penetrates into the fourth quadrant ($\Sigma V < 0$), corresponding to a reversal of stress. The state recovers with the stress reversal, resulting in a relatively softer state ($\alpha_{ij}$) upon reloading. Correspondingly, the creep rate is increased as the state point is returned back into the primary creep regime (Arnold 1987). The net effect of this dynamic recovery is a reduction in the critical time to buckling of approximately 30%.

Load histories involving larger stress reversals produce increased state recovery. In fact, for the given ramp rate $|\dot{\Sigma}_V| = 1,276$ MPa/s in this full reversal case (insert c, Fig. 6) the effect of dynamic recovery at each unloading/reloading cycle is to return the state point well back into the primary creep regime upon reloading. Clear evidence of this increased creep rate following reloading is visible in curve c in Fig. 6, with the cumulative effect of diminishing the critical time to buckling by approximately 35%.

Note, however, that the amount of dynamic recovery occurring for a given stress reversal is strongly dependent not only upon the amount of reversal,
but also upon the rate of reversal (unloading/reloading) and the material parameter $G_0$; see Arnold (1987).

As the period of application of the rapid loading cycles ($\tau_0 = 0.15$) was chosen quite arbitrarily, the effect of the frequency of load cycles is now investigated. Fig. 9 shows the buckling response curves $\eta(\tau)$ for load histories involving a full reversal, i.e., $\sigma_v = -103$ MPa to $\sigma_v = +103$ MPa (as in insert c, Fig. 6), with periods $\tau_0$, $\tau_0/2$, $\tau_0/4$, and $\tau_0/8$. The pronounced effect of more frequent (shorter period) stress reversals is evident. Reversals with period $\tau_0/8$ reduce the time to buckling by more than a factor of 3.5.

**Thermal Recovery**

Now, the influence of load reductions of extended duration on the creep buckling time is examined. The loading histories considered are shown in the inserts of Fig. 10. Case a (i.e., insert a and response curve a) is the previously considered case (insert a of Fig. 6) where the time duration at the reduced load is effectively zero ($\sigma_v = 0$). This history, as noted earlier, produces no change in the buckling response over the constant load case (i.e., no state recovery).

Insert b depicts a history where the load, having been applied for period $\tau_0$, is abruptly removed (at a rate $|\dot{\sigma}_v| = 1,276$ MPa/s) and held at zero for a period $\tau_1$ comparable to the actual critical time ($t_0 = 0.8$ h) of the shell under constant axial load. This sequence is then repeated, reducing the critical time as shown in curve b, by about 28%. Here, the time includes only
the time for which the compressive load $\sigma_v = -103$ MPa is applied.

As before, this behavior is best visualized in the state space $(\Sigma, V)$ shown in Fig. 11. State recovery is observed in that the state point moves, at approximately zero average stress, toward smaller $V$ ($\alpha_m$), e.g., on the first load reduction, from point D to point E. Reloading returns the state point to a softer state and correspondingly to a higher creep rate. Repetition of this sequence thereby causes acceleration of the creep buckling process.

Inserts c and d of Fig. 10 show loading histories in which the time at zero stress is increased an order of magnitude $10\tau_1$ and a hundredfold to $100\tau_1$, respectively. Correspondingly, due to significantly increased recovery with the tenfold and hundredfold increase in hold time, response curves c and d show further decreases in critical time of approximately 36% and 38%, respectively.

**Comparison with an Equivalent Odqvist Law**

In this section, a comparison of the previously presented results with those obtained by a classical multiaxial creep law of the Norton type [Odqvist law (Timoshenko and Gere 1961)],

$$\dot{\epsilon}_i = C(J_2)^m S_{ii} \quad \text{subjected to identical loading histories, is made.}$$

A roughly equivalent (shell)
FIG. 11. Invariant State Space for Inner Membrane Showing State Path Corresponding to History (Hold Period of $\tau_1$) of Insert b, Fig. 10

representation was obtained by calculating the steady state creep rates for various stress levels using the uniaxial form of Eqs. 1–3 and "fitting" the uniaxial form of Eq. 44, that is

$$\dot{\epsilon}' = B\sigma^N \text{sgn}(\sigma)$$  \hspace{1cm} (45)

to these "data" by choosing optimal values of $B$ and $N$ in a least squares sense. This process yielded the following:

$$B = 2.2 \times 10^{-3}$$  \hspace{1cm} (46)

$$N = 8.397$$  \hspace{1cm} (47)

which are consistent with units of $\sigma$ in MPa and $\dot{\epsilon}'$ in h$^{-1}$. These constants $N$ and $B$ in Eq. 45 relate to $M$ and $C$ in Eq. 44 as

$$M = \frac{1}{2} (N - 1)$$  \hspace{1cm} (48)

$$C = \frac{1}{2} (3^{(N+1)/2})B$$  \hspace{1cm} (49)

Eq. 44 is then utilized to calculate the inelastic strain in Eq. 34. The critical time to buckling corresponding to the constant load case of curve a in Fig. 5 is $t_0' = 0.933$ h, which is slightly greater than the earlier reference $t_0 = \ldots$
CONCLUSIONS

The influence of state recovery on the creep buckling response of an idealized circular sandwich shell has been examined. Although the sandwich shell model employed is highly idealized, the constitutive model is quite comprehensive, in that a representation of both dynamic and thermal (state) recovery is incorporated. Experimental evidence suggests that many structural alloys embody internal mechanisms at elevated temperatures that allow inelastic strain rates to increase (recover) following periods of hardening. In particular, this is believed to be true for the representative copper alloy Narloy-Z characterized here.

The loading sequences examined are not intended to represent prototypical loading histories for any particular structural component, but instead are chosen to best illustrate the generic influence of both dynamic and thermal recovery in the presence of a creep-induced instability. It is expected that, at
least qualitatively, similar behavior will accompany more realistic variable loading conditions (and more realistic structures) and that neglecting state recovery effects in such cases may lead to nonconservative predictions of the critical time to creep buckling.

The following conclusions are drawn from this study:

1. State recovery (dynamic and thermal) can have a significant effect on creep buckling behavior, i.e., on the critical time to buckling.
2. Failure to account for state recovery in the constitutive equations can lead to nonconservative predictions of the critical buckling time under variable loading.
3. A classical (multiaxial) Norton–Baily type creep law, as is commonly used in creep buckling analyses, does not account for state recovery and therefore may significantly overpredict the time to creep buckling under variable loading histories.
4. Similar qualitative trends were found in a previous study of a sandwich column. The present study suggests that state recovery will have similar influence on creep buckling behavior under more complex (real) states of stress.

**APPENDIX I. REFERENCES**


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**APPENDIX II. NOTATION**

The following symbols are used in this paper:

\[ a, \delta, d, H = \text{dimensions describing shell geometry}; \]
\[ E = \text{Young's modulus}; \]
\[ e = \text{total strain}; \]
\[ i, o = \text{(sub-subscripts) inner and outer shell membrane, respectively}; \]
\[ P(t) = \text{applied axial load}; \]
\[ S_{ij} = \text{deviatoric stress tensor}; \]
\[ t = \text{temperature}; \]
\[ t_0 = \text{actual critical time to buckling}; \]
\[ W = \text{radial deformation}; \]
\[ x, h = \text{(subscripts) axial and circumferential (hoop) components, respectively}; \]
\[ \alpha = \text{internal or back stress}; \]
\[ \epsilon' = \text{inelastic strain}; \]
\[ \zeta = \text{normalized initial imperfection}; \]
\[ \eta = \text{normalized radial displacement}; \]
\[ \lambda = \text{half-wavelength in axial direction}; \]
\[ \nu = \text{Poisson's ratio}; \]
\[ \rho = \text{stress ratio}; \]
\[ \Sigma = \text{deviatoric stress invariant}; \]
\[ \sigma = \text{stress}; \]
\[ \sigma_a = \text{average applied normal stress}; \]
\[ \tau = \text{normalized time to buckling}; \]
\[ \nabla = \text{internal stress invariant}. \]