Aiming at the performance-enhancement in coarse mesh modeling, we utilize a number of closed form solutions of a class of torsionally loaded thin-walled bars to formulate a two-noded element for spatial buckling analysis. The key in this relates to the use of the “exact” solution for the displacement fields (as oppose to the more conventional finite element approach where polynomial/Lagrangian-type interpolation is employed). That is, in addition to the well known “exact” solution for the coupled flexure/transverse-shear problem, we utilize a new “exact” solution for the more difficult case of coupled system of differential equations governing a torsionally loaded thin-walled beam using the higher-order theories of non-uniform twist/bi-moment with coupled warping-shear deformations. For the linear analysis, convergence and accuracy study indicated that the proposed model to be rapidly convergent, stable and computationally efficient; i.e. one element is sufficient to exactly represent an end loaded part of the beam. Such model has been extended to account for nonlinear analysis, in particular, the flexural torsional buckling of thin-walled structures. To this end, the effect of finite rotations in space is accounted for as per the modern theories of spatial buckling, resulting in second-order accurate geometric stiffness matrices. Compared with the classical theory of thin-walled structures, the present approach is more general in that all significant modes of stretching, bending, shear (due to both flexure and torsional/warping), torsion, and warping are accounted for. The inclusion of non-uniform torsion is accomplished through adoption of the principle sectorial area. This requires incorporation of a warping degree of freedom in addition to the conventional six degrees of freedom at each node. The element is derived for general cross sections including the Wagner-effect contributions. The model’s properties and performance, particularly with regard to the resulting (significant) improvements in mesh accuracy, are assessed in a fairly complete set of numerical simulations.

Keywords: Buckling; thin-walled sections; closed form solution; coarse/mesh accuracy; warping/shear effects; Wagner-effect; mono-symmetric section.
1. Introduction

The past few years have witnessed research efforts directed towards the development of effective nonlinear models for thin-walled beams of different configurations and cross-sectional shapes. Currently, a large number of theoretical studies as well as numerical simulations utilizing the finite element exist for nonlinear analysis of one-dimensional structures. However, for the most part, these studies have focused on cases involving solid cross-sections (i.e. no warping). In this, three main approaches being identified for analysis: Total Lagrangian, Updated Lagrangian and Euclidian. In the former approach system variables are referred to the initial configuration, which leads to complex strain-displacement relationships. In the second approach, system quantities are referred incrementally to the last known equilibrium configuration, which results in simpler strain-displacement relationships. In the latter approach, quantities are referred to the current unknown configuration. Adopting this Euclidian finite element approach with rotation parameters having the traditional meaning of non-commutative orthogonal transformations in Euclidean space, the “consistently-linearized” weak (variational) form derives was shown to generally exhibit a non-symmetric geometric stiffness, even for conservative loading. On the other hand, investigations for more general problems with arbitrary thin-walled sections (i.e. involving the effect of non-uniform or warping torsional behavior) are rather limited. Further, most developed models of this latter case include a simplest form of approximation (shape) functions for warping/torsional deformations.

From both theoretical and numerical stand points; a number of fundamentals issues are called for the “consistent” development of general finite element models for spatial stability analysis of thin-walled elastic beams. Among many others, the most important issues reported in the literature are as follows: (1) Careful selection of the shape (interpolation) functions to account for the coupled stretching-flexural-torsional response, and to avoid the so-called locking phenomena for the limited case of thin beams when the shear deformations are considered in the formulation. (2) Effect of finite nodal rotations on the derivation of the second-order-accurate geometric stiffness components (e.g. Refs. 2, 12 and 13) to model the complete spectrum of the significant instability modes. Details of studies aiming at item (1) constitute our first major objective here, i.e. through the development of new “exact” sets of displacement fields for coupled stretch/flexure/shear/torsion/warping. In particular, a novel one-dimensional formulation for large displacement analysis has been developed, based on the updated Lagrange approach. Interpolation functions for torsional/warping displacements are obtained from the solution of the governing differential equations of a torsionally loaded thin-walled beam with warping restraint. Shear deformations associated with shear/flexure as well as torsion/warping modes are accounted for. The formulation is valid for both open and closed sections; and this is accomplished by utilizing the kinematical description accounting for both flexural and torsional warping effects.

Handling of issues pertinent to item (2) follows along the well known lines of modern theories of spatial buckling; i.e. see pioneering works in Ref. 14; see also
Ref. 3 for detailed results, motivations and significance of these issues in the context of traditional finite element interpolation functions. Only an outline for these parts is given here for completeness.

Our second major objective is to report on the results of an extensive set of problems including linear as well as nonlinear elastic stability. The results obtained by the proposed model are compared to those produced by the hybrid/mixed finite element model (early studies by the author3) to evaluate and assess its performance. The solutions obtained by the new developed model was found to be very rapidly convergent, stable and computationally efficient.

In an out-line-form, the remainder of this paper includes the following sections: weak (variational) formulation, governing equations for thin-walled structures, finite element formulation, verification examples and applications, followed by conclusions. For convenience, tensors as well as their counterpart vector/tensor representations are used interchangeably in all subsequent sections.

2. Weak Formulation-General Form of Nonlinear Analysis

As a starting point, the incremental form of the displacements based variational principle in the step-by-step, non-linear solution has been utilized. For the general continuum case, this takes the following form in the updated Lagrangian description, with the configuration at time $t$ as reference

$$\Delta \pi(u) = \pi(t + \Delta t) - \pi(t)$$

$$\Delta \pi = \int \left[ \frac{1}{2} \Delta \varepsilon^T C \Delta \varepsilon + \sigma^T \Delta e \right] dv - \Delta W$$

where

$$\Delta e = \Delta \varepsilon (\text{linear}) + \Delta \eta (\text{non-linear})$$

$$\Delta \varepsilon_{i,j} = \frac{1}{2} (u_{i,j} + u_{j,i}); \quad \Delta \eta_{i,j} = \frac{1}{2} u_{k,i} u_{k,j} ,$$

are respectively, geometric (from displacement derivatives) Green–Lagrange strain increments; $C$ the material stiffness; $\Delta \sigma = C \Delta \varepsilon$ the stress increments, $\sigma$ the true (Cauchy) stress (initial stresses); and $\Delta W$ the work of prescribed surface/body forces. Equation (1d) gives the tensor components for the linear and non-linear (geometric strains) $\Delta \varepsilon$ and $\Delta \eta$ in terms of the incremental displacement field $u$ (with the reference to rectangular cartesian co-ordinates). It can be noted that the use is made of the summation convention over repeated indices, and the “comma” subscript indicates the differentiation with respect to the spatial co-ordinate following.

The above expression can be now specialized and used as a basis for general non-linear (incremental) analysis of thin-walled structures, i.e. accounting for the effect of initial pre-buckling displacements, instability states, as well as post-buckling...
response. However, restricting the scope of the paper to linearized buckling Eq. (1a) can be now written as:

$$
\Delta \pi = \pi(\text{buckling state}) - \pi(\text{initial state}).
$$

Once the finite element discretizations are introduced for incremental displacement \( \mathbf{u} \) in \( \Delta \mathbf{e} \), the stationarity condition; i.e. \( \delta \Delta \pi = 0 \), with respect to displacement parameters will the yield the governing stiffness equations. The first term in Eq. (1b) yields the element linear stiffness; whereas the geometric stiffness results from the contribution of non-linear (quadratic) strains in the second term. Specific forms of the various arrays for thin-walled element are given later.

3. Governing Equations for Thin-Walled Structures

3.1. Kinematics

The classical theory of thin-walled structures with arbitrary cross-section is based on the classical work of Timoshenko\(^1\) on shear-deformation effects and Vlasov\(^2\) on out-of-plane warping of cross-sections of beams. The following basic assumptions are utilized in the geometric/kinematic descriptions:

(A2): Small strains but large displacements and cross-section rotations.
(A3): Small warping displacement relative to typical lateral beam dimensions.
(A4): Elastic, isotropic and homogenous material.

Consider a typical straight, two-noded, element whose centroidal axis is taken as the beam reference line. The rectangular right hand orthogonal coordinate system \( x, y, \) and \( z \) is chosen such that \( (y, z) \) are the principle centroidal axes of the element. The shear center is located at distances \( e_y \) and \( e_z \) from the \( z \)- and \( y \)-axes, respectively, as shown in Fig. 1.

Fig. 1. Beam element; generalized forces, displacements, and reference axes.
Based on assumption (A1) above, the local incremental displacement field $u$ at any point on the beam cross-section can be expressed in terms of incremental translations $u_o$, the rigid cross-section rotation $\theta$, as well as the “superposed” local warping displacement $\chi$, i.e. with $u = (u, v, w)$,

$$
\begin{align*}
  u &= u_o - y\theta_z + z\theta_y - \omega \chi + y^*\theta_x \theta_y + z^*\theta_x \theta_y \\
  v &= v_o - z^*\theta_x + z\theta_y \theta_z - y^*\theta_x^2 - y\theta_x^2 \\
  w &= w_o + y^*\theta_x + y\theta_y \theta_z - z^*\theta^2_x - z\theta^2_y
\end{align*}
$$

(3a)

where

$$
\begin{align*}
  y^* &= (y - e_y) \\
  z^* &= (z - e_z).
\end{align*}
$$

(3b)

In the above, the axial displacement $u_o$, as well as the cross-section rotations $\theta_y$ and $\theta_z$ are referred to the centroid of the cross-section; while the transverse displacements $v_o$, $w_o$, cross-section rotation $\theta_x$, and warping displacement $\chi$ are referred to the shear center. The warping displacement $\chi$ is assumed to be independent on the derivative of the angle of twist to account for the shear deformations due to warping/torsion effects. The warping function $\omega(y^*, z^*)$ is local prescribed out-of-plane displacement and depends on the cross-sectional shape. Expressions for the generalized warping function (e.g. Refs. 11, 15–17), giving the predefined distribution of warping displacements over typical cross sectional shapes, are available in the literature. These include thickness and contour warping contributions in the open sections,\textsuperscript{11,17} as well as the additional contribution associated with St. Venant torsion in closed and mixed sections.\textsuperscript{11,17}

It is worth to notice that the rotational terms in Eq. (3) are the second-order approximation of the incremental rotational motion. For emphasis the term arising from the large-rotation effect is shown underlined. This is crucial for attaining a sufficient accurate geometric stiffness to represent the whole spectrum of significant instability modes.

### 3.2. Generalized strains and associated stresses

For the present one-dimensional beam element, only three strain components and the associated stresses are significant. The non-vanishing components of the Green–Lagrange strain tensor are

$$
\Delta e = [\Delta e_{xx}, \Delta e_{xy}, \Delta e_{xz}]^T = \Delta \varepsilon + \Delta \eta
$$

(4)

where $\Delta \varepsilon$ and $\Delta \eta$ are the corresponding linear and non-linear parts, respectively. For beam model, it is more convenient to utilize the generalized strains ($\Delta \varepsilon_R$ and $\Delta \eta_R$) instead. The incremental generalized linear strain vector $\Delta \varepsilon_R$ can be defined in the following “ordered” components:

$$
\Delta \varepsilon_R = [\varepsilon_o, \gamma_{xy}, \gamma_{xz}, k_y, k_z, k_w, \gamma_{se}, \gamma_{sw}, \gamma_{sw}]^T
$$

(5)
where
\[ \varepsilon_o = u'_o; \quad \gamma_{xy} = v'_o - \theta_z; \quad \gamma_{xz} = u'_o + \theta_y \]
\[ k_y = \theta'_y; \quad k_z = \theta'_z; \quad k_\omega = \chi' \]
\[ \gamma_{sv} = \theta'_y - \chi \]
\[ \gamma_\omega = \theta'_z - \chi \]
in which the prime indicates the differentiation w.r.t. the coordinate \( x \).

I n t h e above, \( \varepsilon_o \) is the axial stretch, \( \gamma_{xy} \) and \( \gamma_{xz} \) the (average) transverse shear strains due to flexure; \( k_y, k_z \) and \( k_\omega \) the bending and warping curvatures; and \( \gamma_{sv} \) and \( \gamma_\omega \) the torsional shear strains associated with the St. Venant (uniform torsion) and warping (non-uniform torsion) response components, respectively.

Similarly, one can write the generalized non-linear strain components, \( \Delta \eta_R \), as
\[ \Delta \eta_R = [\eta_o, \eta_{xy}, \eta_{xz}, k_y^N, k_z^N, k_\omega^N, \eta_{sv}, \eta_\omega]^T \]

where
\[ \eta_o = \frac{1}{2} [u_o'^2 + v_o'^2 + w_o'^2 + \varepsilon_z^2 + 2(e_z v'_o - e_y w'_o) \theta'_z]
\[ -e_y (\theta'_z \theta'_y + \theta_z \theta'_y) - e_z (\theta'_z \theta'_z + \theta_z \theta'_z)] \]
\[ \eta_{xy} = -u'_o \theta'_z + w'_o \theta'_x + \frac{1}{2} \theta_x \theta_y \]
\[ \eta_{xz} = u'_o \theta'_y - v'_o \theta'_x + \frac{1}{2} \theta_x \theta_z \]
\[ k_y^N = u'_o \theta'_y - v'_o \theta'_x + \frac{1}{2} \theta_x \theta_y \]
\[ k_z^N = u'_o \theta'_z - w'_o \theta'_x + e_y \theta'_z - \frac{1}{2} (\theta'_z \theta'_z + \theta_z \theta'_z) \]
\[ k_\omega^N = u'_o \chi' \]
\[ \eta_{sv} = \frac{1}{2} (\theta'_y \theta'_z - \theta_y \theta'_z) \]
\[ \eta_\omega = \eta_{sv} - u'_o \chi \]

with
\[ r_y^2 = \left[ \frac{I_y + I_z}{A} + (e_y^2 + e_z^2) \right] . \]
The fourth item in Eq. (8a) is often referred to as “uniform stretching” Wagner (effect) parameter. The underlined terms in Eq. (8) arising from the effect of large rotation [see Eq. (3)] are indicated here for further discussions.

The incremental stress resultant \( \Delta \sigma_R \) can be defined as
\[ \Delta \sigma_R = [\Delta F_x, \Delta F_y, \Delta F_z, \Delta M_y, \Delta M_z, \Delta M_\omega, \Delta T_{sv}, \Delta T_\omega]^T \]
in which $\Delta F_x$, $\Delta F_y$ and $\Delta F_z$ are the normal and shear force components; $\Delta M_y$, $\Delta M_z$, $\Delta M_\omega$ the bending and bi-moments; $\Delta T_{sv}$ and $\Delta T_\omega$ the contribution to the twist moment $\Delta M_x$, i.e. the St. Venant and warping (or “bitwist”) torsional moments, respectively. It is worthy to mention that the shear forces, torsional moments, and bi-moment are referred to the shear center; while the normal force, and bending moments are referred to the centroid of the section. Such incremental components can be defined as

$$\Delta F_x = \int_A \Delta \sigma_{xx} dA; \quad \Delta F_y = \int_A \Delta \sigma_{xy} dA; \quad \Delta F_z = \int_A \Delta \sigma_{xz} dA$$

$$\Delta M_y = \int_A \Delta \sigma_{xx} z dA; \quad \Delta M_z = -\int_A \Delta \sigma_{xy} y dA; \quad \Delta M_\omega = -\int_A \Delta \sigma_{xz} \omega dA$$

$$\Delta T_{sv} = \int_A [\Delta \sigma_{xy}(z^* + \omega_{,y}) + \Delta \sigma_{xz}(y^* - \omega_{,z})] dA;$$

$$\Delta T_\omega = \int_A [(\Delta \sigma_{xy} \omega_{,y} + \Delta \sigma_{xz} \omega_{,z})] dA.$$  \hfill (10)

Similar expressions are used for the eight components of the total generalized stress vector of the initially stressed beam, but with the increments $\Delta \sigma$ replaced with their “total” counterparts $\sigma$.

Using the above equations; i.e. Eqs. (1c), (3), (6) and (10), one can arrive at the “resultant-type” constitutive expression; i.e.

$$\Delta \sigma = C \Delta \varepsilon_R$$  \hfill (11)

where the symmetric $(8 \times 8)$ matrix $C$ is the spatial elasticity tensor, i.e. section-rigidities (or moduli) matrix; i.e.

$$C = \text{Diag.}[EA, GA_{xy}, GA_{xz}, EI_y, EI_z, EI_\omega, GJ, G(I_p - J)]$$  \hfill (12)

where $\text{Diag.}[\ ]$ denotes a diagonal matrix. In addition to the well known axial, shear and bending stiffness coefficients in the first five diagonal terms in Eq. (12) ($A_{si} = \alpha A_i$, for $i = y, z$ are the flexural shear correction factors$^{2,3,11}$), the following St. Venant and warping torsion rigidities are defined as in Refs. 15–17

$$J = \int_A [(y^* - \omega_{,z})^2 + (z^* + \omega_{,y})^2] dA, \quad I_p = \int_A \rho^2 dA, \quad I_\omega = \int_A \omega^2 dA$$  \hfill (13)

where $\rho$ is the perpendicular distance from the shear center to the tangent to the sectorial profile at considered point.$^{11}$

4. Finite Element Formulation

4.1. Interpolation functions for displacements

For the present two-node one dimensional element, designated here as DEB2, the incremental displacement fields, $\mathbf{u}$, within the element are approximated in terms
of the incremental nodal displacements, $\mathbf{q}$, as

$$ u = [h_1 \ h_2] \mathbf{q}_u ; $$

$$ \begin{bmatrix} \mathbf{v} \\ \theta_z \end{bmatrix} = \begin{bmatrix} h_3 & h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 & h_{10} \end{bmatrix} \mathbf{q}_{\theta_z} ; $$

$$ \begin{bmatrix} \theta_x \\ \chi \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{15} & h_{16} & h_{17} & h_{18} \end{bmatrix} \mathbf{q}_{\theta_x \chi} $$

(14)

where $h_i$ is the $i$th shape function, $\mathbf{q}_u$, $\mathbf{q}_{\theta_z}$, and $\mathbf{q}_{\theta_x \chi}$ are the column vectors containing the nodal displacements for stretching, shear/flexure, and torsion/warping respectively. The axial displacement is interpolated utilizing the linear shape function, i.e.

$$ h_1 = 1 - \xi ; \quad h_2 = \xi $$

(15a)

with

$$ \xi = x/L . $$

(15b)

The Hermite shape functions are used to interpolate the transverse displacement $v$ and associated rotation $\theta_z$ as

$$ h_3 = \frac{1}{1 + \phi_z} \left[ 1 - 3 \xi^2 + 2 \xi^3 + \phi_z (1 - \xi) \right] ; $$

$$ h_4 = \frac{L}{1 + \phi_z} \left[ \xi - 2 \xi^2 + \xi^3 + \frac{1}{2} \phi_z \xi (1 - \xi) \right] $$

(16a)

$$ h_5 = \frac{1}{1 + \phi_z} \left[ 3 \xi^2 - 2 \xi^3 + \phi_z \xi \right] ; \quad h_6 = \frac{L}{1 + \phi_z} \left[ - \xi^2 + \xi^3 - \frac{1}{2} \phi_z \xi (1 - \xi) \right] $$

(16b)

$$ h_7 = \frac{1}{1 + \phi_z} \left[ - 6 \xi^2 + 6 \xi^3 \right] ; \quad h_8 = \frac{1}{1 + \phi_z} \left[ 1 - 4 \xi + 3 \xi^2 - \phi_z (1 - \xi) \right] $$

(16c)

$$ h_9 = \frac{L}{1 + \phi_z} \left[ 6 \xi^2 - 6 \xi^3 \right] ; \quad h_{10} = \frac{1}{1 + \phi_z} \left[ - 2 \xi + 3 \xi^2 + \phi_z \xi \right] $$

(16d)

where $\phi_z$ is the contribution of the shear deformation effect for flexure in the $x$-$y$ plane and is defined as

$$ \phi_z = \frac{12 EI_z}{GA_{xy} L^2} . $$

(17)

Similar expression to that in the second term of Eq. (14) can be utilized for the transverse displacement $w$ and its associated rotation $\theta_y$. In this case, the interpolation functions are identical to those in Eq. (16) except the two expressions of $h_8$ and $h_{10}$ must multiply by negative sign.

The interpolation functions for the twist rotation and warping displacement are obtained from the solution of the differential equations of the torsionally loaded thin-walled beam with warping restraint. Such governing differential equations of
equilibrium for thin-walled beam under uniform distributed torsional moment $m_x$ can be written as

$$GJ\theta''_x + G(I_p - J)(\theta''_x - \chi') = -m_x \quad (18a)$$

$$EI_w\chi'' + G(I_p - J)(\theta'_x - \chi) = 0. \quad (18b)$$

The above-coupled equations are generally applicable to all thin-walled beams of open, closed and multi-cellular cross-sections.

Differentiating Eq. (18a) twice and Eq. (18b) once w.r.t beam axis $x$, one can get

$$G(I_p - J)\chi''' = GI_p\theta''_x + \frac{d^2m_x}{dx^2} \quad (19a)$$

$$EI_w\chi'' + G(I_p - J)(\theta''_x - \chi') = 0. \quad (19b)$$

Substituting from Eq. (19a) into Eq. (19b) and employing Eq. (18a), one can arrive at

$$\theta''_x - k^2\theta''_x + \frac{1}{GI_p} \frac{d^2m_x}{dx^2} - \frac{k^2}{GJ}m_x = 0 \quad (20)$$

where

$$k^2 = \frac{GJ (I_p - J)}{EI_w I_p}. \quad (21)$$

The above factor $k$ is a measure of the rate of decay of torsional warping effects along the girder and away from the point of warping restraint. Substituting from Eq. (18a) into Eq. (19b) yields

$$\chi''' - k^2 \left( \chi' + \frac{m_x}{GJ} \right) = 0. \quad (22)$$

Equations (20) and (22) are the two differential equations for a torsionally loaded thin walled beam with warping restraint. Assuming that the beam has no distributed twist (i.e. $m_x = 0$), the solution of Eq. (20) can be obtained in the form

$$\theta_x = D_1 + D_2x + D_3 \sinh kx + D_4 \cosh kx \quad (23)$$

where $D_i, (i = 1, 2, 3, 4)$ are constants to be determinate. The warping displacement $\chi$ can now be obtained in terms of the rotation, $\theta_x$, from Eq. (18a) as

$$\chi' = \frac{I_p}{(I_p - J)} \theta''_x. \quad (24)$$

Employing Eq. (23) into Eq. (24) and integrate, one can arrive at an expression of warping displacement as

$$\chi = \frac{GJ}{EI_w k} [D_3 \cosh kx + D_4 \sinh kx + D_5]. \quad (25)$$
In the above, \( D_5 \) is another constant of integration. Substituting from Eq. (23) and Eq. (25) into Eq. (18b) and collecting terms, one can write the constant \( D_5 \) in terms of \( D_2 \) as

\[
D_5 = \frac{EI_w}{GJ} k D_2. \tag{26}
\]

Substituting from Eq. (26) into Eq. (25), one can write the warping displacement, in addition to the rotation \( \theta_x \), in terms of the generalized coordinates \( D_i \), \( (i = 1, 2, 3, 4) \), in a matrix form as

\[
\begin{bmatrix}
\theta_x \\
\chi
\end{bmatrix} = \begin{bmatrix} 1 & x & \sinh kx & \cosh kx \\
0 & 1 & \zeta \cosh kx & \zeta \sinh kx
\end{bmatrix} D \tag{27}
\]

where

\[
M = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & \zeta & 0 \\
1 & L & \sinh kL & \cosh kL \\
0 & 1 & \zeta \cosh kL & \zeta \sinh kL
\end{bmatrix}. \tag{31}
\]

Solving Eq. (30) in \( D \) and substituting into Eq. (27), one can arrive at the expressions for the interpolation functions for the torsion/warping displacements (i.e. \( h_i \) \( (i = 11 \rightarrow 18) \) in Eq. (14)) as

\[
h_{11} = 1 + \frac{\delta}{\alpha \lambda} \mathcal{I} + 2 \mathcal{R} \mathcal{E}; \quad h_{12} = \xi L + \frac{1}{\alpha} \left( \frac{\delta L}{\lambda} - \frac{1}{\zeta} \right) \mathcal{I} + \left( \mathcal{R} L - \frac{\delta}{\zeta \alpha \lambda} \right) \mathcal{E} \tag{32a}
\]

\[
h_{13} = -\frac{\delta}{\alpha \lambda} \mathcal{I} - 2 \mathcal{R} \mathcal{E}; \quad h_{14} = \frac{1}{\zeta \alpha} \mathcal{I} + \frac{\delta}{\zeta \alpha \lambda} \mathcal{E} \tag{32b}
\]

\[
h_{15} = -\mathcal{R} \mathcal{I} + \frac{\delta}{\alpha \lambda} \zeta \sinh kx; \quad h_{16} = 1.0 - \beta \mathcal{I} + \Psi \sinh kx \tag{32c}
\]

\[
h_{17} = \mathcal{R} \mathcal{I} - \frac{\zeta}{\alpha \lambda} \delta \sinh kx; \quad h_{18} = -\frac{\delta}{\alpha \lambda} \mathcal{I} + \frac{1}{\alpha} \sinh kx \tag{32d}
\]
where

\[ I = \cosh kx - 1; \quad \epsilon = \xi x - \sinh kx \]  
\[ \delta = \cosh kL - 1; \quad \lambda = \sinh kL - \zeta L \]  
\[ \alpha = \sinh kL - \frac{\delta^2}{\alpha}; \quad \mathcal{R} = \frac{1}{\lambda} \left( 1 + \frac{\delta^2}{\alpha^2} \right) \]  
\[ \beta = \frac{1}{\lambda} \left( L\zeta + \frac{L\zeta}{\alpha}\delta^2 - \frac{\delta}{\alpha} \right); \quad \Psi = \frac{1}{\alpha} \left( \frac{\zeta L\delta}{\lambda} - 1 \right). \]  

4.2. Element stiffness equations

Specializing for the present beam case, a “consistently linearized” expression for a displacement-based variational principle can be typically be written as (using updated Lagrange approach)

\[ \Delta \pi = \int \left[ \frac{1}{2} \Delta \varepsilon_R^T C \Delta \varepsilon_R + \sigma_R^T \Delta \eta_R \right] dx + U_{NL}^{\eta} - \Delta W \]  

where

\[ U_{NL}^{\eta} = \int \int_A \left( \sigma_{xx} - \frac{F_x}{A} \right) (y^2 + z^2) dA \right] \theta^2_2 dx \]  
\[ \Delta W = \Delta \bar{W} - \int \sigma_R^T \Delta \varepsilon_R dx. \]  

In the above, the integration is carried out over the total length, \( l \), of the beam and all stress/strain vectors are expressed in the resultant form, with ordering of components as in Eq. (5). The term in Eq. (34) accounts for the so-called Wagner effect associated with the torsional warping actions in beam with unsymmetrical sections (i.e. it vanishes for the bi-symmetric case). The first parenthetical term in Eq. (34b) is obtained from the well-known expression (i.e. \( \frac{M_y L_z^2 L_x - M_x L_z^2 L_y - M_z L_x L_y}{I_y I_z} \)). The second term in the bracket of Eq. (34a) gives the geometric stiffness for bi-symmetric case, which can be expand as

\[ \int \sigma_R^T \Delta \eta_R dx = \int \left[ F_x \eta_0 + F_y \eta_{y} + F_z \eta_{z} + M_y \kappa_y^N + M_z \kappa_z^N + M_\omega \kappa_\omega^N + T_{sv} \eta_{sv} + T_\omega \eta_\omega \right] dx \]  

where the generalized incremental non-linear strains are given in Eq. (8), and the associated internal stress resultants (at the initial state) can be obtained from the linear analysis as (providing that the beam is unloaded between its two nodes)

\[ F_x = F_{x2}; \quad F_y = F_{y2}; \quad F_z = F_{z2} \]  
\[ M_i = -M_{i2} \left( 1 - \frac{x}{L} \right) + M_{i2} \frac{x}{L}, \quad (i = y, z) \]
where \((.\,)_1\) and \((.\,)_2\) are the quantities \((.\,)\) calculated at the first and the second nodes, respectively.

The external potential \(\omega W\) in Eq. (34c), for conservative end loads, is given by

\[
\Delta W = \sum_{i=1}^{2} |\mathbf{Q}_i|_{x=x_i},
\]

where (see Fig. 1)

\[
\mathbf{Q} = [F_x, F_y, F_z, M_x, M_y, M_z]^T; \quad \mathbf{q} = [u, v, w, \theta_x, \theta_y, \theta_z]^T
\]

are the nodal forces and the displacements, respectively. The over-bar in Eq. (37a) signifies a prescribed quantity. Note that the kinematic boundary conditions here correspond simply to specifying any of the components in \(\mathbf{q}\).

Substituting Eq. (14) into Eqs. (6) and (8), then using them into linearized form of variational principle (i.e. Eq. (34), and invoking the stationary condition w.r.t. the nodal parameters, \(\mathbf{q}\), leads to the desired stiffness relationships

\[
(K_L + K_{NL}) \cdot \mathbf{q} = \bar{Q} - Q_I
\]

where

\[
K_L = \int_b B^T C B d\mathbf{x}; \quad Q_I = \int_b B^T \sigma_R d\mathbf{x}
\]

\[
\frac{1}{2} \mathbf{q}^T K_{NL} \mathbf{q} = \text{the second term in Eq. (34a)} ; \quad \varepsilon_R = B\mathbf{q}.
\]

Here, \(K_L\) is the element linear stiffness, \(K_{NL}\) its geometric stiffness, \(Q_I\) the internal force vector and \(B\) the strain-displacement operator obtained from Eq. (6). Once assembled for entire structure, the global stiffness provides the following criterion for linearized buckling, i.e.

\[
\|K_L + K_{NL}\| = 0
\]

where \(\|\cdot\|\) stands for the determinant, and where \(K_{NL}\) is now calculated for pre-defined initial stress distribution corresponding to the pre-buckling loading state (load factors). The lowest eigen-value and corresponding mode will then define the critical state.
5. Verification Examples and Applications

The performance of the developed model, DEB2, is examined in a number of numerical simulations for linear as well as nonlinear analyses. For linear cases, we focus on several applications involving complex torsional-warping responses, aiming at demonstrating superior accuracy of DEB2 (e.g., compared to more conventional finite elements in the recent literature [e.g., Refs. 11 and 14]). For the nonlinear problems one, attention is given to spatial buckling of beams and frames, in particular, the lateral torsional buckling of beams with symmetrical as well as mono-symmetrical cross sections. Effect of large rotation in space on the nonlinear of kinematic relations is also investigated. Attention in these applications is given to the fact that the nonlinear response of the new element DEB2 maintained its superiority (relative to other conventional finite elements) in the nonlinear regimes.

The present results are compared with a number of well-documented solutions available in the literature. For better appreciation of these comparisons, we also show results from other state-of-art elements; i.e. the mixed element HMB2 in Ref. 3, noting additionally that this element is very comparable to the recent reduced-integration elements in Ref. 14.

5.1. Linear-analysis results

A number of test problems are considered in this section to assess the performance of the developed element in the linear analysis. These include a variety of thin-walled open as well as closed beams under different flexural-torsional loading conditions. Special emphasis is given to the warping shear effect. Results that are always obtained utilizing a single DEB2 element for an end loaded part of beam are compared with those obtained with HMB2 element developed earlier by the authors [i.e. based on the traditional polynomial-type finite element interpolation]. Note that for the special case of linear flexure/torsion problems, one can easily show that the mixed element in Ref. 11 is equivalent to the reduced integration, displacement-type element in Ref. 14. Also comparisons are made with analytical (when it is available) as well as other numerical solutions.

5.1.1. Continuous beam under concentrated torsional moments

A four-span continuous crane girder shown in Fig. 2 is subjected to torsional moments occur in the first span due to a horizontal forces $H = 6.7$ Kips (29.815 KN) exerted by the wheels at the top of the crane rail. The resulting torque at each wheel location is 56.046 Kips.in (3.064 KN.m) referred to the shear center. The assumed values for material properties are $E = 29.00$ Kips/in$^2$ (199.810 MPa), and $G = 11.154$ Kips/in$^2$ (16851 MPa). The girder is modeled with only six DEB2 elements (i.e. 3 elements for the first span, and one element for each of the others). The simply supported boundary conditions corresponding to no-twist-but-free-warping at all supports are considered.
The results of bimoment, and St. Venant and warping torsional moments obtained utilizing the six-DEB2 element-model are identical to those obtained using 36 HMB2 elements.\textsuperscript{11} The torsional warping moment and the bimoment obtained by the DEB2 element are depicted in Fig. 3 along with the analytical results reported in Ref. 18. As evident from this figure, the DEB2 predictions are identical to those by Stefan in Ref. 18.

5.1.2. An end-loaded symmetric perforated core

Two 10-storey perforated core structures with different cross-sections are considered, see Fig. 4. The storey height is 3.5 m; the depth of lintel beam to its span ($d/l_b$) ranges between 0.1–0.7. The Young’s modulus and Poisson’s ratio are $3.0 \times 10^6$ t/m$^2$.
The torsional rotation along the core versus \((\frac{d}{l_b})\) ratios are shown in Figs. 4(a) and (b) for a single DEB2 element, some of the analytical and numerical methods given in Ref. 20, and also the solution obtained using 40 HMB2 elements in Ref. 11. In addition to that obtained by HMB2 element, several solutions are included for comparisons in Fig. 4. For instance, the so-called method 1 signifies an analysis neglecting the shear deformation effects, while method 2 is based on Omanshy–Benscoter\(^{17}\) theory considering such effects; a finite element model and wide-column frame analogy given in Ref. 21 are also depicted in such figure; Berdt’s formula\(^ {20}\) is for an equivalent continuum replacing the lintel beams; and finally alternating open-closed-box section model is designated as method 3 in Fig. 4(a).

The results obtained with a single DEB2 element are identical to those obtained using 40 HMB2 elements in Ref. 11 and are in good agreement with those given by method 2 specially when the \(\frac{d}{l_b}\) ratio gets bigger (i.e. with significant shear deformations).

Fig. 4. Angle of twist of perforated cores with lintel beam; (a) rectangular cross section. (b) square cross section.
5.2. Stability-analysis results

It is worthy to mention that although the performance of the proposed model is “exact” for the linear analysis case (i.e. a single DEB2 element is sufficient to accurately present a non-loaded segment of a beam), such privilege is no longer valid in the stability case. However, much fewer DEB2 elements (compared to the other displacement elements, which are based of the polynomial-type interpolation assumptions) are adequate to yield the converged solution.

5.2.1. A convergence study

Three sets of beam lateral torsional buckling problems are utilized to assess the convergence properties of the developed model. The first is a transversely loaded cantilever with a rectangular cross section; designated as Cant-Rec. The geometrical data as well as material properties are given in Table 1. Two cases of loading conditions are considered: (1) a tip vertical load along the z-axis; i.e. \( F_z \), and (2) an end moment about the y-axis, i.e. \( M_y \). The buckling load/moment predicted by the DEB2 element are compared to the analytical solutions given by Timoshenko and Gere\(^{15}\) and the percentage of errors are depicted in Fig. 5(a).

The analytical solutions for these two cases of loading are 0.10033 N and 7.854 N.cm, respectively. Comparisons are also presented using the results obtained with the HMB2 element\(^3\) and these are also depicted in Fig. 5(a). As evident from this figure, two DEB2 elements produce solutions with errors of 2.9% and 2.1%, respectively, for the two cases of loading. Compared to the results obtained by the HMB2 element,\(^3\) such errors are 40% and 27%, respectively, when the same number of elements is used. Using more number of elements, the errors are reduced to reach 0.05% and 0.2%, respectively, with 12 DEB2 elements.

The second set deals with a (flexural) buckling of a torsionally loaded simply supported circular shaft; designated as SS-Shaft in Table 1. The results for the critical values are compared with the closed form solution obtained by Zigler\(^{22}\) and depicted in Fig. 5(a), along with those produced by the HMB2 element.\(^3\) The value of buckling moment reaches its convergence using 4 DEB2 elements with an error of 0.35%. Such error is 11.4% when four HMB2 elements are used.

In the third set, we considered a problem of a transversely loaded simply supported beam with \( I \)-shaped cross section;\(^{23}\) designated as SS-Isec (see Table 1). Two cases of loading conditions are considered here: (1) a uniform moment about

<table>
<thead>
<tr>
<th>Section</th>
<th>Length (cm)</th>
<th>Area (cm(^2))</th>
<th>( I_y ) (cm(^4))</th>
<th>( I_z ) (cm(^4))</th>
<th>( I_p ) (cm(^4))</th>
<th>( I_w ) (cm(^6))</th>
<th>( J ) (cm(^4))</th>
<th>( E ) (N/cm(^2))</th>
<th>( G ) (N/cm(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cant-Rec</td>
<td>100</td>
<td>1.0</td>
<td>1.0</td>
<td>0.125</td>
<td>1.125</td>
<td>–</td>
<td>0.01</td>
<td>10E3</td>
<td>5E3</td>
</tr>
<tr>
<td>SS-Shaft</td>
<td>100</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>2.0</td>
<td>–</td>
<td>2</td>
<td>10E3</td>
<td>5E3</td>
</tr>
<tr>
<td>SS-Isec</td>
<td>640</td>
<td>30.62</td>
<td>7939</td>
<td>85.50</td>
<td>6048</td>
<td>3224</td>
<td>6.0</td>
<td>30E6</td>
<td>99.4E5</td>
</tr>
</tbody>
</table>
Fig. 5. Convergence study; (a) Solid cross sections. (b) Open cross sections.

$y$-axis; i.e. $M_y$; and (2) a transverse mid-span load along the $z$-axis; i.e. $F_z$. The results for both cases are compared to the analytical solutions reported in Ref. 23 and the percentage of differences are depicted in Fig. 5(b), along with those given by the HMB2 element. As evident from this figure, the buckling values reach their
convergence using 6 DEB2 elements where the differences are 0.05% and 0.9%, respectively. On the other hand, such values of differences are 2.9% and 3.9% when the same number of the HMB2 elements is utilized.

It is seen from the above results that the DEB2 model exhibits an excellent convergence. In all cases a mesh of two to four elements yields the converged solution.

5.2.2. Comparison with shell models

This example concerns the out of plane buckling of a simply supported beam with a T-shaped cross section. The beam is transversely loaded at is mid span. Unlike the problem considered previously, the Wagner effect contribution to the geometric stiffness becomes important here. The beam length is 550 mm, top and flange thickness is 1.0 mm, flange width is 38 mm, and web height is 65 mm. The beam is made of material with Young’s modulus $E = 70960$ MPa, and Poisson’s ratio $\nu = 0.321$. The beam is analyzed using 6 DEB2 elements and its lateral torsional buckling is compared with those obtained using two complicated shell models in Table 2. Note that the difference is less than 0.2%.

5.2.3. Effect of load location on lateral torsional buckling of mono-symmetric beam

The flexural torsional buckling of a cantilever with a mono-symmetrical I-section investigated experimentally by Anderson and Trahair has been considered here. The cantilever is made of high strength aluminum, and it is subjected to tip load. The cantilever length is 65 in; and its cross sectional dimensions are as follows: large flange = 1.241 in, small flange = 0.625 in, flange thickness = 0.1231 in, depth = 2.975 in, and web thickness = 0.0862 in. The effect of the location of the load height, $a$, above or below the shear center can be incorporated by adding the term $(F_z a)$ into the diagonal of the geometric stiffness at the degree of freedom associated with $\theta_x$ and $\theta_y^2$ at the node where the load is applied. The lateral torsional buckling obtained using 4 DEB2 elements for different value of the load height is depicted in Fig. 6 for two different cases: (1) large flange in compression; and (2) small flange in compression.

<table>
<thead>
<tr>
<th>Method</th>
<th>Discretization</th>
<th>Buckling Load (kN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Strip(*)</td>
<td>15 20</td>
<td>1.87</td>
</tr>
<tr>
<td>Finite Element(**)</td>
<td>7 20</td>
<td>1.874</td>
</tr>
<tr>
<td>DEB2</td>
<td>– 6</td>
<td>1.873</td>
</tr>
</tbody>
</table>

(*) Van Erp and Menken (1990)
(**) Chin et al. (1993)
The experimental results obtained by Anderson and Trahair\textsuperscript{26} along with the analytical solution reported by Chan and Kitipornchai\textsuperscript{4} are also presented in this figure. It can be seen that the results produced by the DEB2 element are in very good agreement with both experimental and analytical buckling loads. It is worth to notice that the bucking load increases when the larger flange is in compression. The closer the applied load to the compression flange, the higher value of the lateral torsional buckling is predicted.

5.2.4. Large-versus small-rotation formulations

The accuracy of the resulting buckling model depends on the non-linear kinematic relations in Eq. (4). This is quantified here by comparing the results of two different models: (1) the complete model as described previously (designated as large rotation formulation); and (2) another model based on the small-rotation assumption of Eq. (3). The latter is simply obtained from the former model by discarding the underlined terms in Eqs. (3) and (8). The bifurcation instability of a simply supported frame under end moment $M_z$\textsuperscript{12} has been investigated. Both ($-/+$) and ($+/-$) moments w.r.t. the respective axes are considered (see Fig. 7 for illustration where $M_z(-/+)$ is indicated).

Making use of symmetry, only one leg of the frame is idealized using 4 DEB2 elements. The variations of the critical values for the end moments with the frame subtended angle $\phi$, for both large- and small-rotation analyses have been depicted in Fig. 8.

These are different to varying degrees except for the special case $\phi = 0^\circ$ or $180^\circ$; the latter corresponds actually to a planar problem. On the other words, the small
rotation assumptions often lead to the totally erroneous results as shown in Fig. 8. The results presented in Fig. 8 are identical to those reported in Ref. 12.

5.2.5. Flexural torsion buckling of continuous mono-symmetric I-beams

The flexural-torsional buckling of two-equal span continuous beam has been investigated. Such continuous beam had been investigated earlier by Trahair. The
Fig. 9. Buckling of mono-symmetrical two-span continuous beams.

the significant of buckling interaction between adjacent beam spans had been reported. The beam is made of Aluminum of Young’s modulus $E = 9400$ Kip/in$^2$, and Poisson’s ratio $\nu = 0.214$. The span length is 60 in; and the cross sectional dimensions are: web height and thickness are 2.742 and 0.084 in, respectively; flange width and thickness are 1.242 and 0.1224 in, respectively. Two concentrated loads are located at mid-span points and applied at the top flanges. Three different types of beam cross-section have been investigated: (1) equal flanges, (2) unequal flanges; (3) T-shaped section. The buckling loads predicted by 12 DEB2 elements are depicted in Fig. 9 in the form of interaction-buckling diagram along with those reported by Chan and Kitipornchai.$^4$

The special case of equal flange $I$-beam is also compared with Trahair’s experimental results in this figure. As evident from this figure, the buckling loads predicted with DEB2 elements are in very good agreement with those reported in Ref. 4 for all three cases. However, these are slightly below the experimental values by Trahair$^{27}$ for the case of symmetrical cross-section.

5.2.6. Buckling of simply supported beam-column with mono-symmetric section

Our final example concerns the out-of-plane buckling of a simply supported beam with unequal-flanged $I$-section. This beam is manufactured by reducing the width
of one flange of a doubly symmetric section having the following geometrical data: flange width = 20 cm, web height = 31.3 cm, and flange/web thickness = 1.0 cm. The Young's modulus and Poisson’s ratio are $210 \times 10^3$ MPa and 0.26, respectively.

For purpose of discussion we defined the following parameters

$$
\rho = \frac{I_{zC}}{I_{zC} + I_{zT}}; \quad \mu = \frac{A_{FC}}{A_{FC} + A_{FT}}; \quad r_z = \sqrt{\frac{I_z}{A}}
$$

where $A_{FC}$ and $A_{FT}$ are the area of compression and tension flanges, respectively; $I_{zC}$ and $I_{zT}$ are the moment of inertia of compression and tension flanges, respectively, about the $z$-axis; and $r_z$ is the slenderness ratio.

A mono-symmetrical section beam with $A_w/A = 0.48$ (where $A_w$ is the area of the web) has been analyzed under pure in-plane moment ($-/+$) $M_y$. Such cross section gives $\rho = 0.75$ and $\mu = 0.59$ when the large flange on top; and $\rho = 0.25$ and $\mu = 0.41$ for reversed beam; i.e. large flange in the bottom. The beam has been analyzed using 4 DEB2 elements and the values for the lateral torsional buckling for different slenderness ratio $r_z$ are compared with the analytical solutions in Fig. 10.

The maximum difference obtained is of order of 0.2% for the entire range. Note that, for the same slenderness ratio, the buckling load is higher when the larger flange is in compression zone.

Effects of the ratio of compression flange area to the total flange areas; i.e. $\mu$, on the lateral torsional buckling of a simply supported beam is shown in Fig. 11.

![Fig. 10. Simply supported mono-symmetrical beam under uniform moments.](image)
Two cases of loading conditions are considered: (1) uniform moment $M_y$; and (2) mid-span transverse load $F_z$. Taking the buckling load of the doubly symmetric section as a reference for comparison, the buckling load increases due to reducing the width of the flange in tension. It does not seem logical for the buckling load to increase, than its value for the doubly symmetric section, with reducing the area of the tension flange. However, this fact is due to the centroid of the section is located near the compression flange in this case, and consequently the maximum compression stress is less than its associated value for the doubly symmetric case for the same value of applied loading condition.

The lateral torsional buckling of mono-symmetrical $I$-section beam columns has been investigated next. Beam columns with large flange at the top ($B_t > B_b$) or at the bottom ($B_t < B_b$) have been studied. Two loading conditions are considered: (1) equal and opposite end axial load and moments; and (2) end axial loads and a mid-span concentrated load. The results obtained using 4 DEB2 elements for the two cases of loading conditions are depicted in Figs. 12 and 13, respectively.

In these figures, axial load/moment, and axial/transverse loads envelopes are normalized with respect to the buckling values for the case of pure axial load $F_{x0}$, end moments $M_{y0}$, and transverse mid-span load $F_{z0}$. Several values of $B_b/B_t$ are considered, but only three cases are reported here in Figs. 12 and 13; that is, 0.5, 1.0 and 2.0. In all cases the slenderness ratio $r_z$ is kept constant at 300. It can be seen from these figures that the doubly symmetric case is a good representative to the behavior of the $I$-beam with unequal flanges. That is, the maximum deviations
between any of the unequal flange cases and the doubly symmetric case for the axial load/moment, and axial/transverse loads are 8% and 3%, respectively.

6. Conclusions

The "exact" (non-polynomial-type) solution of the governing differential equations for the torsional displacements of thin-walled beams, with warping restraints and accounting for (higher-order) shear deformation effects, has been adopted to de-
velop the shape functions in a two-noded beam element for spatial analysis of linear and nonlinear (buckling) problems. The overall formulation is capable of predicting all significant modes of deformations; i.e. stretching, bending, shear, torsion and warping. It is also valid for different types of sections (i.e. open, closed, or mixed), and this is accomplished by utilizing the kinematical description accounting for combined flexural and torsional-warping effects, Wagner-effect contributions for the unsymmetric cross-sections, etc. Even in coarse meshes, the solutions obtained with the developed model were shown to be “very” rapidly convergent, stable, and computationally efficient. A large number of numerical examples have also demonstrated the versatility of such model in practical applications. In particular, the new element has consistently demonstrated its superiority compared to the other state-of-art finite element modeling techniques (i.e. using the more common polynomial/Lagrangian-type interpolations).

References
