ON THE THERMODYNAMIC FRAMEWORK OF GENERALIZED COUPLED THERMOELASTIC-VISCOPLASTIC-DAMAGE MODELING

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Abstract—A complete potential based framework utilizing internal state variables is put forth for the derivation of reversible and irreversible constitutive equations. In this framework, the existence of the total (integrated) form of either the (Helmholtz) free energy or the (Gibbs) complementary free energy are assumed a priori. Two options for describing the flow and evolutionary equations are described, wherein option one (the fully coupled form) is shown to be over restrictive and the second option (the decoupled form) provides significant flexibility. As a consequence of the decoupled form, a new operator, that is, the compliance operator, is defined, which provides a link between the assumed Gibb's and complementary dissipation potential and ensures a number of desirable numerical features, for example, the symmetry of the resulting consistent tangent stiffness matrix. An important conclusion reached is that although many theories in the literature do not conform to the general potential framework outlined, it is still possible in some cases, by slight modifications of the employed forms, to restore the complete potential structure.

I. INTRODUCTION

In general, the inelasticity exhibited by the thermomechanical response of engineering materials is related to irreversible thermodynamic processes. These involve energy dissipations and material stiffness variations due to physical changes in the microstructure. Consequently, thermodynamic arguments have often been utilized as a foundation on which phenomenological constitutive laws may be formulated, that is, the so-called internal variable formalism (COLEMAN & GURTIN [1967]; RICE [1971]; LUBLINER [1972,1973]; GERMAIN et al. [1983]; ONAT & LECKIE [1988]; LEMAITRE & CHABOCHIE [1990]; LEHMANN [1991]). Material descriptions like elastoplastic, viscoplastic, continuum-based damage, and so forth (all falling into this framework) have been subjects of extensive research over the years (RICE [1971]; MARTIN [1975]; MILLER [1976]; PONTER & LECKIE [1976]; CHABOCHIE [1977]; KACHANOV [1986]; LECKIE [1986]; MILLER [1987]; KRAJCINOVIC [1983,1989]; ROBINSON et al. [1987]; ROBINSON & DUFFY [1990]; LEMAITRE & CHABOCHIE [1990]; FREED et al. [1991]).

From a strict mathematical standpoint, however, the “thermodynamic-admissibility” restriction associated with the dissipative mechanisms underlying the above material models reduce to the well-known (local) Clausius–Duhem or dissipation inequality (COLEMAN & GURTIN [1967]; LUBLINER [1973]). Indeed, this is the form that is typically employed in discussing the general structure of the “thermodynamically based” constitutive equations developed (COLEMAN & GURTIN [1967]; LUBLINER [1972,1973]; GERMAIN
et al. [1983]; Krajcinovic [1989]; Lemaitre & Chaboche [1990]; Lehmann [1991]), whereby the mathematical constructs such as the flow or dissipation potential and the attendant normality/generalized normality relations, that is, the $\Omega$-form (Rice [1970]; Ponte & Leckie [1967]; Ponte [1976]; Onat & Leckie [1988]; Robinson et al. [1987]; Robinson & Duffy [1990]; Lemaitre & Chaboche [1990]; Freed et al. [1991]) are introduced solely for convenience in satisfying the above constraints based on simple properties of non-negativeness and convexity of these functions. We emphasize, however, that such forms do not presuppose or automatically imply the existence of the total (integrated) forms of the associated thermodynamic potentials; for example, the (Helmholtz) free energy, or the (Gibbs) complementary free energy. When the latter are utilized a priori, the corresponding formulations will be described as exhibiting a complete potential-based structure; on the other hand, those derived from an assumed $\Omega$-form are referred to as incomplete potential-based models.

The (complete) potential-based class of inelastic constitutive equations possesses a number of distinct and important attributes, from both a theoretical and a computational standpoint. First, they constitute the cornerstone of numerous regularity properties and bounding (or limit) theorems in plasticity and viscoplasticity (Hodge [1966]; Ponte [1976,1979,1980]; Carter & Martin [1976]). Second, they result in a sufficiently general variational structure, whose properties can be exploited to derive a number of useful material conservation laws (Eshelby [1951,1956]; Rice [1968]). Third, on the numerical side, the discrete form of this same variational structure is of great advantage in the development of efficient algorithms for finite element implementation; for example, symmetry-preserving material tangent stiffness operators are easily obtained in implicit solution schemes (Cohn & Maier [1979]; Martin et al. [1987]; Ortiz & Popv [1985]; Saleeb et al. [1990]; Saleeb & Wilt [1993]; Maier & Novati [1990]). Finally, this complete potential-based framework conveniently lends itself to intelligent application of symbolic manipulation systems that facilitate the construction, implementation, and analysis of new deformation and damage models (Arnold & Tan [1990]; Arnold et al. [1990]).

With the above background in mind, our objective is to study the general form and the ensuing restrictions imposed by a complete potential-based viscoplastic formulation, in terms of the Gibb's thermodynamic potential, particularly in regard to several of the currently used forms of the flow and evolutionary laws. It appears that although many of these theories (e.g. Chaboche [1977]; Germain et al. [1983]; Miller [1987]; Robinson et al. [1987]; Robinson & Duffy [1990]; Onat & Leckie [1988]; Freed et al. [1991]) do not conform to the general potential framework outlined, it is still possible in some cases, by modifications of the employed forms, to restore the complete potential structure. However, the question still remains as to whether these modifications will significantly affect the predictive capabilities achieved by the original forms. This is not addressed here, but is addressed in a subsequent article by Arnold and Saleeb [1993].

II. THEORETICAL FRAMEWORK

We begin by summarizing the basic thermodynamic equations governing the thermomechanical behavior of an initially isotropic material element. An assumed expression for the Gibb's thermodynamic potential function (Coleman [1967]; Lubliner [1972, 1973]) in terms of a number of state and internal variables characterizing the changing internal structure of the material, is taken. For conciseness, the discussion is limited to the case of small deformations, in which the initial state is assumed to be stress-free
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throughout. Also, a cartesian reference frame and index notation (with summation implied on repeated roman subscripts) is utilized. It is noticed that all subsequent equations and discussions can be equivalently rephrased, using appropriate Legendre transformations, in terms of the (Helmholtz) free energy and the associated complementary variables.

In its volumetric differential form, the Gibb’s thermodynamic potential \( G(\sigma_{ij}, \xi_\beta, T) \), can be written as follows (LUBLINER [1973]; PONTER [1979]):

\[
dG = -e_{ij}d\sigma_{ij} - SdT - X_\beta d\xi_\beta, \tag{1}
\]

where \( S \) denotes the entropy, \( \xi_\beta \) the internal state variables, \( X_\beta \) the thermodynamic affinities corresponding to \( \xi_\beta \) (where \( \beta = 1, \ldots, N \) and the tensorial character of these internal variables will be defined in the particular context), \( e_{ij} \) the total strain and \( \sigma_{ij} \) the cauchy stress tensor. It follows from eqn (1) then that

\[
e_{ij} = \frac{-\partial G}{\partial \sigma_{ij}} \quad \text{(total strain),} \tag{2}
\]

\[
S = \frac{-\partial G}{\partial T} \quad \text{(entropy),} \tag{3}
\]

and

\[
X_\beta = \frac{-\partial G}{\partial \xi_\beta} \quad \text{(affinity),} \tag{4}
\]

are defined as the equations of state (MALVERN [1969]; LEMAITRE & CHABOCHE [1990]) and \( \sigma_{ij} \), \( T \), and \( \xi_\beta \) are the “force-like” thermodynamic state variables while \( e_{ij}, S, \) and \( X_\beta \) are the conjugate “displacement-like” variables.

Now considering the following general form for the Gibb’s potential:

\[
G = G(\sigma_{ij}, \xi_\beta, T),
\]

where initially no restrictions are imposed on the functional dependence other than as declared above. The most general expression for the total strain rate can then be obtained by differentiating eqn (2). That is

\[
\dot{e}_{ij} = \frac{d}{dt}\left( \frac{-\partial G}{\partial \sigma_{ij}} \right) = \frac{-\partial^2 G}{\partial \sigma_{ij} \partial \sigma_{rs}} \delta_{rs} + \frac{-\partial^2 G}{\partial \sigma_{ij} \partial \xi_\beta} \dot{\xi}_\beta + \frac{-\partial^2 G}{\partial \sigma_{ij} \partial T} \dot{T}. \tag{5}
\]

With the above in mind, two options are actually available for describing the flow and evolutionary equations. The first option assumes a fully coupled form, that is, one in which the irreversible strain rate is intimately linked to the thermodynamic internal state, hence its functional dependence (evolution) is completely defined once \( G \) is assumed. The second option is a decoupled Gibb’s form, in which the selected internal state variables are grouped a priori by separation of the inelastic strain as an independent state parameter and suppressing all stress dependency of the remaining associated internal state groups in the selected \( G \) function. Both of these options are described separately, below.
II.1. Fully coupled form

Consider the following coupled form for the Gibb’s potential:

\[ G = E(\sigma_{ij}) + M(\sigma_{ij}, \xi_\beta) + H(\xi_\beta) - Z(T) - \frac{\sigma_{kk}}{3} \gamma(T - T_0), \]  

(6)

where initially certain restrictions (coming from the affinities \( A_\beta \), discussed subsequently) are imposed on the functions \( E \) and \( M \), \( T_0 \) is a reference temperature, and \( \gamma \) is the volumetric coefficient of thermal expansion. The above special coupled form has been frequently utilized in earlier applications (Robinson et al. [1987]; Pontier & Leckie [1976]; Pontier [1976, 1979, 1980]; Arnold [1987]). Substitution of eqns (4) and (6) into (5) yields a particular expression for the total strain rate, that is,

\[ \dot{e}_{ij} = \left[ \frac{-\partial^2 E}{\partial \sigma_{ij} \partial \sigma_{rs}} + \frac{-\partial^2 M}{\partial \sigma_{ij} \partial \sigma_{rs}} \right] \dot{\sigma}_{rs} + \left[ \frac{\partial X_\beta}{\partial \sigma_{ij}} \right] \dot{\xi}_\beta + \dot{\delta}_{ij} \frac{\gamma}{3} \dot{T}, \]  

(7)

in which the above three terms may be identified as the reversible, irreversible, and thermal expansion components of the total strain rate, respectively. Thus,

\[ \dot{e}_{ij} = \dot{e}_{ij}^R + \dot{e}_{ij}^I + \dot{e}_{ij}^T, \]  

(8)

with

\[ \dot{e}_{ij}^R = \left[ \frac{-\partial^2 E}{\partial \sigma_{ij} \partial \sigma_{rs}} + \frac{-\partial^2 M}{\partial \sigma_{ij} \partial \sigma_{rs}} \right] \dot{\sigma}_{rs} \quad \text{(reversible)}, \]  

(9)

\[ \dot{e}_{ij}^I = \left[ \frac{\partial X_\beta}{\partial \sigma_{ij}} \right] \dot{\xi}_\beta \quad \text{(irreversible)}, \]  

(10)

\[ \dot{e}_{ij}^T = \dot{\delta}_{ij} \frac{\gamma}{3} \dot{T} \quad \text{(thermal)}. \]  

(11)

Furthermore, expressions for the entropy and “displacement-like” affinities can be obtained from eqns (3), (4), and (6), that is,

\[ S = \frac{\partial Z}{\partial T} + \frac{\sigma_{kk}}{3} \gamma \]  

(12)

and

\[ X_\beta = -\frac{\partial M}{\partial \xi_\beta} - \frac{\partial H}{\partial \xi_\beta}, \]  

(13)

thus implying the functional dependence of \( S = S(T, \sigma_{kk}) \) and \( X_\beta = X_\beta(\sigma_{rs}, \xi_\beta) \). Consequently, in its present coupled form, certain restrictions now apply to the functions \( M \) and \( H \) in order for \( X_\beta \) to have the above functional dependence. For instance, \( M \)
must always be at least linear in stress, and furthermore, given an \( H \) that is at least quadratic in internal state will require that \( M \) be at least linear in internal state \( \xi_\beta \); alternatively, given an \( H \) that is assumed to be linear (or zero) in internal state requires that \( M \) be at least quadratic in internal state.

Now assuming the existence of a dissipation potential, \( \varphi(X_\beta) \), the rate of change of the internal state variables (\( \xi_\beta \)) can be expressed in terms of their corresponding affinities. That is, the evolution or kinetic law becomes:

\[
\dot{\xi}_\beta = \frac{\partial \varphi}{\partial X_\beta}.
\] (14)

Furthermore, considering eqns (12) and (13), it is clear that a correspondence between \( \varphi \) and a complementary dissipation potential \( \Omega = \Omega(\sigma_{ij}, \xi_\beta, T) \) exist, because \( \varphi \) is an implicit function of \( \sigma_{ij} \) and \( \xi_\beta \) through \( X_\beta \), that is, \( \varphi = \varphi(X_\beta(\sigma_{ij}, \xi_\beta), T) \). Note, defining the dependence of \( \Omega \) on \( \sigma_{ij} \), \( \xi_\beta \) and \( T \) does not preclude the inclusion (or dependence) of the associated thermodynamic affinities as parameters (as opposed to "true" variables), for example, \( X_\beta \) or \( \epsilon \) (cf. Germain et al. [1983]; Lemaître & Chaboche [1990]).

Here, in the fully coupled option, we make use of eqns (10), (13), and (14) directly to obtain expressions for the flow, that is,

\[
\dot{\epsilon}_{ij} = \frac{\partial X_\beta}{\partial \sigma_{ij}} \dot{\xi}_\beta = \frac{\partial \varphi}{\partial \sigma_{ij}}
\] (15)

and evolutionary laws, cf. eqn (14),

\[
\dot{\xi}_\beta = \frac{\partial \varphi}{\partial X_\beta},
\]

where \( X_\beta \) is obtained from eqn (13). Note that \( X_\beta \) and thus the functional dependence of \( \varphi \) is completely defined once the functions \( M \) and \( H \) are assumed.

It is evident that the present form is too restrictive (see Onat & Leckie [1988] for a number of examples) in that it requires proportionality between the inelastic strain and the rate of change of internal state to maintain consistency between the two expressions for inelastic strain rates in eqns (10) and (15). As a consequence of this proportionality, the classical definition of steady state creep (i.e., an evolving inelastic strain at constant internal structure) cannot be attained in the present coupled form. This inadequacy cannot be overemphasized in light of the strong experimental evidence to the contrary for metals at high temperature. In addition, if used in isolation, it becomes rather restrictive under complex cyclic loading conditions. Consequently, many alternative approaches for loading/unloading treatments have been proposed in the literature (e.g., discontinuity regions as discussed by Robinson & Duffy [1990]).

II.2. Decoupled form

Here, a less restrictive, yet equivalent, decoupled form can be derived by assuming a priori a form for the Gibb's potential, whereby the inelastic strain is assumed to be an independent parameter (whose evolution is defined separately) relative to two other
groupings of internal “force-like” state variables, \( \xi_\beta \), associated with irreversibility. These two groups of internal variables are separated into those associated with the material inelasticity (\( \alpha_\beta \)) and those associated with the degradation of the material, \( D_\beta \), due to some damage mechanisms (Kachanov [1986]; Krajcinovic [1989]; Lemaitre & Chaboche [1990]). Indeed, it is this separation that allows for inclusion of the classical notation of steady state creep. As an example, one may write

\[
G = E(\sigma_{ij}) - \sigma_{ij} \varepsilon_{ij}^f + M(\sigma_{ij}, D_\beta) + H(\alpha_\beta) - Z(T) - \sigma_{kk} \frac{\gamma}{3} (T - T_0) .
\]

Where, once again, differentiating eqn (2) with the above assumed form for \( G \), an expression defining the total strain rate can be obtained, that is,

\[
\dot{\varepsilon}_{ij} = \left[ -\frac{\partial^2 E}{\partial \sigma_{ij} \partial \sigma_{rs}} + \frac{\partial^2 M}{\partial \sigma_{ij} \partial \sigma_{rs}} \right] \sigma_{rs} + \dot{\varepsilon}_{ij}^r + \left[ \frac{\partial \Lambda_\beta}{\partial \sigma_{ij}} \right] \dot{D}_\beta + \delta_{ij} \frac{\gamma}{3} \dot{T} ,
\]

in which the above four terms may be identified as the reversible, inelastic, irreversible, and thermal expansion components of the total strain rate, respectively. Note that \( \Lambda_\beta \) are the “displacement-like” thermodynamic affinities associated with the internal material degradation “force-like” variables \( D_\beta \). Thus,

\[
\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^R + \dot{\varepsilon}_{ij}^I + \dot{\varepsilon}_{ij}^P + \dot{\varepsilon}_{ij}^T ,
\]

with

\[
\dot{\varepsilon}_{ij}^R = \left[ -\frac{\partial^2 E}{\partial \sigma_{ij} \partial \sigma_{rs}} + \frac{\partial^2 M}{\partial \sigma_{ij} \partial \sigma_{rs}} \right] \sigma_{rs} \text{ (reversible)},
\]

\[
\dot{\varepsilon}_{ij}^P = \left[ \frac{\partial \Lambda_\beta}{\partial \sigma_{ij}} \right] \dot{D}_\beta \text{ (irreversible)},
\]

\[
\dot{\varepsilon}_{ij}^T = \delta_{ij} \gamma \dot{T} \text{ (thermal)},
\]

and \( \dot{\varepsilon}_{ij}^I \) (inelastic strain rate) is to be defined separately, subsequently. Note also that the definition of \( \Lambda_\beta \) follows that of eqn (13) (with the suppression of \( H \) and equivalence of \( \xi_\beta \) and \( D_\beta \)) and the evolution of \( D \), that of eqn (14) given an assumed form of a damage dissipation potential \( \Psi = \Psi(\Lambda_\beta) \). Various coupled and decoupled deformation and damage formulations may thus be investigated using this framework.

To provide a framework for the definition of the inelastic strain rate (\( \dot{\varepsilon}_{ij}^I \)) and ensure the thermodynamic admissibility of it, the requirement that the Clausius-Duhem or dissipation inequality be satisfied is introduced. That is,

\[
\Omega(\sigma_{ij}, \alpha_\beta) = \sigma_{ij} \dot{\varepsilon}_{ij} - \Lambda_\beta \alpha_\beta \geq 0 .
\]

Thus differentiating once with respect to stress (\( \sigma_{ij} \)) and once with respect to internal state gives:

\[
\dot{\varepsilon}_{ij}^I = \frac{\partial \Omega}{\partial \sigma_{ij}}
\]
and

\[ \dot{A}_\beta = -\frac{\partial \Omega}{\partial \alpha_\beta}. \]  

(24)

Returning to eqn (4) and differentiating with respect to time, an expression for the rate of change of the affinities \( A_\beta \) in terms of the rate of change of the corresponding internal state can be obtained, that is,

\[ \dot{A}_\beta = \frac{d}{dt} \left[ -\frac{\partial G}{\partial \alpha_\beta} \right] = -\frac{\partial^2 H}{\partial \alpha_\epsilon \partial \alpha_\beta} \dot{\alpha}_\epsilon \]

(25)

where

\[ Q_{\beta \ell} = -\frac{\partial^2 H}{\partial \alpha_\beta \partial \alpha_\ell}, \]

(26)

given the assumed form of the Gibb's potential in eqn (16). Due to the fact that the operator \( Q \) relates the "force-like" state variables to the "displacement-like" variables and is completely defined once a Gibb's potential is chosen, this operator will be defined as the compliance operator henceforth. Furthermore, it is interesting to note that this operator provides information relative to the curvature of the Gibb's potential as well as the relaxation trajectories in the associated state space (see ARNOLD [1987]).

Now substituting eqn (25) into (24) and rearranging terms, gives,

\[ \dot{\alpha}_\ell = -[Q_{\beta \ell}] \frac{\partial \Omega}{\partial \alpha_\beta}, \]

(27)

which defines the evolution of internal state. Thus eqns (23) and (27) represent the flow and evolutionary laws, respectively, for an assumed \( \Omega = \Omega(\sigma_{ij}, \alpha_\beta, T) \) and Gibb's potential wherein both potentials are directly linked through the internal state variables \( \alpha_\beta \).

It is interesting to note that only under special conditions (e.g., uniaxial states of stress and scalar internal state variables) does eqn (27) reduce to those discussed by PONTER [1976,1979] and PONTER & LECKIE [1976].

Clearly, this framework provides a structure in which the flow and evolution laws are associative, satisfying eqn (25) explicitly, and convenient to integrate. Furthermore, this option provides significant freedom (i.e., \( \Omega \geq 0 \)) in the definition of the complementary dissipation potential so that various complete potential-based formulations can be expressed. The price for this additional freedom, however, is in general the nontrivial inversion of the compliance operator.

III. DISCUSSION OF SPECIAL CASES

Equations (14) and (15), and (23) and (27), provide the general thermodynamic framework whereby most, if not all, potential-based models with associated flow and evolutionary laws can be derived. To illustrate this, a number of classical theories as well as
a few nonclassical forms will be recovered, depending of course upon the assumed form for the Gibb's potential, that is, functions $E$, $M$, and $H$.

III.1. **Elasticity**

Here, if $E$ is assumed to be quadratic in stress, that is,

$$E = -\frac{1}{2} C_{rskl} \sigma_{rs} \sigma_{kl},$$  \hspace{1cm} (28)

and $M$ is assumed to be a separable function linear in stress and internal state (where, $\alpha_\beta$ is taken as a symmetric second order tensor),

$$M = L(\sigma_{rs}) P(\alpha_{\beta})$$

$$= \sigma_{rs} \alpha_{rs},$$  \hspace{1cm} (29)

and $H$ is taken as zero, then from eqn (9);

$$\dot{\varepsilon}_{ij} = C_{ijrs} \sigma_{rs},$$  \hspace{1cm} (30)

the classical form describing generalized Hooke's Law (Chen & Saleeb [1982]) is obtained. If an expression describing nonlinear elastic behavior is desired any scalar form, linear or nonlinear, in stress for $E$ and/or $M$ can be chosen.

III.2. **Coupled damage elasticity**

If $\alpha_\beta$ is associated with material degradation, that is, "damage," classical forms of an elastically damaging material can be recovered. Here, $E$ is assumed as before to be quadratic in stress (see eqn 28), while $M$ and $H$ are taken as follows:

$$M = \frac{1}{3} \mathcal{D} C_{rskl} \sigma_{rs} \sigma_{kl},$$  \hspace{1cm} (31)

and

$$H = h \mathcal{D}^n,$$  \hspace{1cm} (32)

where the internal state variable ($\alpha_\beta = \mathcal{D}$) is the scalar "force-like" damage affinity.

Now calculating the total strain rate, as given in eqn (7), we find

$$\dot{\varepsilon}_{ij} = [1 - \mathcal{D}] C_{ijrs} \sigma_{rs} - C_{ijrs} \sigma_{rs} \mathcal{D} + \delta_{ij} \frac{\gamma}{3} \mathcal{T},$$  \hspace{1cm} (33)

wherein the term $[1 - \mathcal{D}] C_{ijrs}$ is the classical effective compliance tensor (Lemaitre & Chaboche [1990]), and $C_{ijrs} \sigma_{rs} \mathcal{D}$ is the irreversible strain measure. Note that the "displacement-like" affinity (eqn 13) is

$$D = -\frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} - hn \mathcal{D}^{n-1},$$  \hspace{1cm} (34)
and that if \( n = 0 \), we see that the corresponding affinity is the elastic strain energy for an equivalent undamaged solid. Now with regard to the evolution of \( \mathcal{D} \) eqn (14) gives,

\[
\dot{\mathcal{D}} = \frac{\partial \Psi(D)}{\partial D}.
\]

III.3. Viscoplasticity

To illustrate the potential of this framework for both the coupled and decoupled options, a specific, yet general, \( J_2 \) (second invariant of the deviatoric stress) forms for the functions \( M \) and \( H \) of the Gieb's potential shall be assumed (see eqns 6 and 16, respectively).

\[
E = -\frac{1}{2} C_{rskl} \sigma_{rs} \sigma_{kl}
\]

\[
H = -A \delta_2^n
\]

\[
M = L(S_{rs}) P(\alpha_\beta)
= BS_{rs} \alpha_{rs} \delta_2^m,
\]

with

\[
\delta_2 = \frac{1}{2} \alpha_{ij} \alpha_{ij}
\]

and

\[
S_{ij} = \sigma_{ij} - \frac{1}{2} \sigma_{kk} \delta_{ij}.
\]

Note that in the assumed form for \( M \) and \( H \), the internal state variable \( (\alpha_\beta) \) is taken as a second-order and traceless tensor and is typically identified as the back stress tensor and associated with kinematic like hardening (MILLER [1987]; ONAT & LECKIE [1988]; ROBINSON et al. [1987]). Also in \( M \) and \( H \) the constants \( A, B, m, \) and \( n \) are taken to be material parameters, and finally when examining the decoupled form only the functions \( E \) and \( H \) are assumed to be non-zero.

III.3.1. Coupled form. Considering the coupled form, and the above definitions, the affinity \( A_{ij} \) takes on the form:

\[
A_{ij} = c(S_{ij} - b \alpha_{ij}),
\]

where

\[
c = -B \delta_2^m
\]

and

\[
b = \frac{Am \delta_2^{m-1}}{B \delta_2^m} - \frac{n}{\delta_2} S_{rs} \alpha_{rs},
\]
while the flow and evolutionary laws come directly from eqns (15) and (14), respectively, given an assumed dissipation potential \( \varphi \). Note that the typical effective stress definition found in the literature is recovered (within a factor \( c \)) when \( n = 0, m = 1, \) and \( A = B, \) that is,

\[
A_{ij} = c \{ S_{ij} - \alpha_{ij} \}.
\]

Furthermore, note that for the above assumed forms, the Gibb's potential is a homogeneous quadratic function which is known to be convex.

For convenience, the dissipation potential \( \varphi \) is assumed to have the following form;

\[
\varphi = \int f(F(A_{ij})) \, dF,
\]

where \( F \) is a scalar function, for example, \( F = A_2/\kappa^2 - 1 \) and \( A_2 = \frac{1}{2} A_{ij} A_{ij} \), whereby it is easily shown that

\[
\frac{\partial \varphi}{\partial \sigma_{kl}} = \bar{F} \delta_{rk} \delta_{sl} A_{rs}
\]

\[
\frac{\partial \varphi}{\partial A_{rs}} = \bar{F} A_{rs}
\]

where

\[
\bar{F} = f(F)/\kappa^2,
\]

such that the flow and evolutionary laws become,

\[
\dot{\epsilon}_{ij} = \bar{F} c A_{ij},
\]

\[
\dot{\alpha}_{ij} = \frac{1}{c} \dot{\epsilon}_{ij},
\]

respectively.

After examining the above flow and evolution laws one might observe that only a purely hardening material may be modeled with the above complete coupled form, because in order for a classical steady state condition \( (\dot{\alpha}_{rs} = 0) \), in the present class of internal variable based models, to occur, the \( \dot{\epsilon}_{ij} \) must be also zero. Such a state is more appropriately referred to mathematically as a fully bounded material response (e.g., the classical Kelvin model). Thus, no secondary creep rate could be predicted. However, as shown in the following section, if one assumes a decoupled but consistent form, a complete potential-based, yet general, form can still be obtained (e.g., one with a Bailey-Orowan (PONTER & LECKIE [1976]) competing mechanism in its evolution law).

III.3.2. Decoupled form. Now considering a decoupled form, we can assume the complementary dissipation potential to take the form
\[ \Omega = \int f(F(\Sigma_{ij})) \, dF + \frac{R}{H} \int g(G(\alpha_{ij})) \, dG, \]  

(45)

where \( F \) and \( G \) are scalar functions (e.g., \( F = J_2/\kappa^2 - 1 \) and \( G = \gamma_2/\kappa^2 \), see Arnold [1987]) and \( R, H, \) and \( \kappa \) are material constants. Thus, it is easily shown that

\[ \frac{\partial \Omega}{\partial \sigma_{kl}} = F \Sigma_{kl}, \]

\[ \frac{\partial \Omega}{\partial \alpha_{ij}} = -\left\{ \frac{\partial \Omega}{\partial \sigma_{ij}} - \frac{R}{H\kappa^2} g(G)\alpha_{ij} \right\}, \]

where

\[ F = f(F)/\kappa^2, \]

so that the flow (eqn 23) and evolutionary (eqn 27) laws become,

\[ \dot{\varepsilon}_{ij} = F \Sigma_{ij}, \]

(46)

and

\[ \dot{\alpha}_{ij} = -[Q_{rs}^{-1}] \frac{\partial \Omega}{\partial \alpha_{rs}}, \]

\[ = [Q_{rs}^{-1}] \left\{ \frac{\partial \Omega}{\partial \sigma_{rs}} - \frac{R}{H\kappa^2} g(G)\alpha_{rs} \right\}, \]

\[ \dot{\alpha}_{ij} = [Q_{rs}^{-1}] \left\{ \dot{\varepsilon}_{rs} - \frac{R}{H\kappa^2} g(G)\alpha_{rs} \right\}, \]

(48)

respectively. Where with the above definition for \( H \) (see, eqn 36) the compliance operator is defined as

\[ Q_{klij} = -\frac{\partial^2 H}{\partial \alpha_{kl}\partial \alpha_{ij}}, \]

\[ Q_{klij} = Amg_2^{m-1} \left[ \frac{(m - 1)}{g_2} \alpha_{kl}\alpha_{ij} + I_{ijkl} \right], \]

(49)

with

\[ I_{ijkl} = \delta_{ik}\delta_{jkl}. \]

Thus, considering the case when \( m = 1 \), the classically (i.e., first term below) assumed linear kinematic evolutionary law (Lemaitre & Chaboche [1990]) can be recovered, that is,
in which the evolution of back stress is proportional to the evolution of the irreversible strain rate minus a thermal recovery term. Clearly, this analytical form allows for steady state (i.e., \( \dot{\alpha}_{ij} = 0 \)) without the added requirement that the inelastic strain be zero.

Considering the more general case given above, that is, \( m > 1 \), the commonly assumed evolution equation for the back stress \( (\alpha_{ij}) \) containing a strain-induced recovery term can be recovered under a multiaxial state of stress. However, as opposed to the conventional approach in which an arbitrary function is multiplied by \( \| \dot{\epsilon}_l \| \), the present form of the thermodynamically based derivation is seen to lead to the inclusion of a specific scaling factor \( \eta \) (i.e., the projected back stress length) for that same term (see equations below). This can be seen by substituting eqn (49) into eqn (48), that is,

\[
\dot{\alpha}_{ij} = \frac{1}{q} \left[ \frac{(m-1)}{\delta_2} \alpha_{kl} \alpha_{ij} + I_{klj} \right]^{-1} \left\{ \dot{\epsilon}_l^I - \frac{R}{H \kappa^2} g(G) \alpha_{kl} \right\},
\]

where

\[
q = A m \delta_2^{m-1}
\]

An important and not trivial next step, is the inversion (Saleeb & Wilt [1993]) of the compliance operator. This inversion may not always be available in analytical form, although in the above case it is and eqn (51) becomes,

\[
\dot{\alpha}_{ij} = \frac{1}{q} \left[ I_{klj} - \frac{\dot{m}}{\delta_2} \alpha_{kl} \alpha_{ij} \right] \left\{ \dot{\epsilon}_l^I - \frac{R}{H \kappa^2} g(G) \alpha_{kl} \right\},
\]

where

\[
\dot{m} = \frac{(m-1)}{(2m-1)}.
\]

Realizing that

\[
\dot{\epsilon}_l^I = n_{kl} \| \dot{\epsilon}_l^I \|,
\]

it is easily shown that

\[
\dot{\alpha}_{ij} = \frac{1}{q} \left[ \dot{\epsilon}_{ij} - \eta \alpha_{ij} \| \dot{\epsilon}_l^I \| - \frac{R}{H \kappa^2} (1 - 2 \dot{m}) g(G) \alpha_{ij} \right],
\]

where

\[
\eta = \frac{\dot{m}}{\delta_2} n_{kl} \alpha_{kl}.
\]

Therefore, the fully thermodynamically consistent evolutionary law associated with a complementary dissipation potential whose functional dependence is taken as in eqn (36) is eqn (53).
Consequently, comparing eqn (53) with similar forms found in the literature (Chaboche [1977]; Miller [1976, 1987]; Lemaître & Chaboche [1990]; Freed et al. [1991]), one can see a difference in the leading term typically taken as a constant and here shown to be a function of the back stress, that is, \( (1/q) \), and the lack of an appropriate scaling factor in front of the special strain-induced recovery term, that is, \( \alpha_{ij} \| \dot{\varepsilon}_{ij} \| \); the origin of which is the nonlinear kinematic hardening characteristic of the present model. The implications of this scaling factor \( \eta \) on the response prediction due to multiaxial states of stress is anticipated to be significant (Arnold & Saleeb [1993]). These differences are related to the fact that here the flow and evolutionary laws are associative and directly linked to the Gibb's potential, through the compliance operator. Another distinction is the resulting desirable numerical features, for example the symmetry in the consistent tangent stiffness matrix produced by the present form, eqn (52), and discussed at length by Saleeb & Wilt [1993]. The special strain-induced recovery term will of course disappear under a uniaxial state of stress, thus implying no detract from the uniaxial predictive capability of the model.

Inclusion, however, of both a uniaxial as well as multiaxial strain-induced recovery term can of course still be achieved in the present framework by a careful selection of the assumed form of the dissipation potential (Lemaître & Chaboche [1990]; Arnold & Saleeb [1993]), thereby suggesting a theory with not only the desired uniaxial predictive capability but all of the theoretical and numerical niceties discussed previously as well. In particular, this is in the scope of the present framework provided that a "nonclassical" form of the dependency of \( F \), in eqn (45), on the internal state variable \( \alpha \) is adopted. A case in point is the direction-determining function utilized by Lemaître and Chaboche [1990]. However, note that this was in the context of nonassociative inelastic plasticity and viscoplasticity. Indeed, utilizing a newly introduced nonclassical form for \( F \), we were able to construct a special complete potential-based model of the type described here, that is capable of capturing most of the well-established experimental features of metals (Arnold & Saleeb [1993]).

Finally, examining the form proposed by Robinson et al. [1987] and Robinson and Duffy [1990], it is clear that this form is obviously a truncation (though an inconsistent one) of the special strain-induced recovery term discussed above. Thus, this form as well can easily be modified to be consistent with a complete potential framework by the multiplication of a fourth order tensor (i.e., the inverse of the compliance operator), that is

\[
\dot{\alpha}_{ij} = \frac{1}{q} N_{ijkl} \left( \dot{\varepsilon}_{kl} - \frac{R}{Hk^2} g(G) \alpha_{ij} \right),
\]

where

\[
N_{ijkl} = \left[ I_{ijkl} - \frac{\dot{m}}{\hat{\alpha}} \alpha_{kl} \alpha_{ij} \right],
\]

without altering any of its uniaxial predictive capabilities.

IV. DISCUSSION AND CONCLUSION

A careful consideration of the literature reveals that two schools of thought exist when considering nonlinear kinematic hardening. In the first school, a form of the Gibbs or
Helmholtz free energy is assumed as the starting point, whereas in the second school no direct connection is assumed between the thermodynamic potential and the associated dissipation potential. An example of the first type can be found in PONTER [1979] and ARNOLD [1987], although, as mentioned previously, these forms lack the consistently derived compliance operator. Indication of the implied presence of such a tensorial compliant operator, \( Q \), (in the context of eqn 25) can be found in CHABOCHE [1979]; where somewhat abstract forms for the tensor operators were noted but no explicit applications were discussed.

In the second school of thought, the need for the nonlinear compliance operator is entirely bypassed in eqn (25) (that is, a linear Prager type model is used) because the nonlinearity is built into the nonassociative flow and evolutionary equations through the inclusion of a dynamic recovery term. Models of this type appear to be the most popular in the literature, see WALKER [1981]; CHAN et al. [1984]; MILLER [1987]; LEMAITRE and CHABOCHE [1990], and FREED et al. [1991].

With an emphasis on the mathematical structure, the present framework can be considered to be a generalization of Prager's linear kinematic hardening framework to the nonlinear regime, while still preserving the \textit{complete} (or fully associative) format. The key is a number of important interrelationships between the selected thermodynamic and dissipation potential functions and their dependence on the state variables chosen a priori. From the practical viewpoint, successful application of this framework will require careful investigation, particularly under cyclic loading conditions (LEMAITRE & CHABOCHE [1990]; ARNOLD & SALEEB [1993]).

In conclusion, we have presented here a \textit{complete} potential-based framework utilizing internal state variables for the derivation of reversible and irreversible constitutive equations. This framework assumes the existence of the total (integrated) form of either the (Helmholtz) free energy or the (Gibbs) complementary free energy \textit{a priori}. To closely assess the literature, two forms for describing the flow and evolutionary equations have been discussed; in the \textit{fully coupled} form, the irreversible strain rate is intimately linked to the thermodynamic internal state, hence its functional dependence is completely defined once the complementary energy is assumed. This form was shown to be overrestrictive. The second and recommended form for general application is the \textit{decoupled form}. This form provides significant flexibility because the inelastic strain is taken as an independent state parameter. Here the definition of the inelastic strain rate requires the identification of a complementary dissipation potential. As a consequence of the decoupled form, a new explicit operator, that is, the \textit{compliance operator}, is defined, which provides a link between the assumed Gibb's and complementary dissipation potential and ensures a number of desirable theoretical and numerical features.

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