Three-dimensional stress intensity factors of a central square crack in a transversely isotropic cuboid with arbitrary material orientations

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A B S T R A C T

In this paper, we present the dual boundary element method (dual-BEM) or single-domain BEM to analyze the mixed three-dimensional (3D) stress intensity factors (SIFs) in a finite and transversely isotropic solid containing an internal square crack. The planes of both the transverse isotropy and square crack can be oriented arbitrarily with respect to a fixed global coordinate system. A set of four special nine-node quadrilateral elements are utilized to approximate the crack front as well as the outer boundary, and the mixed 3D SIFs are evaluated using the asymptotic relation between the SIFs and the relative crack opening displacements (COD) via the Barnett–Lothe tensor.

Numerical examples are presented for a cracked cuboid which is transversely isotropic with any given orientation and is under a uniform vertical traction on its top and bottom surfaces. The square crack is located in the center of the cuboid but is oriented arbitrarily. Our results show that among the selected material and crack orientations, the mode-I SIF reaches the largest possible value when the material inclination angle $\psi_1 = 45^\circ$ and dig angle $\beta_1 = 45^\circ$, and the crack inclination angle $\psi_2 = 0^\circ$ and dig angle $\beta_2 = 0^\circ$. It is further observed that when the crack is oriented vertically or nearly vertically, the mode-I SIF becomes negative, indicating that the crack closes due to an overall compressive loading normal to the crack surface. Variation of the SIFs for modes II and III along the crack fronts also shows some interesting features for different combinations of the material and crack orientations.

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1. Introduction

Rock fracture mechanics is essentially an extension of the classic fracture mechanics in solids [1,2], which recognizes the importance of the stress intensity near a crack tip. Irwin [2] introduced the stress intensity factors (SIFs) to describe the stress and displacement fields near a crack tip. As is well known, there are three basic crack propagation modes in the fracture process: opening (mode I), sliding (mode II), and tearing (mode III), and furthermore, mechanical safety of a solid elastic structure can be analyzed based on these SIFs. Therefore, determination of SIFs near the crack front in linear elastic fracture mechanics has been always an interesting but challenging task. While most previous studies in SIFs were focused on one or two fracture modes, mixed three-dimensional (3D) modes need to be considered as materials could mostly fail under combined tensile/compressive, shearing, and tearing loads. For 3D isotropic elastic materials, Singh et al. [3] obtained the SIFs using the concept of a universal crack closure integral. For transversely isotropic, orthotropic, and anisotropic solids, Pan and Yuan [4] presented the general relationship between the SIF and the relative crack opening displacement (COD). Lazarus et al. [5] compared the calculated SIFs with experimental results for brittle solids under mixed mode I–III or I–II–III loadings. The 3D SIFs were also calculated by Zhou et al. [6] using the variable-order singular boundary element. More recently, Yue et al. [7] employed the dual boundary element method (dual-BEM) in their calculation of the 3D SIFs of an inclined square crack within a bi-material cuboid. We point out that the dual-BEM was originally proposed by Hong and Chen [8] as reviewed in [9]. Other recent representative works in this direction are those by Ariza and Dominguez [10], Liu et al. [11], Hatzigeorgiou and Beskos [12], Farthymuller et al. [13], Popov et al. [14], Zhao et al. [15], Lo et al. [16], dell’Erba and Aliabadi [17], and Blackburn [18]. The weakly singular and weak-form integral equation method recently proposed by Rundamornrat [19] and Rungamornrat and Mear [20] is also very efficient in crack analysis in anisotropic media. Besides the analytical (integral equation) and BEM methods [21], other common methods, such as the finite difference (FD) [22–24] and the finite element (FE) [25,26], were also applied for 3D SIF analysis. Since both the FD and FE methods require discretization of the whole problem domain,
these methods could be time consuming and more expensive than the BEM in fracture analysis.

In this paper, we apply the dual-BEM or the single-domain BEM to analyze the mixed 3D SIFs in a finite and transversely isotropic solid containing an internal square crack. Both the transversely isotropic plane and square crack plane can be oriented arbitrarily with respect to a fixed global coordinate system. A set of special nine-node quadrilateral elements are utilized to approximate the crack surface as well as the outer boundary, and the mixed 3D SIFs are evaluated using the asymptotic relation between the SIFs and the relative CODs via the Barnett–Lothe tensor. Our numerical examples show clearly the strong dependence of 3D SIFs on both the material and crack orientations, and these results could be useful in fracture analysis and design of anisotropic elastic solids.

The paper is organized as follows: in Section 2, we briefly present the required basic equations, including the related local and global coordinate systems. In Section 3, the two important BEM equations are presented for the modeling of a cracked 3D anisotropic cuboid: One is the displacement BEM and another is the traction BEM. The special crack front elements and the corresponding formulation for the mixed SIF calculation are presented in Section 4, and detailed numerical results are discussed in Section 5. Finally, conclusions are drawn in Section 6.

2. Basic equations

We consider a transversely isotropic elastic finite domain with arbitrarily oriented transverse isotropy plane. Inside this domain, there is a central square crack, which is also oriented arbitrarily. First, shown in Fig. 1 is the relation between the global coordinates \((x,y,z)\) or \((x_1,x_2,x_3)\) and the local transversely isotropic material coordinate system \(x', y', z'\), where \(z'\) is along the symmetry axis of the material, and \((x',y')\) is parallel to the isotropic plane. The inclined angle \(\psi_1\) is defined as the angle between the global horizontal plane and the isotropic plane of the material, and the dip orientation \(\beta_1\) is defined as the angle between the inclined angle plane and the global y-axis.

![Fig. 1. Relation between the local \((x',y',z')\) and global \((x,y,z)\) coordinate systems where the local \(z'\)-axis is along the symmetry axis of the transversely isotropic material. In other words, the local \((x',y')\)-plane is parallel to the isotropic plane of the material; \(\psi_1\) is the inclined angle between the global horizontal plane \((x,y)\) and the local isotropic plane \((x',y')\); \(\beta_1\) is the dip orientation between the global y-axis and the include plane.](image)

It is obvious that the transformation between the local \((x',y',z')\) and global \((x,y,z)\) coordinates can be described by the following relation:

\[
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix} = 
\begin{bmatrix}
\cos \beta_1 & -\sin \beta_1 & 0 \\
\cos \psi_1 \sin \beta_1 & \cos \psi_1 \cos \beta_1 & -\sin \psi_1 \\
\sin \psi_1 \sin \beta_1 & \sin \psi_1 \cos \beta_1 & \cos \psi_1
\end{bmatrix} 
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\tag{1}
\]

To solve the problem using the BEM formulation, we first present the governing equations in linear elasticity.

(1) Equations of equilibrium:

\[
\sigma_{ij} + b_i = 0, \quad i,j = 1,2,3
\tag{2}
\]

where \(\sigma_{ij}\) is the stress tensor; \(b_i\) the body force component; and subscript “\(j\)” denotes partial differentiation with respect to the global coordinates \(x, y, z\).

(2) Constitutive relation:

\[
[a] = [a][e]
\tag{3}
\]

where

\[
[a] = [e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{13}, 2e_{12}]
\tag{4}
\]

is the strain in the column matrix form, and

\[
[e] = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}]
\tag{5}
\]

is the stress in the column matrix form. Also in Eq. (3), \([a]\) is the elastic compliance matrix of the anisotropic elastic solid. For the transversely isotropic material, there are five independent elastic parameters \((E, E', v, v', G)\) in the matrix \([a]\). The definitions of these moduli are: \(E\) and \(E'\) are the Young’s moduli in the plane of transverse isotropy and in a direction normal to it, respectively; \(v\) and \(v’\) are the Poisson’s ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel and normal to it, respectively; \(G\) is the shear modulus normal to the plane of transverse isotropy. Furthermore, the shear modulus \(G\) in the plane of transverse isotropy is equal to \(E/(2(1+v))\). For the transversely isotropic material oriented arbitrarily with respect to the global coordinate system, the matrix \([a]\) will also be a function of the inclined angle \(\psi_1\) and dip orientation \(\beta_1\). As an example, Appendix A lists the elements of \([a]\) when \(\beta_1 = 0\) \([27]\). It is further observed from Appendix A that the coefficients \(a_{15}, a_{25}, a_{35}\), and \(a_{46}\) are zero when \(\psi_1\) is equal to 0° or 90°.

(3) Strain–displacement relation:

\[
ev_{ij} = 0.5(u_{ij} + u_{ji}), \quad i,j = 1,2,3
\tag{6}
\]

where \(u_i\) are the elastic displacements.

3. Single-domain boundary integral equations

We now present the basic relations in the boundary element analysis for fracture problems in linear elasticity. It is based on the dual-BEM \([8,9]\) or the single-domain BEM \([28]\) approach.

We assume that the finite domain under consideration is free of any body force (i.e., \(b_i = 0\) in Eq. (2)) and is bounded by an outer boundary \(S\) with given boundary conditions. Inside its domain, there is a crack described by its surface \(\Gamma\) (where \(\Gamma = \Gamma^+ - \Gamma^-\), with superscripts “\(^+\)” and “\(^-\)” denoting the positive and negative sides of the crack). We further assume that the tractions on both sides of the crack are equal and opposite. Then the single-domain BEM formulation consists of the following displacement and traction boundary integral equations \([29]\).
(1) Displacement boundary integral equations

\[ b_j u_j(y_s) + \int_S T_j^i(y_s, x_0) u_i(x_0) dS(x_0) - \int_S U^i_j(y_s, x_0) T_j(y_0) dS(x_0) = - \int_T T_j(y_s, x_{ij}) [u_i(x_{ij}) - u_i(x_{ij}^-)] d\Gamma(x_{ij}^-) \]  

(7)

where \( b_j \) are coefficients that depend only on the local geometry of the uncracked boundary at \( y_s \). A point on the positive (or negative) side of the crack is denoted by \( x_{ij} \) (or \( x_{ij}^- \)), and on the uncracked boundary by both \( x_0 \) and \( y_s \). Also in Eq. (7), \( u_i \) and \( T_j \) represent the displacements and tractions on the boundary (or crack surface), and \( U^i_j \) and \( T_j^i \) are the Green’s functions for displacements and tractions in a general anisotropic elastic solid [4], respectively. A practical material case that will be studied in detail in this paper is the transversely isotropic material with arbitrary material orientation with respect to the global coordinates. We also point out that the first integral on the left-hand side of Eq. (7) has a strong singularity, which will be treated by the rigid-body motion method. At the same time, the calculation of \( b_j \) can also be avoided. The second term on the left-hand side has only a weak singularity, and thus, is integrable.

(2) Traction boundary integral equations

\[ \frac{T_i(y_{ij}^-) - T_i(y_{ij}^+)}{2} + n_{nm}(y_{ij}^-) \times \int_S c_{\text{n}m} T^i_k(y_{ij}, x_0) u_j(x_0) dS(x_0) + n_{nm}(y_{ij}^-) \times \int_T c_{\text{n}m} T^i_k(y_{ij}, x_{ij}) [u_j(x_{ij}) - u_j(x_{ij}^-)] d\Gamma(x_{ij}^-) \]

\[ = n_{nm}(y_{ij}^+) \int_S c_{\text{n}m} U^i_j(y_{ij}, x_0) T_j(y_0) dS(x_0) \]  

(8)

where \( n_{nm} \) is the unit outward normal of the positive side of the crack surface at \( y_{ij}^- \), and \( c_{\text{n}m} \) is the fourth-order stiffness tensor of the anisotropic medium; \( U^i_j \) and \( T^i_k \) are the derivatives of the Green’s displacements and tractions with respect to the source point, respectively [30].

Eqs. (7) and (8) form a pair of boundary integral equations, called single-domain BEMs, and they can be applied to generally anisotropic media. They can be discretized and solved numerically for unknown boundary displacements (or the relative CODs on the crack surface) and tractions. However, before we apply these single-domain BEMs to calculate the mixed SIFs, we first briefly present the special elements and the approach for the evaluation of 3D SIF.

4. Different types of boundary/crack elements and SIF expressions

In order to discretize both the boundary and crack surface, the nine-node quadrilateral curved elements are used [4]. There are four types of elements, with type I for the uncracked boundary or the interior of the crack surface, and the other three types for different crack fronts (types II–IV), as shown in Fig. 2.

First, the global coordinates \( x_i \) at any point within the element are expressed as

\[ x_i = \sum_{j=1}^{9} \phi_j x_j, \quad i = 1, 2, 3 \]  

(9)

where the subscript \( i \) denotes the component of nodal coordinates and the superscript \( j \) denotes the number of nodes. The shape functions \( \phi_j \) (\( j = 1–9 \)) are functions of the intrinsic coordinates \((\xi_1, \xi_2)\), and their expressions for different elements are listed in Appendix B.
three SIFs can be expressed as follows:

\[
\begin{bmatrix}
K_{\text{II}} \\
K_{\text{I}} \\
K_{\text{III}}
\end{bmatrix} = 2\sqrt{\frac{\pi}{T}} \begin{bmatrix}
\Delta U_{\text{I}} \\
\Delta U_{\text{II}} \\
\Delta U_{\text{III}}
\end{bmatrix},
\]

where \( L \) is the Barnett–Lothe tensor \([31]\), which depends only on the anisotropic properties of the solid in the crack front coordinates. The normalized SIFs \((F_1, F_{\text{II}}, \text{and } F_{\text{III}})\) can be calculated as follows:

\[
\begin{bmatrix}
F_1 \\
F_{\text{II}} \\
F_{\text{III}}
\end{bmatrix} = T^{-1} \begin{bmatrix}
K_1 \\
K_{\text{II}} \\
K_{\text{III}}
\end{bmatrix}
\]

where \( a \) is the half crack length and \( T \) the applied vertical traction in the problem to be discussed below. We also point out that \( r \) in Eq. (14) was selected to be 0.00001\( a \) in our numerical calculation.

5. Numerical results and discussion

Consider a linearly elastic, homogeneous, and transversely isotropic cuboid with dimension \( W \times W \times H \), as shown in Fig. 3. Let \( x, y, \) and \( z \) (or \( x_1, x_2, \) and \( x_3 \)) be the global Cartesian coordinates.

Fig. 3. A central square crack \((ABCD: 2a \times 2a)\) within a finite cuboid \( W \times W \times H \) under a uniform normal stress \( T \) in the vertical direction. The orientation of the square crack is described by the inclined angle \( \phi_2 \) and dip orientation \( \beta_2 \).

Fig. 4. The normalized mode-I SIF along the square-shaped crack fronts \( AB, BC, CD, \) and \( DA \) for different crack orientation \((\phi_2, \beta_2)\) within a finite transversely isotropic cuboid. The material orientation is fixed at \((\phi_1 = 0^\circ, \beta_1 = 0^\circ)\).

Fig. 5. The normalized mode-II SIF along the square-shaped crack fronts \( AB, BC, CD, \) and \( DA \) for different crack orientation \((\phi_2, \beta_2)\) within a finite transversely isotropic cuboid. The material orientation is fixed at \((\phi_1 = 0^\circ, \beta_1 = 0^\circ)\).

Fig. 6. The normalized mode-I SIF along the square-shaped crack fronts \( AB, BC, CD, \) and \( DA \) for different crack orientation \((\phi_2, \beta_2)\) within a finite transversely isotropic cuboid. The material orientation is fixed at \((\phi_1 = 45^\circ, \beta_1 = 45^\circ)\).
with their origin in the center of the cuboid. A square crack of 
2a \times 2a is also located in the center of the cuboid, and a local 
coordinate system \((x', y', z')\) is attached to it with its \(z'\)-axis
normal to the crack surface. The crack orientation is described by 
two angles: the inclined angle \(\psi_2\) and dip orientation \(\beta_2\), where \(\psi_2\) 
is defined as the angle between the global horizontal \((x', y')\) plane 
and the crack plane \((z', z')\), and \(\beta_2\) as the angle between the global 
y-axis and the inclined angle plane.

The cracked finite cuboid is under a uniform normal tensile 
stress \(T\) applied at the top and bottom faces, as shown in Fig. 3. In 
the numerical example, the cuboid size is chosen to be \(H/W = 2\) 
and the square size \(2a/W = 0.5\). The material is a transversely 
isotropic marble and its elastic properties were obtained experimentally [32] as \(E = 90\) GPa, \(E' = 55\) GPa, \(\nu = \nu' = 0.3\), \(G = 35\) GPa, 
and \(G' = 21\) GPa. After checking our program for a couple of 
special cases for accuracy, 40 and 36 nine-nodal quadrilateral

- **Fig. 7.** (a) The normalized mode-II SIF along the square-shaped crack fronts AB, BC, CD, and DA for different crack orientation \((\psi_2, \beta_2)\) within a finite transversely isotropic cuboid. The material orientation is fixed at \(\psi_1 = 45^\circ, \beta_1 = 45^\circ\). (b) The normalized mode-III SIF along the square-shaped crack fronts AB, BC, CD, and DA for different crack orientation \((\psi_2, \beta_2)\) within a finite transversely isotropic cuboid. The material orientation is fixed at \(\psi_1 = 45^\circ, \beta_1 = 45^\circ\).
observed that the SIF $F_1$ is the same along the crack fronts ABC and CDA, and that its maximum value for this material type is larger than that for the material case with $\psi_1 = 0^\circ$ and $\beta_1 = 0^\circ$ (1.4 in Fig. 6 vs. 1.12 in Fig. 4). Furthermore, similar to Fig. 4, the maximum SIF value along each side decreases with increasing dip angle $\psi_2$. It reaches zero when the dip angle is somewhere between $60^\circ$ and $90^\circ$; then, it comes negative, corresponding to the crack closure. Figs. 7a and b show, respectively, the variation of the normalized SIFs $F_{II}$ and $F_{III}$ in the material with angles $\psi_1 = 45^\circ$ and $\beta_1 = 45^\circ$, along the square crack front, for different crack angles $\psi_2 = 0^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, and $90^\circ$ ($\beta_2 = 0^\circ$). It is observed that the SIF values for the shear and tearing modes along the crack front ABC are antisymmetric as compared to those along CDA. It is further noticed that the maximum SIF for $F_{II}$ is reached along the crack fronts AB and CD, whilst for $F_{III}$, the maximum is along BC and DA.

The variation of the SIFs along the crack front for the other material case is shown in Figs. 8 and 9, where the material orientation angles are $\psi_1 = 90^\circ$ and $\beta_1 = 0^\circ$, and the crack angles are $\psi_2 = 0^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, and $90^\circ$ ($\beta_2 = 0^\circ$). Comparing Fig. 8 to Fig. 4, we noted that while in Fig. 4, the SIFs of mode-I are the same along each side of the square, in Fig. 8, they are the same along sides AB and C, and the same along BC and DA. Furthermore, the maximum SIF along AB and CD is larger than that along BC and DA ($1.185$ vs. $0.94$), with these two values being, respectively, larger and smaller than those in Fig. 4. This different phenomenon is due to the fact that in Fig. 8 the material isotropic plane is normal to the crack plane, whilst in Fig. 4 both planes are parallel to each other.

Figs. 9a and b show, respectively, the corresponding variation of the normalized SIFs $F_{II}$ and $F_{III}$ with material angles $\psi_1 = 90^\circ$ and $\beta_1 = 0^\circ$, along the square crack front, for different crack angles $\psi_2 = 0^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, and $90^\circ$ ($\beta_2 = 0^\circ$). Again, we observed that the SIF values for the shear and tearing modes along the crack front ABC are antisymmetric as compared to those along CDA. It is further noticed that the maximum SIF for $F_{II}$ is reached along the crack fronts AB and CD, whilst for $F_{III}$, the maximum is along BC and DA. Furthermore, the magnitude of the maximum in $F_{II}$ is larger than that in $F_{III}$ (0.474 vs. 0.287), due to the different orientation relations between the crack fronts and the material isotropic plane.

We have also run our program for another case where the material orientation angles are $\psi_1 = 45^\circ$, but the crack angles are $(\psi_2, \beta_2) = (30^\circ, 0^\circ), (30^\circ, 30^\circ), (30^\circ, 60^\circ), (60^\circ, 0^\circ), (60^\circ, 30^\circ)$, and $(60^\circ, 60^\circ)$ (Figs. 10 and 11a, b). It is observed from Fig. 10 that the crack inclined angle $\psi_2$ is an important factor in the mode-I SIF. It is obvious that the SIF values are much larger for $\psi_2 = 30^\circ$ than those for $\psi_2 = 60^\circ$, independent of the crack dip orientation $\beta_2$. Furthermore, for fixed $\psi_2 = 30^\circ$, the SIF value increases with increasing dip orientation $\beta_2$. Figs. 11a and b show the corresponding $F_{II}$ and $F_{III}$ along the crack front. As can be seen, the variation of $F_{II}$ and $F_{III}$ is much more complicated than that of $F_{I}$, although they still possess the antisymmetric feature. In other words, the SIF values along the crack front ABC have the same magnitude but opposite sign as compared to those along CDA.

Finally, shown in Figs. 12a-c are the relative CODs. Fig. 12a is for the material angles $\psi_1 = \beta_1 = 0^\circ$ and crack angles $\psi_2 = \beta_2 = 0^\circ$. It is apparent that for this case, the crack is under a uniform pure tensile stress state.
solid containing an internal square crack. A set of four special nine-node quadrilateral elements are employed to approximate the crack surface as well as the outer boundary. The mixed 3D SIFs are evaluated using the asymptotic relation between the SIFs and the relative CODs via the Barnett–Lothe tensor. Numerical examples of the mixed 3D SIFs are presented for a transversely isotropic and cracked rock cuboid with any given material orientation. The cuboid is under a uniform vertical traction along its top and bottom surfaces and the central square crack is arbitrarily oriented. Our results show that among the selected material and crack orientations, the mode-I SIF reaches the largest possible value when the material inclined angle $\psi_1 = 45^\circ$ and dig angle $\beta_1 = 45^\circ$, and the crack inclined angle $\psi_2 = 0^\circ$ and dig angle $\beta_2 = 0^\circ$. It is further observed that when the crack is oriented vertically or nearly vertically, the mode-I SIF becomes negative, indicating that the crack closes due to an overall compressive loading normal to the crack surface. Variation of the SIFs for modes II and III along the crack fronts also shows some interesting features for different combinations of the material and crack orientations, which could be useful in the future failure analysis and design of cracked anisotropic solids.

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**Appendix A**

The anisotropic elastic compliance [$\mathcal{C}$] as functions of the elastic constants ($E$, $F$, $\nu$, $\nu'$, and $G$) and the inclined angle ($\psi_1$) with fixed $\beta_1 = 0^\circ$:

\[
a_{11} = \sin^2 \psi_1 \left( \frac{\sin^2 \psi_1}{E} + \frac{\nu}{E} \cos^2 \psi_1 \right) + \cos^2 \psi_1
\]

\[
a_{12} = -\frac{\nu}{E} \sin^2 \psi_1 \left( \frac{\nu}{E} \sin^2 \psi_1 \right) + \cos^2 \psi_1
\]

\[
a_{13} = \sin^2 \psi_1 \left( \frac{\cos^2 \psi_1}{E} - \frac{\nu}{E} \sin^2 \psi_1 \right) + \cos^2 \psi_1
\]

\[
a_{15} = -\sin 2 \psi_1 \sin^2 \psi_1 \left( \frac{1}{E} + \frac{\nu'}{E} \right) + \sin 2 \psi_1 \cos^2 \psi_1
\]

\[
a_{22} = \frac{1}{E} + \frac{\nu'}{E}
\]

\[
a_{23} = -\frac{\nu}{E} \cos^2 \psi_1 - \frac{\nu}{E} \sin^2 \psi_1
\]

\[
a_{25} = \sin 2 \psi_1 \left( \frac{\nu'}{E} - \frac{\nu}{E} \right)
\]

\[
a_{33} = \cos^2 \psi_1 \left( \frac{\cos^2 \psi_1}{E} - \frac{\nu}{E} \sin^2 \psi_1 \right) + \sin^2 \psi_1
\]

\[
a_{35} = -\sin 2 \psi_1 \cos^2 \psi_1 \left( \frac{1}{E} + \frac{\nu'}{E} \right) + \sin 2 \psi_1 \sin^2 \psi_1
\]

\[
a_{55} = \frac{1}{E} + \frac{\nu'}{E}
\]

6. Conclusions

We applied the dual-BEM or the single-domain BEM to the analysis of mixed 3D SIFs in a finite and transversely isotropic
Appendix B

Shape functions for the four special elements
Shape functions for type-I element

\[
\begin{align*}
\phi_1 &= 0.25 \xi_1 \xi_2 (\xi_1 - 1)(\xi_2 - 1) \\
\phi_2 &= 0.5 \xi_2 (1 - \xi_1^2)(\xi_2 - 1) \\
\phi_3 &= 0.25 \xi_1^2 (\xi_1 + 1)(\xi_2 - 1) \\
\phi_4 &= 0.5 \xi_1 (\xi_1 - 1)(1 - \xi_2^2) \\
\phi_5 &= (1 - \xi_1^2)(1 - \xi_2^2) \\
\phi_6 &= 0.5 \xi_1 (\xi_1 + 1)(1 - \xi_2^2)
\end{align*}
\]

\[
\begin{align*}
\phi_7 &= 0.25 \xi_1 \xi_2 (\xi_1 - 1)(\xi_2 + 1) \\
\phi_8 &= 0.5 \xi_2 (1 - \xi_1^2)(\xi_2 + 1) \\
\phi_9 &= 0.25 \xi_1 \xi_2 (\xi_1 + 1)(\xi_2 + 1)
\end{align*}
\]

(B.1)

Shape functions for type-II element

\[
\begin{align*}
\phi_1 &= 0.45 \xi_1 \xi_2 (\xi_1 - 1)(\xi_2 - 1) \\
\phi_2 &= 0.9 \xi_2 (1 - \xi_1^2)(\xi_2 - 1) \\
\phi_3 &= 0.45 \xi_1 \xi_2 (\xi_1 + 1)(\xi_2 - 1) \\
\phi_4 &= 0.75 \xi_1 (\xi_1 - 1)(1 - \xi_2)(\xi_1 + \xi_2) \\
\phi_5 &= 1.5(\xi_1^2 - 1)(\xi_2 - 1)(\xi_1 + \xi_2) \\
\phi_6 &= 0.75 \xi_1 (\xi_1 + 1)(1 - \xi_2)(\xi_1 + \xi_2) \\
\phi_7 &= 0.3 \xi_1 \xi_2 (\xi_1 - 1)(\xi_1 + \xi_2) \\
\phi_8 &= 0.6 \xi_2 (1 - \xi_1^2)(\xi_1 + \xi_2) \\
\phi_9 &= 0.3 \xi_1 \xi_2 (\xi_1 + 1)(\xi_1 + \xi_2)
\end{align*}
\]

(B.2)

Shape functions for type-III element

\[
\begin{align*}
\phi_1 &= 0.81 \xi_1 \xi_2 (\xi_1 - 1)(\xi_2 - 1) \\
\phi_2 &= 1.35 \xi_2 (1 - \xi_1^2)(\xi_2 - 1) \\
\phi_3 &= 0.54 \xi_1 \xi_2 (\xi_1 + 1)(\xi_2 - 1)
\end{align*}
\]
\[ \begin{align*}
\phi_1 & = 1.35\xi_1^2(\xi_1 - 1) - \xi_2^2) \left( \xi_1 + \xi_2 \right) \\
\phi_2 & = 2.25(1 - \xi_1)(\xi_1 + \xi_2)(1 - \xi_2)(\xi_1 + \xi_2) \\
\phi_3 & = 0.9\xi_1(\xi_1 + \xi_2)(1 - \xi_2)(\xi_1 + \xi_2) \\
\phi_4 & = 0.54\xi_1^2(\xi_1 - 1)(\xi_1 + \xi_2) \\
\phi_5 & = 0.9\xi_2(1 - \xi_1)(\xi_1 + \xi_2)(\xi_1 + \xi_2) \\
\phi_6 & = 0.36\xi_1^2(\xi_1 + \xi_2)(\xi_1 + \xi_2) \\
\end{align*} \]

Shape functions for type-IV element

\[ \begin{align*}
\phi_1 & = 0.54\xi_1\xi_2(\xi_1 - 1) - (\xi_2 - 1) \\
\phi_2 & = -1.35\xi_2(1 - \xi_1)(\xi_1 - 1)(\xi_2 - 1) \\
\phi_3 & = 0.81\xi_1\xi_2(\xi_1 + 1)(\xi_2 - 1) \\
\phi_4 & = 0.9\xi_1(\xi_1 - \frac{1}{2})(1 - \xi_2)(\xi_1 + \xi_2) \\
\phi_5 & = -2.25(1 + \xi_1)(\xi_1 - \frac{1}{2})(1 - \xi_2)(\xi_1 + \xi_2) \\
\phi_6 & = 1.35\xi_1(\xi_1 + 1)(1 - \xi_2)(\xi_1 + \xi_2) \\
\phi_7 & = 0.36\xi_1\xi_2(\xi_1 - \frac{1}{2})(\xi_1 + \xi_2) \\
\phi_8 & = -0.9\xi_2(1 + \xi_1)(\xi_1 - \frac{1}{2})(\xi_1 + \xi_2) \\
\phi_9 & = 0.54\xi_1\xi_2(\xi_1 + 1)(\xi_1 + \xi_2) \\
\end{align*} \]

References