Analytical solution of a semi-permeable crack in a 2D piezoelectric medium based on the PS model

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This paper considers a straight but nonlinear crack in a two-dimensional piezoelectric plane. Different to the existing theoretical solution of the well-known polarization saturation (PS) model, we assume the crack to be semi-permeable. By introducing the dislocation density along the crack line, we derive the analytical solution for the field quantities. Numerical results show that the effect of different boundary conditions on the electric yielding zone and the stress intensity factor is significant and should not be ignored. The influence of the saturated electric displacement on the stress intensity factor and the electric displacement in the crack cavity is also demonstrated.

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1. Introduction

Due to the wide use of piezoelectric ceramics in smart structures, research on fracture of this type of materials becomes extremely important (i.e., Suo et al., 1992; Pan, 1999; Zhang et al., 2002; Jin and Zhong, 2002; Lin et al., 2003; Fang et al., 2004; Kuna, 2010; Zhong and Zhang, 2010). As is well known, the extension of the current fundamental fracture concepts or criteria in pure elasticity to piezoelectricity is not straightforward since the coupling between the mechanical and electric fields is complicated (Suo et al., 1992; Zhang et al., 2002).

Gao et al. (1997) extended the classical Dugdale model (Dugdale, 1960) to a strip polarization saturation (PS) model in piezoelectricity by assuming that the electric displacement is constant on a strip adjacent to a crack tip. The piezoelectric fields and fracture features predicted based on the PS model are in broad agreement with experimental observations (Park and Sun, 1995). McMeeking (2001) pointed out, from the energy point of view, that the PS model actually corresponds to a mechanical Dugdale model in which the strain remains constant. For this reason, Zhang et al. (2005) proposed a strip dielectric breakdown (DB) model assuming that the electric field strength should be constant in a strip adjacent to a crack tip. It is found that the DB model gives the same results as the PS model in predicting the effects of an applied electric field on the fracture of piezoelectric media (Zhang et al., 2005). The study of the PS and DB models was also conducted by Ru and Mao (1999), Wang (2000), Beom et al. (2006), Loboda et al. (2010), Gao et al. (2006), among others. Recently, Fan et al. (2009) developed a numerical method, the nonlinear hybrid extended displacement discontinuity-fundamental solution (NHEDD-PS) method, where both the PS and DB models can be considered, to study the effect of the electric boundary condition on the field quantities. The numerical results by Fan et al. (2009) show that the electric displacement in the crack cavity was approximately constant and that the calculated result of the field quantities under the semi-permeable electric crack condition was very close to that under the impermeable condition for the given loadings and material parameters. However, the influence of the electric boundary condition on the fracture features could be significant as demonstrated by Loboda et al. (2010) and Fan et al. (2011), among others.

Several models were proposed in the literature to consider the effect of the electric field in the crack cavity, e.g., the self-consistent, energetically consistent, electrostatic traction (Zhang et al., 2002; Landis, 2004; Ricoeur and Kuna, 2009) and semi-permeable boundary conditions (Hao and Shen, 1994). Fan et al. (2011) studied these models and found that the self-consistent one would lead to almost the same result as the semi-permeable boundary condition model. As such, the simple boundary condition, i.e., the semi-permeable boundary condition will be adopted in this paper in order to obtain...
the analytical solution of the problem. In other words, in this paper, we assume a semi-permeable crack for the opening part of the crack and the PS model for the electric yielding zone (EYZ). The dislocation density concept is then applied to derive the integral equations. Simple numerical solutions of these equations show significant influence of the semi-permeable crack model on the field quantities as compared to the simple impermeable crack model. The paper is organized as follows: In Section 2, we present the basic equations; In Section 3, the semi-permeable as well as the impermeable crack models are described along with the corresponding boundary conditions; The boundary integral equations are derived in Section 4, and the expressions for the field intensity factors and local J-integral are given in Section 5; Numerical examples are carried out in Section 6, and conclusions are drawn in Section 7.

2. Basic equations

In the absence of body force and free electric charge, the extended equilibrium equations, geometric relations, and the constitutive relations of piezoelectric media are given by

\[
\sigma_{ij} = 0, \quad D_{ik} = 0, \quad i, j = 1, 2, 3 \text{ or } x, y, z,
\]

\[
\varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), \quad E_i = -\varphi_j,
\]

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad k, l = 1, 2, 3 \text{ or } x, y, z,
\]

\[
D_k = \varepsilon_{ik} E_i + \kappa_{ik} \varphi_i,
\]

where \(\sigma_{ij}\) and \(D_{ik}\) are the stresses and electric displacements, respectively; \(\varepsilon_{ij}\) and \(E_i\) are the strain and electric fields, respectively; \(u_{ij}\) is the strain; \(\varphi_j\) is the potential; \(C_{ijkl}\) and \(\kappa_{ik}\) stand for the elastic constants, the piezoelectric constants and the dielectric constants, respectively. A subscript comma denotes the partial differentiation with respect to the coordinate along the comma.

For a generalized two-dimensional deformation in which the extended displacement vector \(\mathbf{u} = (u_1, u_2, u_3, \varphi)^T\) and the extended stress function vector \(\mathbf{\Phi} = (\varphi_1, \varphi_2, \varphi_3, \varphi)^T\) depends only on \(x_1\) and \(x_2\), the general solution takes the form (Suo et al., 1992; Zhang et al., 2002):

\[
\mathbf{u} = \mathbf{A}f(z) + \mathbf{B}f(z),
\]

\[
\mathbf{\Phi} = \mathbf{B} \mathbf{f}(z),
\]

where \(\mathbf{A} = (a_1, a_2, a_3, a_4)\) and \(\mathbf{B} = (b_1, b_2, b_3, b_4)\) with \(a_i\) and \(b_i\) \((i = 1, 2, 3, 4)\) being both the four-dimensional eigenvectors; \(f(z) = (f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4))^T\) is an analytic function vector; \(z_i = x_1 + p_i x_2\) with \(p_i\) being the eigenvalue with a positive imaginary part. The eigenvalues and eigenvectors are determined by the following eigenvalue relations

\[
[\mathbf{Q} + p_i (\mathbf{R} + \mathbf{T}^T) + p_i^2 \mathbf{T}] \mathbf{a}_i = 0,
\]

\[
\mathbf{b}_i = [\mathbf{R} + p_i \mathbf{T}] \mathbf{a}_i = -\frac{1}{p_i} [\mathbf{R} + p_i \mathbf{T}] \mathbf{a}_i,
\]

where

\[
\mathbf{Q} = \mathbf{C}_{ijk1}, \quad \mathbf{R} = \mathbf{C}_{ijk2}, \quad \mathbf{T} = \mathbf{C}_{2ijk2}.
\]

The extended stress vectors are calculated from the extended stress function vector:

\[
\Sigma_2 = (\sigma_{11}, \sigma_{22}, \sigma_{23}, D_2)^T = \mathbf{\Phi}_{,1},
\]

\[
\Sigma_1 = (\sigma_{11}, \sigma_{12}, \sigma_{13}, D_1)^T = -\mathbf{\Phi}_{,2},
\]

In addition, matrix \(\mathbf{H}\), which was defined in Zhang et al. (2005), will be used in the following analyses

\[
\mathbf{H} = 2\text{Re}[\mathbf{AB}^{-1}], \quad \mathbf{H}^{-1} = \left(\begin{array}{cccc}
F_1 & F_2 & F_3 & F_4 \\
F_2 & F_3 & F_4 & F_1 \\
F_3 & F_4 & F_1 & F_2 \\
F_4 & F_1 & F_2 & F_3 \\
\end{array}\right),
\]

where

\[
F_1 = (F_{11}, F_{12}, F_{13}), \quad F_2 = (F_{41}, F_{42}, F_{43}),
\]

and \(F_{ij}\) \((i, j = 1, 2, 3, 4)\) are material-related constants (Zhang et al., 2005).

3. The crack models and associated boundary conditions

Fig. 1 shows a crack \(S\) based on the PS model where the poling direction of the piezoelectric medium is along the \(x_2\)-axis, \((-a,a)\) denotes the opening part of the crack, and \((-c,a)\) and \((a,c)\) denote the electric yielding zone (EYZ). The external loading is applied uniformly at infinity,

\[
\Sigma_2^\infty = (\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty)^T.
\]

The extended boundary conditions on the crack surface for the general case require

\[
\Sigma_2 = (\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2)^T = (0, 0, 0, D_2^S(x_1))^T, \quad -a < x_1 < a,
\]

where \(D_2^S\) denotes the electric displacement in the crack cavity. For an electrically impermeable crack model, we have

\[
D_2^S(x_1) = D_2^C(x_1) = 0,
\]

where the superscripts “+” and “−” denote the upper and lower crack surfaces, respectively.

For a semi-permeable crack model, the electric displacement in the crack cavity along the crack is related to the crack opening displacement and electric potential jump by (Hao and Shen, 1994)

\[
D_2^S(x_1) = -\kappa^e \left[ \varphi^e(x_1) - \varphi^e(x_1) \right] u_2^e(x_1) - u_2^e(x_1) \left[ \varphi^e(x_1) - \varphi^e(x_1) \right],
\]

where \(\kappa^e\) is the dielectric constant of the medium inside the opening crack, \(\varphi^e(x_1)\) is the potential jump, and \(\left| u_2^e(x_1) \right|, \left| u_2^e(x_1) \right| = u_2^e(x_1) - u_2^e(x_1)\) is the crack opening displacement.
Eqs. (13)–(15) give the boundary conditions in the opening part of the crack $|x_1| \leq a$. Along the EYZ, we assume the PS model, which takes the following form (Gao et al., 1997),

$$u^*_i(x_1) = u^*_j(x_1), \quad D^*_i(x_1) = D^*_j(x_1) = D_S, \quad a < |x_1| < c, \quad i = 1, 2, 3,$$

where $D_S$ is the saturated electric displacement.

4. Solutions in terms of integral equations

We will solve the crack problem described above via the integral equation method. We introduce four functions to represent the distributed dislocations, $g_j(x_1)$, which are actually associated with the Burgers vector components (the conventional elastic dislocations $b^* = (b_1, b_2, b_3)$, and the electric dislocation $\Delta \varphi$) such that $g_j(x_1) b_1 dx_1$ ($j = 1, 2, 3, 4$; with $b_4 = \Delta \varphi$) represents the strength of the Burgers vector located at $x_1$ in the interval $dx_1$. We now distribute the conventional dislocations from $-a$ to $a$, and the electric dislocation from $-c$ to $c$. Therefore, the crack opening displacements and the potential jump can be expressed by the extended dislocations, as

$$\left| u^*_i(x_1) \right| = u^*_i(x_1) - u^*_j(x_1) = \int_{x_1}^{a} g_j(x_1) b_1 dx_1, \quad |x_1| \leq a,$$

$$\left| \varphi(x_1) \right| = \varphi^*(x_1) - \varphi^*(x_1) = \int_{x_1}^{c} g_4(x_1) \Delta \varphi dx_1, \quad |x_1| \leq c.$$

Making use of the extended dislocation Green’s functions (Zhang et al., 2005) and the boundary conditions in Eqs. (12) and (13), we derive the following integral equations for the general case (Eq. (13)) based on the PS model

$$\int_{-a}^{a} \frac{1}{\pi(x_1-x_1)} F_1 \langle g_i \rangle b^* dx_1 + \int_{-c}^{c} \frac{1}{\pi(x_1-x_1)} F_2^2 g_4 \Delta \varphi dx_1 + T^* = 0, \quad |x_1| \leq a,$$

$$\int_{-a}^{a} \frac{1}{\pi(x_1-x_1)} F_2 \langle g_i \rangle b^* dx_1 + \int_{-c}^{c} \frac{1}{\pi(x_1-x_1)} F_2^2 g_4 \Delta \varphi dx_1 + D_S = D_S^*, \quad |x_1| \leq a,$$

From Eqs. (18a) and (18b), we can obtain

$$\int_{-a}^{a} \frac{1}{\pi(x_1-x_1)} F_1 \langle g_i \rangle b^* dx_1 + T^* = 0, \quad |x_1| \leq a,$$

$$\int_{-c}^{c} \frac{1}{\pi(x_1-x_1)} F_2 \langle g_i \rangle b^* dx_1 + D_S = D_S^*, \quad |x_1| \leq a,$$

where $\langle g_j(x_1) \rangle$ is a $3 \times 3$ diagonal matrix, and

$$\mathbf{T} = \left( \sigma_1^{\alpha_1}, \sigma_2^{\alpha_2}, \sigma_3^{\alpha_3} \right)^T, \quad D^*_n = D^*_p$$

Generally speaking, the potential jump $[\varphi(x_1)]$ and the crack opening displacement $[u^*_i(x_1)]$ depend on the position along the crack surface. As such, $D_S^*(x_1)$ in Eq. (15) or in Eq. (18b) would generally be the function of $x_1$.

However, for a given loading, the crack opening displacement and the potential jump are approximately proportional to each other. Therefore, to the first-order approximation, the electric displacement $D_S^*$ within the opening part of crack can be assumed as constant, except for points near the crack tip (Fan et al., 2009; Loboda et al., 2010). This approximation will be validated numerically later on.

From Eqs. (18a) and (18b), we can obtain

$$\int_{-a}^{a} \frac{1}{\pi(x_1-x_1)} F_1 \langle g_i \rangle b^* dx_1 + T^* = 0, \quad |x_1| \leq a,$$

where

$$F_1 = F_{11} - \frac{F_{12} F_{22}}{F_{44}}, \quad T^* = T^* - \left( \frac{F_{12}}{F_{44}} \right) D_S^* - D_S^*.$$
which determines the size of the EYZ.

For the impermeable crack model (Eq. (14)), we have $D_s^2 = 0$, and thus the size of the EYZ from Eq. (27) is reduced to

$$\frac{a}{c} = \cos \left( \frac{\pi D_s^2}{2D_4} \right),$$

which is the same as in Gao et al. (1997) and Wang (2000). In other words, Eq. (27) extends the non-singularity condition in the impermeable crack model to the general crack surface loading case.

Substituting Eq. (27) into Eq. (26) and Eq. (17), the crack opening displacement and the potential jump at an arbitrary point $(x_1, 0)$ along the crack surface and on the EYZ can be expressed in terms of the extended dislocation. Then, substituting the crack opening displacement and the potential jump into Eq. (15), we obtain the electric displacement $D_s^2$ at any point $(x_1, 0)$ along the crack

$$\frac{D_s^2}{\sigma} = \frac{1}{\pi \epsilon_0 |F_1^-|^2} \left[ \frac{(D_0 - D_s^2)(M_1(D_0, x_1) - M_0(D_0, x_1))}{M_1(x_1)} - \pi \epsilon_0 |F_1^-|^2 \right].$$

where $|F_1^-|^2$ denotes the second row of the matrix, and

$$M_1(D_0, x_1) = \int_{x_1}^{c} \left( \frac{c^2 - ax_1}{c(a - x_1)} \right) dx_1,$$

$$M_2(D_0, x_1) = \int_{x_1}^{c} \left( \frac{c^2 + ax_1}{c(a + x_1)} \right) dx_1,$$

$$M_3(x_1) = \sqrt{c^2 - x_1^2}. $$

Eq. (29) is used to determine the electric displacement $D_s^2$ in the crack cavity for the semi-permeable crack model. It is obvious that, under the condition Eq. (15), the electric displacement would be, in general, not constant but the function of $x_1$. As stated before, we assume it to be a constant in this paper. Substituting the derived $D_s^2$ in Eq. (29) into Eq. (27), we finally obtain the EYZ.

5. Field intensity factors and local $j$-integral

The stress in front of the crack tip on the $x_1$-axis is calculated by

$$\Sigma_2 = \begin{pmatrix} \sigma_{21} & \sigma_{22} & \sigma_{23} \end{pmatrix} = \frac{1}{\pi} \int_{-a}^{a} \left( \frac{F_1}{F_2} \right) \langle g_2(x_1') \rangle b' dx_1' + \int_{-c}^{c} \left( \frac{F_2}{F_4} \right) g_2(x_1') \Delta \phi dx_1' + t.$$ 

Defining the field intensity factor vector,

$$\mathbf{K} = \lim_{x_1 \to a} \sqrt{2\pi(1 - a)} \Sigma_2,$$

we obtain the local field intensity factor vector as

$$\mathbf{K} = \begin{pmatrix} K_1 & K_2 & K_3 \\ K_1 & K_2 & K_3 \\ K_1 & K_2 & K_3 \end{pmatrix} \begin{pmatrix} F_{41} \sigma_{21} \sigma_{22} \sigma_{23} \end{pmatrix},$$

which can be further written as

$$\mathbf{K} = \begin{pmatrix} K_{11}^a & K_{12}^a & K_{13}^a \\ K_{11}^a & K_{22}^a & K_{33}^a \\ K_{11}^a & K_{22}^a & K_{33}^a \end{pmatrix} \begin{pmatrix} F_{41} \sigma_{21} \sigma_{22} \sigma_{23} \end{pmatrix},$$

where

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} F_{41} \sigma_{21} \sigma_{22} \sigma_{23} \end{pmatrix}.$$
increases and eventually reaches the EYZ size of the impermeable crack (at around $\sigma = 450$ MPa). With further increase of the far-field mechanical load, the electric displacement in the crack cavity becomes negative for the semi-permeable crack (as shown in the inserted figure in Fig. 3), which means that the electric potential jump is in the opposite direction. Fig. 4 shows the corresponding EYZ size versus the applied far-field electric displacement for both crack models. It is interesting that the EYZ size of the impermeable and semi-permeable cracks increases monotonically with increasing far-field electric displacement. It is noted that, under a small electric loading (e.g., $D < 0.025$ or $< 12.5\%$ of $D_0$), the electric displacement in the crack cavity corresponding to the semi-permeable crack could not be determined (as shown in the inserted figure for the regular part only). The reason is unclear yet.

**Fig. 2.** Variation of the electric displacement in the crack along the crack using Eq. (15).

**Fig. 3.** Size of the electric yielding zone (EYZ) versus the electric loading $D^2 = D$ for semi-permeable and impermeable crack models with fixed $\sigma_{EYZ}^0 = \sigma = 10$ MPa and $D_0 = 0.2 C/m^2$. Inserted figure shows the electric displacement in the crack cavity versus the mechanical loading.

**Fig. 4.** Size of the electric yielding zone (EYZ) versus the electric loading $D^2 = D$ for semi-permeable and impermeable crack models with fixed $\sigma_{EYZ}^0 = \sigma = 10$ MPa and $D_0 = 0.2 C/m^2$. Inserted figure shows the electric displacement in the crack cavity versus the electric loading.

**Fig. 5.** Normalized stress intensity factor (SIF) versus the mechanical loading $\sigma_{EYZ}^0 = \sigma$ for semi-permeable and impermeable crack models with fixed $D^2 = D = 0.1 C/m^2$ and $D_0 = 0.2 C/m^2$.

It is observed from Fig. 5 that for a given mechanical load, the normalized SIF corresponding to the impermeable crack is larger than that corresponding to the semi-permeable crack and that the SIFs in both crack models decrease monotonically with increasing far-field mechanical load. They both reach the same asymptotical value (about 1.4) under a large mechanical load. This interesting feature implies that, under a relatively large mechanical load...
(say >150 MPa), the simple impermeable crack model can be safely employed to calculate the normalized SIF (with a relative error less than 5%) for \( D_s^* = D = 0.1 \text{C/m}^2 \) and \( \sigma_s^* = 0.2 \text{C/m}^2 \). Fig. 6 shows the variation of the normalized SIF with the applied far-field electric displacement. It is obvious that for a given electric displacement, the SIF of the impermeable crack is much larger than that of the semi-permeable crack. However, the SIF for impermeable crack increases linearly with electric displacement, while the SIF for semi-permeable crack varies nonlinearly with electric displacement under the relatively small electric load.

It is observed from Figs. 3–6 that the difference between the results based on the semi-permeable crack and the impermeable crack is significant. This demonstrates that the effect of the electric field in the crack cavity cannot be ignored except for the case of large mechanical load. These nonlinear features are different from those associated with the impermeable crack (Wang, 2000).

6.2. Influence of saturated electric displacement

Fig. 7 displays, for a semi-permeable crack model, the electric displacement \( D_s^* \) in the crack cavity versus the saturated electric displacement \( D_s \) for different far-field electric displacements but with fixed far-field mechanical load \( \sigma_s^* = \sigma = 10 \text{ MPa} \). It is obvious that for a given saturated electric displacement, the electric displacement \( D_s^* \) in the crack cavity decreases with decreasing far-field \( D \), and that for a given \( D \), the electric displacement \( D_s^* \) decreases with increasing saturated electric displacement \( D_s \) (approaches asymptotically a limit value for larger \( D_s \)). It is further observed from Fig. 7 that the electric displacement \( D_s^* \) in the crack cavity is singular when \( D_s^* = D_s \). Fig. 8 displays the normalized SIF of the semi-permeable crack versus saturated electric displacement \( D_s \). It is seen from Fig. 8 that the normalized SIF slightly increases with increasing \( D_s \) and that there is a singularity in the SIF when the far-field electric displacement is close to certain value of \( D_s \).

7. Concluding remarks

We have proposed a semi-permeable polarization saturation (PS) model. This relatively real boundary value problem of the crack is solved via the integral equation approach by introducing the dislocation density along the crack. Comparing to the simple impermeable crack model, our numerical results show clearly that, under either a far-field mechanical or electric load, the effect of different crack boundary conditions on the size of the electric yielding zone and the stress intensity factor is significant and should not be ignored in the piezoelectric fracture analysis. The influence of the saturated electric displacement on the stress intensity factor and on the electric displacement in the crack cavity is presented for the first time.

We further point out that the proposed approach can be extended to include other nonlinear electric boundary conditions, e.g., the electrostatic traction. However, in so doing, the quadratic terms of \( D_s^* \) would appear in the boundary condition, which makes the analytical solution more difficult. If not impossible. This problem will be attacked in the future via the numerical approaches, e.g., the extended displacement discontinuity method.

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