

Note

**Concentrated Force in an Infinite Space  
of Transversely Isotropic Material**

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**Summary**

An exact closed form solution for the displacements and stresses in a transversely isotropic infinite space due to concentrated point forces is presented, which contains the solution for the corresponding granular material as its special case and the well-known three-dimensional Kelvin solution as its limiting case.

**1. Introduction**

It is well-known that the singular solution for a point source within an infinite medium is the foundation of the integral equation and boundary element methods [1]. The frequently used solutions in elastostatics are the famous Kelvin solutions for point forces in two- and three-dimensional spaces [2]. Extended results of these solutions were obtained by several authors. For example, Benitez and Rosakis [3] presented an analytical solution of the displacements and stresses in an infinite three-dimensional isotropic layer subjected to concentrated force acting upon an arbitrary internal point, Dumir and Mehta [4] extended the two-dimensional Kelvin solution to the corresponding orthotropic half-plane case, and Chowdhury [5] constructed the solution to an axisymmetric boundary value problem of a semi-space of transversely isotropic (granular) material due to a vertical point force. The purpose of this paper is to give an exact closed form solution of the three-dimensional problem of an infinite transversely isotropic elastic medium due to concentrated point forces, which can be reduced to the solution for the corresponding granular material [5], as well as to the three-dimensional Kelvin solution and its associated stresses [6].

## 2. General Solutions

With no loss in generality, we assume that the concentrated point forces

$$f_i(r, \theta, z) = n_i \delta(r) \delta(\theta) \delta(z)/r, \quad i = r, \theta, z \quad (2.1)$$

are applied to the origin of an infinite elastic space which is homogeneous and transversely isotropic. Where  $(n_r, n_\theta, n_z)$  are the direction cosines of the unit force vector in cylindrical coordinates  $(r, \theta, z)$ , and the material axis of symmetry of this medium is chosen as the  $z$ -axis. The solution is to provide expressions for the displacements and stresses throughout the infinite space.

This problem can be solved in terms of the cylindrical system of vector functions  $\mathbf{L}$ ,  $\mathbf{M}$ ,  $\mathbf{N}$  [7]. We first expand formally the unknown displacement and traction vectors, and also the prescribed point forces (2.1), respectively, as

$$\mathbf{u}(r, \theta, z) = \sum_m \int_0^{+\infty} [U_L(z) \mathbf{L}(r, \theta) + U_M(z) \mathbf{M}(r, \theta) + U_N(z) \mathbf{N}(r, \theta)] \lambda d\lambda \quad (2.2)$$

$$\mathbf{T}(r, \theta, z) \equiv \sigma_{rz} \mathbf{i}_r + \sigma_{\theta z} \mathbf{i}_\theta + \sigma_{zz} \mathbf{i}_z \quad (2.3)$$

$$= \sum_m \int_0^{+\infty} [T_L(z) \mathbf{L}(r, \theta) + T_M(z) \mathbf{M}(r, \theta) + T_N(z) \mathbf{N}(r, \theta)] \lambda d\lambda$$

$$\mathbf{F}(r, \theta, z) = \sum_m \int_0^{+\infty} [F_L(z) \mathbf{L}(r, \theta) + F_M(z) \mathbf{M}(r, \theta) + F_N(z) \mathbf{N}(r, \theta)] \lambda d\lambda \quad (2.4)$$

In Eqs. (2.2)–(2.4), the dependence of the vector functions  $\mathbf{L}$ ,  $\mathbf{M}$ ,  $\mathbf{N}$ , and of the expansion coefficients  $U_L$ ,  $U_M$ ,  $U_N$ ,  $T_L$ ,  $T_M$ ,  $T_N$ ,  $F_L$ ,  $F_M$ ,  $F_N$ , on the parameters  $\lambda$ ,  $m$  has been omitted for simplicity.

In order to obtain the expressions for displacement and stress components, we are therefore required to determine their expansion coefficients. Proceeding as in [7], we find that when body forces are present, these coefficients satisfy the following two sets of simultaneous linear differential equations

$$\begin{aligned} dU_L/dz &= \lambda^2 U_M A_{13}/A_{33} + T_L/A_{33} \\ dU_M/dz &= -U_L + T_M/A_{44} \end{aligned} \quad (2.5)$$

$$\begin{aligned} dT_L/dz &= \lambda^2 T_M - F_L \\ dT_M/dz &= \lambda^2 U_M (A_{11} A_{33} - A_{13}^2)/A_{33} - A_{13} T_L/A_{33} - F_M \\ dU_N/dz &= T_N/A_{44} \\ dT_N/dz &= \lambda^2 U_N A_{66} - F_N \end{aligned} \quad (2.6)$$

where  $A_{66} = (A_{11} - A_{12})/2$ , and  $A_{11}, A_{12}, A_{13}, A_{33}, A_{44}$  are the five elastic constants of the medium [7];  $F_L, F_M, F_N$  are the expansion coefficients of the body force vector, which are found, for the concentrated point forces (2.1), as

$$F_L(z) = \delta(z) n_z / (2\pi)^{1/2}, \quad m = 0 \quad (2.7)$$

$$F_M(z) = -\delta(z) (Fn_x + in_y) / [2\lambda(2\pi)^{1/2}], \quad m = \pm 1 \quad (2.8)$$

$$F_N(z) = -\delta(z) (in_x \pm n_y) / [2\lambda(2\pi)^{1/2}], \quad m = \pm 1 \quad (2.9)$$

where  $(n_x, n_y, n_z)$  are the  $(x, y, z)$  components of the unit force vector in the space-fixed Cartesian coordinates, with  $x$ - and  $y$ -directions being taken along  $\theta = 0$  and  $\theta = \pi/2$  of the cylindrical coordinates respectively.

We introduce an imaginary plane  $z = 0$ , which passes through the point of the applied forces and divides the infinite space into two half-spaces ( $z > 0$  and  $z < 0$ ). It is obvious that Eqs. (2.5), (2.6) become homogeneous in these two domains. Further, the general solutions in the  $z > 0$  half-space are derived as

$$[E(z)] = Ce^{-\lambda z} [G(\alpha)] + De^{-\beta z} [G(\beta)] \quad (2.10)$$

$$U_N(z) = BL e^{-\lambda z}, \quad T_N(z)/\lambda = -BL \bar{s} e^{-\lambda z} \quad (2.11)$$

and in the  $z < 0$  half-space as

$$[E(z)] = Ae^{\lambda z} [G(-\alpha)] + Be^{\beta z} [G(-\beta)], \quad (2.12)$$

$$U_N(z) = AL e^{\lambda z}, \quad T_N(z)/\lambda = AL \bar{s} e^{\lambda z}. \quad (2.13)$$

In Eqs. (2.10)–(2.13), the column matrices are defined by

$$[E(z)] = [\dot{U}_L(z), \lambda U_M(z), T_L(z)/\lambda, T_M(z)]^T,$$

$$[G(x)] = [c(x), -d(x), -1/x, 1]^T$$

with  $[-]^T$  denoting the transpose of the matrix  $[-]$ ;  $\alpha^2$  and  $\beta^2$  are two distinct roots of the equation

$$(A_{44}x^2 - A_{11})(A_{33}x^2 - A_{44}) + (A_{13} + A_{44})^2 x^2 = 0, \quad (2.14)$$

and

$$s = (A_{66}/A_{44})^{1/2}, \quad \bar{s} = sA_{44}. \quad (2.15)$$

As  $\alpha^2$  and  $\beta^2$  may be either real or complex conjugates depending upon the elastic constants, we have specified that  $\alpha$  and  $\beta$  always have positive real parts; Functions  $c(x)$  and  $d(x)$  are defined by

$$c(x) = (A_{11} + x^2 A_{13}) / [x^2 (A_{11} A_{33} - A_{13}^2)], \quad (2.16)$$

$$d(x) = (A_{13} + x^2 A_{33}) / [x (A_{11} A_{33} - A_{13}^2)].$$

Finally,  $A, B, C, D, AL, BL$  are the constants which can be determined by the discontinuities or jumps of stresses caused by the concentrated forces at  $z = 0$ .

It is shown that these discontinuities can be equivalently represented by the jumps in the expansion coefficients of traction vector, which can be expressed in the forms [8]

$$\begin{aligned} T_L(+0) - T_L(-0) &= -n_z/(2\pi)^{1/2}, & m &= 0 \\ T_M(+0) - T_M(-0) &= (\mp n_x + in_y)/[2\lambda(2\pi)^{1/2}], & m &= \pm 1 \\ T_N(+0) - T_N(-0) &= (in_x \pm n_y)/[2\lambda(2\pi)^{1/2}], & m &= \pm 1 \end{aligned} \quad (2.17)$$

the expansion coefficients of displacements are continuous across  $z = 0$ .

Once the constants in Eqs. (2.10)–(2.13) are determined, we can obtain the expressions for the displacement and traction vectors at any point of the infinite space by Eqs. (2.2), (2.3), and those for the remaining stresses by the generalized Hooke's law and the strain-displacement relations [7].

### 3. Concentrated Forces

#### 3.1 Concentrated Force in $z$ -Direction

Let  $z \rightarrow +0$  in Eqs. (2.10), (2.11),  $z \rightarrow -0$  in Eqs. (2.12), (2.13), and using the continuity condition at  $z = 0$ , we can determine the constants, and derive the following solutions for the  $z > 0$  and  $z < 0$  half-spaces

$$[E(z)] = A\lambda^{-1}e^{\mp\lambda\alpha z}[G(\pm\alpha)] + B\lambda^{-1}e^{\mp\lambda\beta z}[G(\pm\beta)], \quad z \geq 0, \quad (3.1)$$

$$U_N(z) = T_N(z)/\lambda = 0, \quad z \geq 0 \quad (3.2)$$

where the upper (lower) sign is corresponding to the  $z > 0$  ( $z < 0$ ) domain, and

$$A = \alpha c(\alpha)/[2(c(\alpha) - c(\beta))(2\pi)^{1/2}], \quad B = -\beta c(\beta)/[2(c(\alpha) - c(\beta))(2\pi)^{1/2}]. \quad (3.3)$$

The fundamental solutions for the infinite transversely isotropic medium due to the concentrated force in  $z$ -direction at the origin can therefore be obtained from [7]

$$\begin{aligned} u_r(r, \theta, z) &= \sum_{m=0}^{+\infty} \int [U_M(z) \partial S/\partial r + U_N(z) \partial S/(r \partial \theta)] \lambda d\lambda, \\ u_\theta(r, \theta, z) &= \sum_{m=0}^{+\infty} \int [U_M(z) \partial S/(r \partial \theta) - U_N(z) \partial S/\partial r] \lambda d\lambda, \\ u_z(r, \theta, z) &= \sum_{m=0}^{+\infty} \int U_L(z) S \lambda d\lambda. \end{aligned} \quad (3.4)$$

We see that in the case of a concentrated force in  $z$ -direction, only the terms for  $m = 0$  are present on the right-hand side of Eq. (3.4). Substituting Eqs. (3.1), (3.2) into Eq. (3.4) and making use of the integral formula [9]

$$\int_0^{+\infty} J_m(\lambda r) e^{-\lambda a} d\lambda = [(r^2 + a^2)^{1/2} - a]^m / [r^m (r^2 + a^2)^{1/2}] \quad (3.5)$$

we derive the fundamental solutions as follows

$$\begin{aligned} u_r^z(r, \theta, z) &= -zc(\alpha) c(\beta) [g(\alpha, r, z) - g(\beta, r, z)] / [4\pi r(c(\alpha) - c(\beta))], \\ u_\theta^z(r, \theta, z) &= 0, \\ u_z^z(r, \theta, z) &= [g(\alpha, r, z) c^2(\alpha) - g(\beta, r, z) c^2(\beta)] / [4\pi(c(\alpha) - c(\beta))], \end{aligned} \quad (3.6)$$

where the superscript  $z$  is attached to denote the  $z$ -direction of the concentrated force, and  $g(x, r, z)$  is defined by

$$g(x, r, z) = x / [r^2 + (xz)^2]^{1/2}. \quad (3.7)$$

Similarly, the components of the traction vector due to the concentrated force in  $z$ -direction are found as

$$\begin{aligned} \sigma_{rz}^z(r, \theta, z) &= -r[c(\alpha) y(\alpha, r, z) - c(\beta) y(\beta, r, z)] / [4\pi(c(\alpha) - c(\beta))], \\ \sigma_{\theta z}^z(r, \theta, z) &= 0, \\ \sigma_{zz}^z(r, \theta, z) &= -z[y(\alpha, r, z) c(\alpha) - y(\beta, r, z) c(\beta)] / [4\pi(c(\alpha) - c(\beta))], \end{aligned} \quad (3.8)$$

where  $y(x, r, z)$  is defined by

$$y(x, r, z) = x / [r^2 + (xz)^2]^{3/2}. \quad (3.9)$$

It is noted that Eqs. (3.6), (3.8) can be reduced directly to the solutions for concentrated force in  $z$ -direction in an infinite granular material [5].

The remaining stress components are derived as follows

$$\begin{aligned} \sigma_{rr}^z(r, \theta, z) &= \{zc(\alpha) c(\beta) / [4\pi(c(\alpha) - c(\beta))]\} \\ &\quad \times \{A_{11}(y(\alpha, r, z) - y(\beta, r, z)) + 2A_{66}(g(\alpha, r, z) - g(\beta, r, z)) / r^2 \\ &\quad - A_{13}[\alpha^2 c(\alpha) y(\alpha, r, z) / c(\beta) - \beta^2 c(\beta) y(\beta, r, z) / c(\alpha)]\}, \\ \sigma_{r\theta}^z(r, \theta, z) &= 0, \\ \sigma_{\theta\theta}^z(r, \theta, z) &= \{zc(\alpha) c(\beta) / [4\pi(c(\alpha) - c(\beta))]\} \\ &\quad \times \{A_{12}(y(\alpha, r, z) - y(\beta, r, z)) - 2A_{66}(g(\alpha, r, z) - g(\beta, r, z)) / r^2 \\ &\quad - A_{13}[\alpha^2 c(\alpha) y(\alpha, r, z) / c(\beta) - \beta^2 c(\beta) y(\beta, r, z) / c(\alpha)]\}. \end{aligned} \quad (3.10)$$

### 3.2 Concentrated Force in $x$ -Direction

In this case, the expansion coefficients of displacement and traction vectors are given by

$$[E(z)] = \pm A(m) \lambda^{-1} e^{\mp \lambda \alpha z} [G(\pm \alpha)] \pm B(m) \lambda^{-1} e^{\mp \lambda \beta z} [G(\pm \beta)] \quad z \geq 0 \quad (3.11)$$

$$U_N(z) = iD\lambda^{-2} e^{\mp \lambda \alpha z}, \quad T_N(z)/\lambda = \mp iD\lambda^{-2} \bar{s} e^{\mp \lambda \alpha z} \quad z \geq 0 \quad (3.12)$$

where

$$\begin{aligned} A(m) &= mc(\beta) / [4(2\pi)^{1/2} (c(\alpha) - c(\beta))] \\ B(m) &= -mc(\alpha) / [4(2\pi)^{1/2} (c(\alpha) - c(\beta))] \quad m = \pm 1 \\ D &= -1 / [4(2\pi)^{1/2} \bar{s}]. \end{aligned} \quad (3.13)$$

Following the same procedure as in Section 3.1, we get the following expressions of displacement and traction vectors in the infinite transversely isotropic space due to the concentrated force in  $x$ -direction

$$\begin{aligned} u_r^x(r, \theta, z) &= [\cos \theta / (4\pi)] \left\{ z^2 [g(\alpha, r, z) c^2(\beta) - g(\beta, r, z) c^2(\alpha)] / [r^2 (c(\alpha) - c(\beta))] \right. \\ &\quad \left. + p(s, r, z) / \bar{s} \right\}, \\ u_\theta^x(r, \theta, z) &= [\sin \theta / (4\pi)] \left\{ [p(\alpha, r, z) c^2(\beta) / \alpha - p(\beta, r, z) c^2(\alpha) / \beta] / [c(\alpha) - c(\beta)] \right. \\ &\quad \left. + sz^2 g(s, r, z) / (\bar{s} r^2) \right\}, \end{aligned} \quad (3.14)$$

$$u_z^x(r, \theta, z) = -zc(\alpha) c(\beta) \cos \theta [g(\alpha, r, z) - g(\beta, r, z)] / [4\pi r (c(\alpha) - c(\beta))],$$

$$\begin{aligned} \sigma_{rz}^x(r, \theta, z) &= [z \cos \theta / (4\pi r^2)] \left\{ [y(\alpha, r, z) q_1(\alpha, r, z) c(\beta) - y(\beta, r, z) q_1(\beta, r, z) c(\alpha)] / \right. \\ &\quad \left. [c(\alpha) - c(\beta)] + g(s, r, z) \right\}, \end{aligned}$$

$$\begin{aligned} \sigma_{\theta z}^x(r, \theta, z) &= [z \sin \theta / (4\pi r^2)] \left\{ [g(\alpha, r, z) c(\beta) - g(\beta, r, z) c(\alpha)] / [c(\alpha) - c(\beta)] \right. \\ &\quad \left. + y(s, r, z) q_1(s, r, z) \right\}, \end{aligned} \quad (3.15)$$

$$\sigma_{zz}^x(r, \theta, z) = -[r \cos \theta / (4\pi)] \left\{ [y(\alpha, r, z) c(\beta) / \alpha^2 - y(\beta, r, z) c(\alpha) / \beta^2] / [c(\alpha) - c(\beta)] \right\},$$

where  $p(x, r, z)$  and  $q_1(x, r, z)$  are defined by

$$p(x, r, z) = [r^2 + (xz)^2]^{1/2} / r^2, \quad q_1(x, r, z) = 2r^2 + (xz)^2. \quad (3.16)$$

The remaining stress components are found to be

$$\begin{aligned} \sigma_{rr}^x(r, \theta, z) = & -A_{13}c(\alpha) c(\beta) r \cos \theta (y(\alpha, r, z) - y(\beta, r, z)) / [4\pi(c(\alpha) - c(\beta))] \\ & + [\cos \theta / (4\pi r^3)] \left\{ -z^2 A_{11} [c^2(\beta) y(\alpha, r, z) q_3(\alpha, r, z) \right. \\ & \quad \left. - c^2(\alpha) y(\beta, r, z) q_3(\beta, r, z)] / (c(\alpha) - c(\beta)) \right. \\ & + A_{12} [c^2(\beta) g(\alpha, r, z) q_2(\alpha, r, z) / \alpha^2 \\ & \quad \left. - c^2(\alpha) g(\beta, r, z) q_2(\beta, r, z) / \beta^2] / (c(\alpha) - c(\beta)) \right. \\ & \left. - 2A_{66} g(s, r, z) q_2(s, r, z) / (s\bar{s}) \right\}, \\ \sigma_{\theta\theta}^x(r, \theta, z) = & -[A_{66} \sin \theta / (4\pi r^3)] \left\{ 2[c^2(\beta) g(\alpha, r, z) q_2(\alpha, r, z) / \alpha^2 \right. \\ & \quad \left. - c^2(\alpha) g(\beta, r, z) q_2(\beta, r, z) / \beta^2] \right. \\ & \quad (c(\alpha) - c(\beta)) + y(s, r, z) \\ & \quad \left. \times [r^4 + 6(rsz)^2 + 4(sz)^4] / (s\bar{s}) \right\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \sigma_{\theta\theta}^z(r, \theta, z) = & -A_{13}c(\alpha) c(\beta) r \cos \theta (y(\alpha, r, z) - y(\beta, r, z)) / [4\pi(c(\alpha) - c(\beta))] \\ & + [\cos \theta / (4\pi r^3)] \left\{ -z^2 A_{12} [c^2(\beta) y(\alpha, r, z) q_3(\alpha, r, z) \right. \\ & \quad \left. - c^2(\alpha) y(\beta, r, z) q_3(\beta, r, z)] / (c(\alpha) - c(\beta)) \right. \\ & + A_{11} [c^2(\beta) g(\alpha, r, z) q_2(\alpha, r, z) / \alpha^2 \\ & \quad \left. - c^2(\alpha) g(\beta, r, z) q_2(\beta, r, z) / \beta^2] / (c(\alpha) - c(\beta)) \right. \\ & \left. + 2A_{66} g(s, r, z) q_2(s, r, z) / (s\bar{s}) \right\}, \end{aligned}$$

with  $q_2(x, r, z)$  and  $q_3(x, r, z)$  being defined by

$$q_2(x, r, z) = r^2 + 2(xz)^2, \quad q_3(x, r, z) = 3r^2 + 2(xz)^2. \tag{3.18}$$

### 3.3 Concentrated Force in $y$ -Direction

Though the same procedure as above may be used to derive the solutions for this case, we can simply obtain the expressions of displacements and stresses in the infinite space due to the concentrated force in  $y$ -direction from the following relationships

$$\begin{aligned} u_i^y(r, \theta, z) = u_i^x(r, \theta - \pi/2, z) \\ \sigma_{ij}^y(r, \theta, z) = \sigma_{ij}^x(r, \theta - \pi/2, z) \end{aligned} \quad i, j = r, \theta, z. \tag{3.19}$$

That is, the solutions for the corresponding problem of the concentrated force in  $y$ -direction are the same as those in Eqs. (3.14), (3.15), (3.17) with  $\cos \theta$  and  $\sin \theta$  being replaced by  $\sin \theta$  and  $-\cos \theta$ , respectively.

So far we have obtained the displacements and stresses in a homogeneous and transversely isotropic infinite space due to concentrated forces. They are given by Eqs. (3.6), (3.8), (3.10), (3.14), (3.15), (3.17), (3.19). While the concentrated force vector is decomposed in terms of Cartesian coordinates, the displacements and stresses caused by it are expressed in terms of cylindrical coordinates, as the expressions for displacements and stresses are simpler in the later system than in the former one. However, in order to obtain the results in Cartesian coordinates, we are only required to perform well-known coordinates transforms.

The complete three-dimensional solution for the corresponding granular material is the same as that for the transversely isotropic medium with the five elastic constants being replaced by the appropriate quantities since in this case, Eq. (2.14) still has two distinct roots [5]. In the isotropic case, however,  $\alpha = \beta = 1$ , and accordingly the expressions for displacements and stresses become indefinite. In order to get the result for the corresponding isotropic case, we first let  $\alpha \rightarrow \beta$  and use the rule of de l'Hospital in the expressions of displacements and stresses to derive a result for  $\alpha = \beta$ , and then, substitute the elastic constants for isotropic medium [7] into the middle result with  $\alpha = \beta = s = 1$ . In doing so, we obtain the result in the cylindrical coordinates, and after performing coordinate transforms, we find that our result is exactly the same as the classical three-dimensional Kelvin solution and its associated stresses [6].

Finally we point out that, since the present result is actually a generalized three-dimensional Kelvin solution, it can be used, as a fundamental result, to obtain displacement and stress distributions for a number of problems of practical importance. On the one hand, some nuclei of strain and the generalized Mindlin solution [10] in a transversely isotropic medium may be obtained from the present solution by the method of synthesis and superposition. On the other hand, using it as a basic elementary solution, we may construct the integral equation formulation for the three-dimensional transversely isotropic elastic solid in the same way as in [11], to study the effect of anisotropy on the elastostatic field for various problems in engineering.

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