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Static response of a transversely isotropic and layered half-space to general surface loads

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The propagator matrix method is used to solve the problem of the static deformation of a transversely isotropic and layered elastic half-space under the action of general surface loads. The solution is obtained in two systems of vector functions for different cases of characteristic root determined by the elastic constants of the media. It is shown that this general solution contains, as its special cases, the solutions obtained by previous researchers, such as the solution for isotropic and layered media, and the solutions for the problems of axially symmetric and two-dimensional deformation of transversely isotropic and layered media.

Numerical examples are given to verify the present formulation. It is noted that by using the propagator matrix method in two systems of vector functions, the present analysis method is efficient, convenient and easy to apply in practice.

1. Introduction

Hu (1953, 1954) solved the general problem of three-dimensional deformation of a transversely isotropic and homogeneous half-space using the potential function method. Recently, the double Fourier transformations, the Hankel transforms and the so-called finite layer approach were used by Small and Booker (1984, 1986) to solve the deformation problem of the corresponding layered medium. However, that method requires the solution of a system of simultaneous linear equations with an order proportional to the number of layers and the introduction of auxiliary variables and coordinate transformations for the three-dimensional problem. At the same time, it is difficult to use direct Hankel transforms to solve deformation problems by applying some simple surface loadings (Wang, 1987) and some internal sources (Singh, 1970). Using the propagator matrix method (Gilbert and Backus, 1966) and the generalized Love's strain potential, Singh (1986) solved the problem by assuming axially symmetric deformation of a transversely isotropic and layered half-space by surface loads.

The Cartesian system and cylindrical system of vector functions are introduced simultaneously in association with the propagator matrix method in this paper to solve the problem of static deformation of a transversely isotropic and layered half-space by general surface loads. It is shown that the equilibrium equations are reduced to the same two sets of simultaneous linear differential equations for the two systems, which are called type I and type II. The general solutions and the layer matrices are then obtained from the two sets of differential equations. By using the continuity conditions at the layer interfaces and the boundary conditions at the surfaces, the displacement and stress components at any point of the medium are obtained in the two systems by multiplication of matrices. As the solutions are contained in the two systems in terms of a layer matrix and include different cases of characteristic root, the present

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formulation avoids the complicated nature of the problem on the one hand (Small and Booker, 1986), and on the other hand can be reduced directly to the solutions of the corresponding two-dimensional deformation (Small and Booker, 1984) and axially symmetric deformation (Singh, 1986), and also to the solution of the corresponding isotropic case (Singh, 1970). Three numerical examples are also given, which represent, respectively, three-dimensional, two-dimensional and axially symmetric deformation.

2. Basic equations and systems of vector functions

We choose the axis of symmetry of a homogeneous and transversely isotropic elastic medium as the z axis. The generalized Hooke's law in Cartesian coordinates (x, y, z) can then be expressed as (Lekhnitskii, 1963)

$$\begin{aligned} \sigma_{xx} &= A_{11}e_{xx} + A_{12}e_{yy} + A_{13}e_{zz} & \sigma_{yz} &= 2A_{44}e_{yz} \\ \sigma_{yy} &= A_{12}e_{xx} + A_{11}e_{yy} + A_{13}e_{zz} & \sigma_{xz} &= 2A_{44}e_{xz} \\ \sigma_{zz} &= A_{13}e_{xx} + A_{13}e_{yy} + A_{33}e_{zz} & \sigma_{xy} &= 2A_{66}e_{xy} \end{aligned} \tag{1}$$

where

$$A_{66} = (A_{11} - A_{12})/2 \tag{2}$$

In (1), σ_{xx}, σ_{yy}, etc., are the components of stress and e_{xx}, e_{yy}, etc., are the components of strain. Parameters A_{11}, A_{12}, A_{13}, A_{33} and A_{44} are the five elastic constants of the medium. In the case of an isotropic medium

$$\begin{aligned} A_{11} &= A_{33} = E(1-\nu)/[(1+\nu)(1-2\nu)] \\ A_{12} &= A_{13} = E\nu/[(1+\nu)(1-2\nu)] \\ A_{44} &= E/[2(1+\nu)] \end{aligned} \tag{3}$$

where E is the Young's modulus and ν is the Poisson ratio. On replacing subscript x by r and y by θ, we then obtain the generalized Hooke's law in cylindrical coordinates (r, θ, z).

When the body forces are absent, the equilibrium equations are

$$\begin{aligned} &\partial\sigma_{xx}/\partial x + \partial\sigma_{xy}/\partial y + \partial\sigma_{xz}/\partial z = 0 \\ &\partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{yz}/\partial z = 0 \\ &\partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y + \partial\sigma_{zz}/\partial z = 0 \end{aligned} \tag{4}$$

in Cartesian coordinates, and

$$\begin{aligned} &\partial\sigma_{rr}/\partial r + \partial\sigma_{r\theta}/(r\,\partial\theta) \\ &\quad + \partial\sigma_{rz}/\partial z + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0 \\ &\partial\sigma_{r\theta}/\partial r + \partial\sigma_{\theta\theta}/(r\,\partial\theta) + \partial\sigma_{\theta z}/\partial z + 2\sigma_{r\theta}/r = 0 \\ &\partial\sigma_{rz}/\partial r + \partial\sigma_{\theta z}/(r\,\partial\theta) + \partial\sigma_{zz}/\partial z + \sigma_{rz}/r = 0 \end{aligned} \tag{5}$$

in cylindrical coordinates.

The final basic equations are the strain–displacement relations. It is well known that these relations are

$$\begin{aligned} e_{xx} &= \partial u_x/\partial x & 2e_{xy} &= \partial u_x/\partial y + \partial u_y/\partial x \\ e_{yy} &= \partial u_y/\partial y & 2e_{yz} &= \partial u_y/\partial z + \partial u_z/\partial y \\ e_{zz} &= \partial u_z/\partial z & 2e_{xz} &= \partial u_x/\partial z + \partial u_z/\partial x \end{aligned} \tag{6}$$

in Cartesian coordinates and

$$\begin{aligned} &e_{rr} = \partial u_r/\partial r \\ &\quad 2e_{r\theta} = \partial u_r/(r\,\partial\theta) + \partial u_\theta/\partial r - u_\theta/r \\ &e_{\theta\theta} = \partial u_\theta/(r\,\partial\theta) + u_r/r \\ &\quad 2e_{\theta z} = \partial u_\theta/\partial z + \partial u_z/(r\,\partial\theta) \\ &e_{zz} = \partial u_z/\partial z \quad 2e_{rz} = \partial u_z/\partial r + \partial u_r/\partial z \end{aligned} \tag{7}$$

in cylindrical coordinates. In (6) and (7), (u_x, u_y, u_z) and (u_r, u_θ, u_z) are the (x, y, z) and (r, θ, z) components of the displacement vector.

We now introduce two systems of vector functions (Ulitko, 1979). The first system is based on Cartesian coordinates and is called the Cartesian system of vector functions

$$\begin{aligned} &\mathbf{L}(x, y; \alpha, \beta) = \mathbf{i}_z S(x, y; \alpha, \beta) \\ &\mathbf{M}(x, y; \alpha, \beta) = \text{grad } S = \mathbf{i}_x\,\partial S/\partial x + \mathbf{i}_y\,\partial S/\partial y \\ &\mathbf{N}(x, y; \alpha, \beta) = \text{curl } \mathbf{i}_z S = \mathbf{i}_x\,\partial S/\partial y - \mathbf{i}_y\,\partial S/\partial x \end{aligned} \tag{8}$$

where $(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z)$ are the unit vectors in (x, y, z) directions of Cartesian coordinates. The scalar function

$$S(x, y; \alpha, \beta) = \exp[-i(\alpha x + \beta y)]/(2\pi) \tag{9}$$

satisfies the Holmholtz equation

$$\partial^2 S/\partial x^2 + \partial^2 S/\partial y^2 + \lambda^2 S = 0 \tag{10}$$

where

$$\lambda^2 = \alpha^2 + \beta^2$$

In eqns. (8)–(10), α, β and λ are parameter variables. The second system is based on cylindrical coordinates and is called the cylindrical system of vector functions

$$\mathbf{L}(r, \theta; \lambda, m) = \mathbf{i}_z S(r, \theta; \lambda, m)$$

$$\mathbf{M}(r, \theta; \lambda, m) = \text{grad } S = \mathbf{i}_r\, \partial S/\partial r + \mathbf{i}_\theta\, \partial S/(r\, \partial\theta) \tag{11}$$

$$\mathbf{N}(r, \theta; \lambda, m) = \text{curl } \mathbf{i}_z S = \mathbf{i}_r\, \partial S/(r\, \partial\theta) - \mathbf{i}_\theta\, \partial S/\partial r$$

where $(\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_z)$ are the unit vectors in (r, θ, z) directions of the cylindrical coordinates. The scalar function

$$S(r, \theta; \lambda, m) = J_m(\lambda r) \exp(im\, \theta)/(2\pi)^{1/2} \quad m = 0, \pm 1, \pm 2, \ldots \tag{12}$$

also satisfies the Holmholtz equation

$$\partial^2 S/\partial r^2 + \partial S/(r\, \partial r) + \partial^2 S/(r^2\, \partial^2\theta) + \lambda^2 S = 0 \tag{13}$$

In eqn. (12), $J_m(\lambda r)$ denotes the Bessel function of order m.

Owing to the orthogonality of the systems (8) and (11), any vector functions may be expressed in terms of them. In particular, for the unknown displacement and 'surface' stress vectors, we may have

$$\mathbf{u}(x, y, z) = \iint_{-\infty}^{+\infty} [U_L(z)\mathbf{L}(x, y) + U_M(z)\mathbf{M}(x, y) + U_N(z)\mathbf{N}(x, y)]\, d\alpha\, d\beta \tag{14}$$

$$\mathbf{T}(x, y, z) \equiv \sigma_{xz}\mathbf{i}_x + \sigma_{yz}\mathbf{i}_y + \sigma_{zz}\mathbf{i}_z = \iint_{+\infty}^{+\infty} [T_L(z)\mathbf{L}(x, y) + T_M(z)\mathbf{M}(x, y) + T_N(z)\mathbf{N}(x, y)]\, d\alpha\, d\beta \tag{15}$$

in the Cartesian system and

$$\mathbf{u}(r, \theta, z) = \sum_m \int_0^{+\infty} [U_L(z)\, \mathbf{L}(r, \theta) + U_M(z)\mathbf{M}(r, \theta) + U_N(z)\mathbf{N}(r, \theta)]\, \lambda\, d\lambda \tag{16}$$

$$\mathbf{T}(r, \theta, z) \equiv \sigma_{rz}\mathbf{i}_r + \sigma_{\theta z}\mathbf{i}_\theta + \sigma_{zz}\mathbf{i}_z = \sum_m \int_0^{+\infty} [T_L(z)\mathbf{L}(r, \theta) + T_M(z)\mathbf{M}(r, \theta) + T_N(z)\mathbf{N}(r, \theta)]\, \lambda\, d\lambda \tag{17}$$

in the cylindrical system. In eqns. (14)–(17), the dependence of vector functions **L**, **M** and **N** on the parameters α and β or λ and m have been omitted for simplicity, and we have used the same expansion coefficients U_L, U_M, U_N, T_L, T_M and T_N in these two systems. It will be shown in the next section that these coefficients satisfy the same linear differential equations, whether they are in the Cartesian system or in the cylindrical system.

It is of interest to note from (8), (11), (14) and (16) that while the displacement solutions expressed in terms of **N** have zero dilatation, the solutions expressed in terms of **L** and **M** give zero z component of the curl of the displacement vector.

3. General solutions and layer matrices

The problem that we analyse is shown schematically in Fig. 1. General surface loading $\mathbf{P}(x, y) \equiv \mathbf{P}(r, \theta)$ is applied to the surface $z = 0$ of a layered elastic system, which is composed of parallel, homogeneous and transversely isotropic p layers lying over a homogeneous half-space. The continuity conditions at the layer interfaces are assumed to be in welded contact (with the possible exception to the layer interface $z = z_p$).

In the following, we will only give detailed derivation of the results for the Cartesian system of vector functions, and in the corresponding place give the results only for the cylindrical system of vector functions.

Substituting (14) into (6) and then into (1), we obtain, for any layer k, the stresses expressed in terms of the coefficients of displacement

$$\sigma_{xx} = \left[A_{11}\left(U_M \frac{\partial^2}{\partial x^2} + U_N \frac{\partial^2}{\partial x\, \partial y}\right) + A_{12}\left(U_M \frac{\partial^2}{\partial y^2} - U_N \frac{\partial^2}{\partial x\, \partial y}\right)\right.$$

$$\left. +A_{13}\,\mathrm{d}U_L/\mathrm{d}z \right] S(x,\,y)$$

$$\sigma_{yy} = \left[A_{12}\left(U_M \frac{\partial^2}{\partial x^2} + U_N \frac{\partial^2}{\partial x \partial y} \right) \right.$$
$$+ A_{11}\left(U_M \frac{\partial^2}{\partial y^2} - U_N \frac{\partial^2}{\partial x\,\partial y} \right)$$
$$\left. +A_{13}\,\mathrm{d}U_L/\mathrm{d}z \right] S(x,\,y)$$

$$\sigma_{zz} = \left[A_{13} U_M \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right.$$
$$\left. +A_{33}\,\mathrm{d}U_L/\mathrm{d}z \right] S(x,\,y)$$

$$\sigma_{xz} = A_{44}\left[U_L \frac{\partial}{\partial x} + \mathrm{d}U_M/\mathrm{d}z \frac{\partial}{\partial x} \right.$$
$$\left. +\mathrm{d}U_N/\mathrm{d}z \frac{\partial}{\partial y} \right] S(x,\,y)$$

$$\sigma_{yz} = A_{44}\left[U_L \frac{\partial}{\partial y} + \mathrm{d}U_M/\mathrm{d}z \frac{\partial}{\partial y} \right.$$
$$\left. -\mathrm{d}U_N/\mathrm{d}z \frac{\partial}{\partial x} \right] S(x,\,y)$$

$$\sigma_{xy} = A_{66}\left[2U_M \frac{\partial^2}{\partial x\,\partial y} \right.$$
$$\left. + U_N \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right) \right] S(x,\,y) \qquad (18)$$

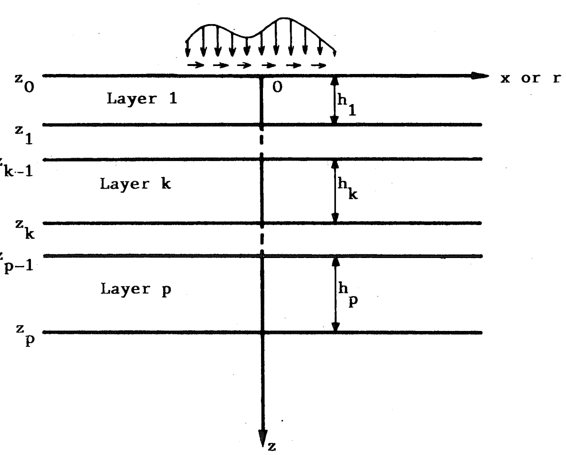

Fig. 1. Scheme of a layered elastic system under general surface loads.

Except for special cases, we will omit the subscript k and the notations $\iint_{-\infty}^{+\infty}[-\ -]\,\mathrm{d}\alpha\,\mathrm{d}\beta$ for the Cartesian system, and $\Sigma_m \int_0^{+\infty}[-\ -]\lambda\,\mathrm{d}\lambda$ for the cylindrical system. Comparing eqn. (15) with the above σ_{xz}, σ_{yz} and σ_{zz}, we have

$$\left(T_M \frac{\partial}{\partial x} + T_N \frac{\partial}{\partial y} \right) S$$
$$= \left[A_{44}\left(U_L \frac{\partial}{\partial x} + \mathrm{d}U_M/\mathrm{d}z \frac{\partial}{\partial x} + \mathrm{d}U_N/\mathrm{d}z \frac{\partial}{\partial y} \right) \right] S$$
$$\left(T_M \frac{\partial}{\partial y} - T_N \frac{\partial}{\partial x} \right) S$$
$$= \left[A_{44}\left(U_L \frac{\partial}{\partial y} + \mathrm{d}U_M/\mathrm{d}z \frac{\partial}{\partial y} - \mathrm{d}U_N/\mathrm{d}z \frac{\partial}{\partial x} \right) \right] S \qquad (19)$$

$$T_L S = \left[A_{13} U_M \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + A_{33}\,\mathrm{d}U_L/\mathrm{d}z \right] S$$

From (19) we can immediately obtain three relations between the expansion coefficients

$$T_L = -\lambda^2 A_{13} U_M + A_{33}\,\mathrm{d}U_L/\mathrm{d}z$$
$$T_M = A_{44}(U_L + \mathrm{d}U_M/\mathrm{d}z) \qquad (20)$$
$$T_N = A_{44}\,\mathrm{d}U_N/\mathrm{d}z$$

Substituting (18) into the equations of equilibrium (4) and by making use of (20), we finally obtain three other relations

$$\mathrm{d}T_L/\mathrm{d}z - \lambda^2 T_M = 0$$
$$\mathrm{d}T_N/\mathrm{d}z - \lambda^2 A_{66} U_N = 0 \qquad (21)$$
$$-\lambda^2 A_{11} U_M + A_{13}\,\mathrm{d}U_L/\mathrm{d}z + \mathrm{d}T_M/\mathrm{d}z = 0$$

Similarly, one may also obtain the relations between the coefficients in eqns. (16) and (17) in the cylindrical system. With considerable algebraic manipulation, it is found that they satisfy exactly the same relations given in (20) and (21). The following results for these coefficients thus hold for these two systems.

It is easy to show that (20) and (21) can be cast into two independent sets of simultaneous linear differential equations. They are called type I and type II, respectively. For type I

$$\begin{aligned} \mathrm{d}U_L/\mathrm{d}z &= \lambda^2 U_M A_{13}/A_{33} + T_L/A_{33} \\ \mathrm{d}U_M/\mathrm{d}z &= -U_L + T_M/A_{44} \\ \mathrm{d}T_L/\mathrm{d}z &= \lambda^2 T_M \\ \mathrm{d}T_M/\mathrm{d}z &= \lambda^2 U_M\left(A_{11}A_{33} - A_{13}^2\right)/A_{33} \\ &\quad - A_{13}T_L/A_{33} \end{aligned} \tag{22}$$

and for type II

$$\begin{bmatrix} \mathrm{d}U_N/\mathrm{d}z \\ \mathrm{d}T_N/\mathrm{d}z \end{bmatrix} = \begin{bmatrix} 0 & 1/A_{44} \\ \lambda^2 A_{66} & 0 \end{bmatrix} \begin{bmatrix} U_N \\ T_N \end{bmatrix} \tag{23}$$

The general solution for type II is found to be

$$\begin{bmatrix} U_N \\ T_N \end{bmatrix} = [\mathbf{Z}^L] \begin{bmatrix} A^L \\ B^L \end{bmatrix} \tag{24}$$

where A^L and B^L are the arbitrary functions of λ, and

$$[\mathbf{Z}^L] = \begin{bmatrix} \exp(\lambda s z) & \exp(-\lambda s z) \\ \lambda \bar{s} \exp(\lambda s z) & -\lambda \bar{s} \exp(-\lambda s z) \end{bmatrix} \tag{25}$$

is the solution matrix with

$$\begin{aligned} s &= \left(A_{66}/A_{44}\right)^{1/2} \\ \bar{s} &= sA_{44} = \left(A_{44}A_{66}\right)^{1/2} \end{aligned} \tag{26}$$

The propagating relation is

$$\begin{bmatrix} U_N(z_{k-1}) \\ T_N(z_{k-1})/\lambda \end{bmatrix} = \left[\mathbf{a}_k^L\right] \begin{bmatrix} U_N(z_k) \\ T_N(z_k)/\lambda \end{bmatrix} \tag{27}$$

where

$$\left[\mathbf{a}_k^L\right] = \begin{bmatrix} \cosh(\lambda s_k h_k) & -\sinh(\lambda s_k h_k)/\bar{s}_k \\ -\bar{s}_k \sinh(\lambda s_k h_k) & \cosh(\lambda s_k h_k) \end{bmatrix} \tag{28}$$

is the layer matrix or the propagator matrix of the layer k. Noting that

$$A_{44} = 1/a_{44}$$
$$A_{66} = 1/a_{66}$$

where a_{44} and a_{66} are the elastic constants used by Singh (1986), the layer matrix (28) is then the same as that obtained by Singh under the assumption of axially symmetric deformation. It is easy to show (Singh, 1986) that (28) can be reduced directly to the layer matrix of the corresponding isotropic case (Singh, 1970).

The analytical derivation of the general solution and layer matrix of type I involves considerable algebra and the derived results may have different forms according to different cases of characteristic root. But with careful treatment they can be cast into a uniform form, i.e. the general solution can be unified into the form

$$[\mathbf{E}(z)] = [\mathbf{Z}(z)][\mathbf{K}] \tag{29}$$

In eqn. (29), the two column matrices are defined by

$$[\mathbf{E}(z)] = \left[U_L(z),\ \lambda U_M(z),\ T_L(z)/\lambda,\ T_M(z)\right]^{\mathrm{T}} \tag{30}$$

and

$$[\mathbf{K}] = [A,\ B,\ C,\ D]^{\mathrm{T}} \tag{31}$$

where A, B, C and D are the arbitrary functions of λ, and $[-\ -]^{\mathrm{T}}$ denotes the transpose of the matrix $[-\ -]$. The elements of solution matrix $[\mathbf{Z}(z)]$ for different cases of characteristic root are given in Appendix A.

From eqn. (29) we obtain the propagating relation

$$[\mathbf{E}(z_{k-1})] = [\mathbf{a}_k][\mathbf{E}(z_k)] \tag{32}$$

where $[\mathbf{a}_k]$ is the propagator matrix of layer k, and its elements for different cases of characteristic root are given in Appendix B. It is noted from Appendix B that when the characteristic roots are all equal to one, the layer matrix is then directly reduced to the isotropic layer matrix (Singh, 1970).

4. Deformation of a layered elastic system by general surface loads

For any surface load $\mathbf{P}(x, y) \equiv \mathbf{P}(r, \theta)$, we may expand it in terms of the Cartesian system of vector functions in the form

$$\begin{aligned} \mathbf{P}(x, y) = \iint_{-\infty}^{+\infty} &\left[P_L(\alpha, \beta)\mathbf{L}(x, y)\right. \\ &+ P_M(\alpha, \beta)\mathbf{M}(x, y) \\ &\left.+ P_N(\alpha, \beta)\mathbf{N}(x, y)\right] \mathrm{d}\alpha\, \mathrm{d}\beta \end{aligned} \tag{33}$$

or in terms of the cylindrical system of vector functions in the form

$$\mathbf{P}(r,\theta)=\sum_m\int_0^{+\infty}[P_L(\lambda,m)\mathbf{L}(r,\theta)+P_M(\lambda,m)\mathbf{M}(r,\theta)+P_N(\lambda,m)\mathbf{N}(r,\theta)]\lambda\,d\lambda \tag{34}$$

In order to find the displacements and the stresses at any point of the medium under the surface load **P**, we need the continuity conditions at the layer interfaces and the relations (27) and (32). By making use of them and by multiplication of matrices, we obtain

$$[U_N(z_0),\ T_N(z_0)/\lambda]^{\mathrm{T}}=[\mathbf{a}_1^L][\mathbf{a}_2^L]\cdots[\mathbf{a}_p^L][U_N(z_p),\ T_N(z_p)/\lambda]^{\mathrm{T}} \tag{35}$$

$$[\mathbf{E}(0)]=[\mathbf{a}_1][\mathbf{a}_2]\cdots[\mathbf{a}_p][\mathbf{E}(z_p)] \tag{36}$$

Substituting the surface traction condition (33) or (34) into eqn. (15) or (17), the left-hand sides of eqns. (35) and (36) then become

$$U_N(z_0)=U_N(0),\ T_N(z_0)/\lambda=P_N/\lambda \tag{37}$$

$$[\mathbf{E}(0)]=[U_L(0),\ \lambda U_M(0),\ P_L/\lambda,\ P_M]^{\mathrm{T}} \tag{38}$$

The unknown quantities contained in eqns. (35) and (36) may be determined according to the behaviour of the homogeneous half-space and the condition at the layer interface $z=z_p$. We discuss here two typical cases.

4.1. *Elastic half-space and welded condition at* $z=z_p$

In this case, using the continuity condition at layer interface z_p, eqns. (35) and (36) become

$$[U_N(0),\ T_N(0)/\lambda]^{\mathrm{T}}=[\mathbf{a}_1^L][\mathbf{a}_2^L]\cdots[\mathbf{a}_p^L][\mathbf{Z}^L(z_p)][A_{p+1}^L,\ B_{p+1}^L]^{\mathrm{T}} \tag{39}$$

$$[\mathbf{E}(0)]=[\mathbf{a}_1][\mathbf{a}_2]\cdots[\mathbf{a}_p][\mathbf{Z}(z_p)][\mathbf{K}_{p+1}] \tag{40}$$

For the same reason as Singh (1986), A_{p+1}^L in eqn. (39) and A_{p+1} and C_{p+1} in eqn. (40) should all be equal to zero. The remaining unknown quantities in eqns. (39) and (40) are thus determined by

$$B_{p+1}^L=P_N/(\lambda F_{22}) \tag{41}$$

$$B_{p+1}=(G_{44}P_L/\lambda-G_{34}P_M)/\Delta \tag{42a}$$

$$D_{p+1}=(G_{32}P_M-G_{42}P_L/\lambda)/\Delta \tag{42b}$$

In eqns. (41) and (42a and b)

$$[\mathbf{F}]=[\mathbf{a}_1^L][\mathbf{a}_2^L]\cdots[\mathbf{a}_p^L][\mathbf{Z}^L(z_p)] \tag{43}$$

$$[\mathbf{G}]=[\mathbf{a}_1][\mathbf{a}_2]\cdots[\mathbf{a}_p][\mathbf{Z}(z_p)] \tag{44}$$

$$\Delta=G_{32}G_{44}-G_{34}G_{42}$$

Knowing B_{p+1}^L, B_{p+1} and D_{p+1} from eqns. (41) and (42a and b), the response at any point of the medium can be obtained from the relations

$$[U_N(z),\ T_N(z)/\lambda]^{\mathrm{T}}=[\mathbf{a}_k^L(z_k-z)][\mathbf{a}_{k+1}^L]\times\cdots[\mathbf{a}_p^L][\mathbf{Z}^L(z_p)][0,\ B_{p+1}^L]^{\mathrm{T}} \tag{45}$$

and

$$[\mathbf{E}(z)]=[\mathbf{a}_k(z_k-z)][\mathbf{a}_{k+1}]\times\cdots[\mathbf{a}_p][\mathbf{Z}(z_p)][0,\ B_{p+1},\ 0,\ D_{p+1}]^{\mathrm{T}} \tag{46}$$

where $z_{k-1}\leqslant z\leqslant z_k$; $[\mathbf{a}_k^L(z_k-z)]$ and $[\mathbf{a}_k(z_k-z)]$ are obtained, respectively, from $[\mathbf{a}_k^L]$ in (28) and $[\mathbf{a}_k]$ in Appendix B on replacing h_k by z_k-z.

4.2. *Rigid half-space and rough or smooth condition at* $z=z_p$

In this case, the continuity condition at z_p may be divided into rough-rigid and smooth-rigid (Small and Booker, 1984, 1986). For rough-rigid, we have

$$u_x(x,y,z_p)=u_y(x,y,z_p)=u_z(x,y,z_p)=0$$

and for smooth-rigid, we have

$$u_z(x,y,z_p)=\sigma_{xz}(x,y,z_p)=\sigma_{yz}(x,y,z_p)=0$$

As an illustration of the rigid half-space case, we give here detailed discussion only for the rough-rigid type. From eqn. (14) or (16), this continuity condition leads to

$$U_L(z_p)=U_M(z_p)=U_N(z_p)=0$$

Substituting these values into the right-hand sides of eqns. (35) and (36), we thus obtain the remaining unknown quantities

$$T_N(z_p)/\lambda = P_N/(\lambda F_{22}) \tag{47}$$

$$T_L(z_p)/\lambda = (G_{44}P_L/\lambda - G_{34}P_M)/\Delta \tag{48a}$$

$$T_M(z_p) = (G_{33}P_M - G_{43}P_L/\lambda)/\Delta \tag{48b}$$

where

$$\Delta = G_{33}G_{44} - G_{43}G_{34}$$

Matrices **[F]** and **[G]** are obtained, respectively, from eqns. (43) and (44) by removing the solution matrices $[\mathbf{Z}^L(z_p)]$ and $[\mathbf{Z}(z_p)]$. Knowing the coefficients T_L, T_M and T_N from eqns. (47) and (48a and b), we therefore find the response at any point of the medium from the relations

$$\begin{aligned}[U_N(z),\ T_N(z)/\lambda]^{\mathrm{T}} &= [\mathbf{a}_k^L(z_k - z)][\mathbf{a}_{k+1}^L] \\ &\quad \times \cdots [\mathbf{a}_p^L][0,\ T_N(z_p)/\lambda]^{\mathrm{T}}\end{aligned} \tag{49}$$

$$\begin{aligned}[\mathbf{E}(z)] &= [\mathbf{a}_k(z_k - z)][\mathbf{a}_{k+1}] \\ &\quad \times \cdots [\mathbf{a}_p][0,\ 0,\ T_L(z_p)/\lambda,\ T_M(z_p)]^{\mathrm{T}}\end{aligned} \tag{50}$$

So far, we have obtained the displacement and 'surface' stress vectors at any point of the medium by the general surface load (eqn. (33) or (34)). They are given by eqns. (14) and (15) or (16) and (17) as the system may be. In eqns. (14)–(17), the expansion coefficients are given by eqns. (45) and (46) or (49) and (50) as the case may be. The final point is to obtain the remaining stress components in (18) at any point of the medium. By making use of the linear differential equations ((22) and (23)), it is found that they can be expressed as the linear combinations of the known quantities in eqns. (45) and (46) or (49) and (50), i.e.

$$\begin{aligned}\sigma_{xx}(x,\ y,\ z) &= \iint_{-\infty}^{+\infty}[T_L A_{13}/A_{33} - \alpha\beta(A_{11} - A_{12})U_N \\ &\quad + (\lambda^2 A_{13}^2/A_{33} - \alpha^2 A_{11} - \beta^2 A_{12})U_M] \\ &\quad \times S(x,\ y;\ \alpha,\ \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\end{aligned}$$

$$\begin{aligned}\sigma_{xy}(x,\ y,\ z) &= A_{66}\iint_{-\infty}^{+\infty}[(\alpha^2 - \beta^2)U_N - 2\alpha\beta U_M] \\ &\quad \times S(x,\ y;\ \alpha,\ \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\end{aligned} \tag{51}$$

$$\begin{aligned}\sigma_{yy}(x,\ y,\ z) &= \iint_{-\infty}^{+\infty}[T_L A_{13}/A_{33} + \alpha\beta(A_{11} - A_{12})U_N \\ &\quad + (\lambda^2 A_{13}^2/A_{33} - \alpha^2 A_{12} - \beta^2 A_{11})U_M] \\ &\quad \times S(x,\ y;\ \alpha,\ \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\end{aligned}$$

in the Cartesian system of vector functions, and

$$\begin{aligned}\sigma_{rr}(r,\ \theta,\ z) &= \sum_m \int_0^{+\infty}[T_L A_{13}/A_{33} + U_N(A_{11} - A_{12})\Delta_1 \\ &\quad + U_M\lambda^2(A_{13}^2 - A_{11}A_{33})/A_{33} \\ &\quad - U_M(A_{11} - A_{12})\Delta_2]S(r,\ \theta;\ \lambda,\ m)\lambda\,\mathrm{d}\lambda\end{aligned}$$

$$\begin{aligned}\sigma_{r\theta}(r,\ \theta,\ z) &= A_{66}\sum_m \int_0^{+\infty}[U_N(\lambda^2 + 2\Delta_2) + 2U_M\Delta_1] \\ &\quad \times S(r,\ \theta;\ \lambda,\ m)\lambda\,\mathrm{d}\lambda\end{aligned} \tag{52}$$

$$\begin{aligned}\sigma_{\theta\theta}(r,\ \theta,\ z) &= \sum_m \int_0^{+\infty}[T_L A_{13}/A_{33} - U_N(A_{11} - A_{12})\Delta_1 \\ &\quad + U_M\lambda^2(A_{13}^2 - A_{12}A_{33})/A_{33} \\ &\quad + U_M(A_{11} - A_{12})\Delta_2]S(r,\ \theta;\ \lambda,\ m)\lambda\,\mathrm{d}\lambda\end{aligned}$$

in the cylindrical system of vector functions. In (52), the surface operators indicate

$$\Delta_1 = \frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial\theta}$$

$$\Delta_2 = \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

5. Numerical examples

In order to verify the above formulation, we have chosen three examples (Small and Booker, 1986). The model is composed of two transversely isotropic and homogeneous layers and the boundary condition at the base $z = z_2$ is assumed

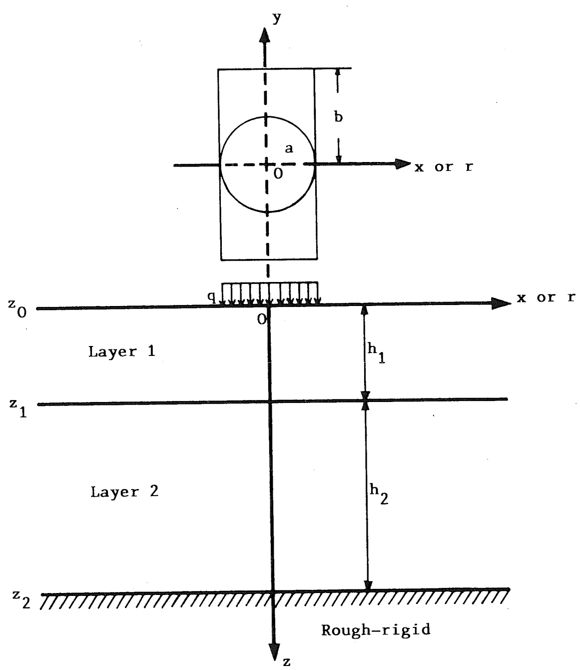

Fig. 2. Models of layers and surface loads used for computation, where $b/a = 2$, $h_1/a = 2$ and $h_2/a = 4$.

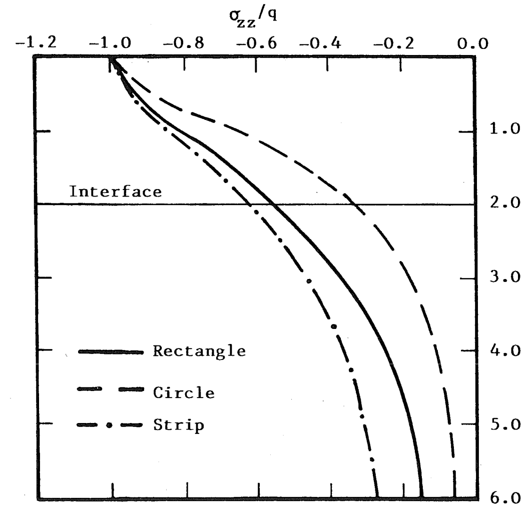

Fig. 3. Vertical stresses beneath the central point of the surface loads.

to be rough-rigid (Fig. 2). Uniform vertical loads are applied to the surface $z = 0$ in rectangle, circle and strip areas. These loadings correspond to three-dimensional, axially symmetric and two-dimensional deformations, respectively. The values of the elastic constants are given in Table I (Small and Booker, 1986), where E_h and E_v are the Young's moduli with respect to directions lying in the plane of isotropy and perpendicular to it; ν_h and ν_{hv} are Poisson ratios which characterize the effects of horizontal strain on complementary horizontal strain and on vertical strain (i.e., the z-direction strain), respectively; ν_{vh} is the Poisson ratio which characterizes the effect of vertical strain on horizontal strain; $f/2$ is the shear modulus for the planes normal to the plane of isotropy.

TABLE I

Elastic constants of a transversely isotropic and two-layered system

	Layer 1	Layer 2
E_h/E_v	1.5	3.0
f/E_v	0.9	1.0
ν_h	0.25	0.1
ν_{hv}	0.3	0.9
ν_{vh}	0.2	0.3
$(E_v)_1/(E_v)_2$		0.25

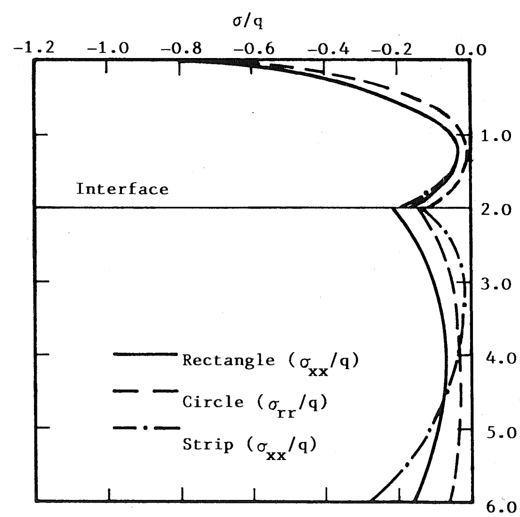

Fig. 4. Horizontal stresses beneath the central point of the surface loads.

$\frac{\nu_{vh}}{E_v} = \frac{\nu_{hv}}{E_h}$

These constants are related to those in eqn. (1) by the formulae (Wardle, 1981)

$$
\begin{aligned}
A_{11} &= E_h(1-\nu_{hv}\nu_{vh})(1+\nu_h)^{-1} \\
&\quad \times(1-\nu_h-2\nu_{hv}\nu_{vh})^{-1} \\
A_{12} &= E_h(\nu_h+\nu_{hv}\nu_{vh})(1+\nu_h)^{-1} \\
&\quad \times(1-\nu_h-2\nu_{hv}\nu_{vh})^{-1} \\
A_{13} &= E_h\nu_{vh}(1-\nu_h-2\nu_{hv}\nu_{vh})^{-1} \\
A_{33} &= E_v(1-\nu_h)(1-\nu_h-2\nu_{hv}\nu_{vh})^{-1} \\
A_{44} &= f/2
\end{aligned}
\tag{53}
$$

For the three types of surface loading to be considered, the expansion coefficients in the systems of vector functions are, respectively

$$P_L(\alpha,\beta) = -2q\sin(\alpha a)\sin(\beta b)/(\alpha\beta\pi) \tag{54a}$$

$$P_L(\lambda,0) = -aqJ_1(\lambda a)(2\pi)^{1/2}/\lambda \tag{54b}$$

$$P_L(\alpha,0) = -2q\sin(\alpha a)/\left[\alpha(2\pi)^{1/2}\right] \tag{54c}$$

It is noted that for these three problems

$$P_M = P_N = 0$$

and therefore the type II has only trivial solution. Equations (54a)–(54c) are obtained by using formulae (33), (34) and (33), respectively. It is obvious that $m=0$ in eqn. (34) corresponds to the axially symmetric deformation and that (8) will represent two-dimensional deformation after replacing 2π by $(2\pi)^{1/2}$ and β by 0 in the scalar function (9). Using eqns. (54a–c) we obtain $T_L(z_p)$ and $T_M(z_p)$ from eqns. (48a and b) and the response at any point of the medium from eqn. (50).

Using these equations, we have calculated the normalized stress components σ_{xx}/q (or σ_{rr}/q for the circular loading) and σ_{zz}/q and displacement component $u_z(E_v)_1/(aq)$ along the z axis. The infinite integrals involved in the numerical computations are carried out by a 16-point Gauss formula. The results are shown schematically in Figs. 3–5. First we note that with the change of signs for σ_{xx}/q (or σ_{rr}/q) and σ_{zz}/q, the curves in Figs. 3 and 4 will be, respectively, the same as those in fig. 5a and b plotted by Small and Booker (1986). The change of signs results from the definition for positive stresses. Second, because the elastic modulus in vertical direction E_v has a four times difference in the two layers, the lower layer is more 'rigid' than the upper layer, as can obviously be seen from Fig. 5. Thirdly, it is interesting to note that for examining the characteristics of the response of a layered medium to surface loads, it is sufficient to choose only some simple surface loads, such as circle or strip loads (Small and Booker, 1984) to do so.

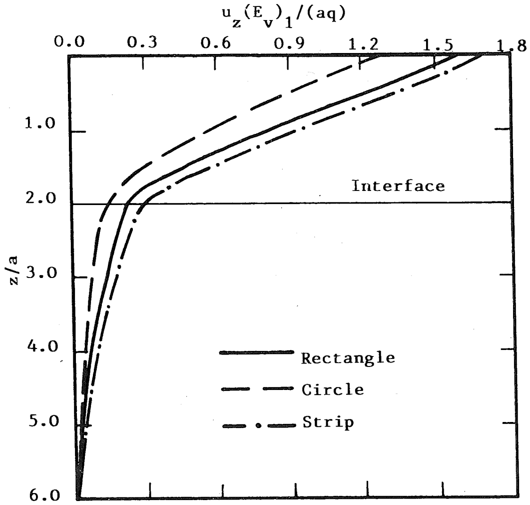

Fig. 5. Deflections beneath the central point of the surface loads.

6. Conclusions

The method of vector functions is introduced in association with the propagator matrix method to solve the deformation of transversely isotropic and layered elastic materials under general surface loads. The formulation is presented so that it can be used directly to perform practical calculations. As the solution is given simultaneously in two systems of vector functions, one can easily solve some problems by choosing the system suitable for the different types of surface loading. It is shown that the formulation given is especially suitable for two-dimensional (Small and Booker, 1984) and axially symmetric (Singh, 1986) deformation. Since the general solution and the propagator matrix for different cases of characteristic root are also given, which includes the isotropic case (Singh, 1970), the present formulation provides a complete solution of deformations by general surface loads of transversely isotropic and layered elastic half-space. Numerical examples are also carried out for three-dimensional, two-dimensional and axially symmetric deformation.

Finally, we point out that, with slight modifications, the present results can be used to analyse some other problems of transversely isotropic and layered media, in particular, the problems of the static response of this medium to displacement dislocation sources (Ben-Menahem and Singh, 1968) and cracks (Parihar and Sowdamini, 1985).

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Appendix A

The elements of the solution matrix [**Z**(z)] in eqn. (29) are:

(1) When $x_1 \neq x_2$

$$\begin{aligned} Z_{11} &= c(x_1)\exp(\lambda x_1 z) \\ Z_{12} &= c(x_1)\exp(-\lambda x_1 z) \\ Z_{21} &= d(x_1)\exp(\lambda x_1 z) \\ Z_{22} &= -d(x_1)\exp(-\lambda x_1 z) \\ Z_{31} &= x_1^{-1}\exp(\lambda x_1 z) \\ Z_{32} &= -x_1^{-1}\exp(-\lambda x_1 z) \\ Z_{41} &= \exp(\lambda x_1 z) \\ Z_{42} &= \exp(-\lambda x_1 z) \end{aligned} \tag{A1}$$

where x_1 and x_2 are the characteristic roots of the following equation

$$(A_{44}x^2 - A_{11})(A_{33}x^2 - A_{44}) + (A_{13} + A_{44})^2 x^2 = 0$$

Z_{i3} and Z_{i4} are obtained from Z_{i1} and Z_{i2}, respectively, on replacing x_1 by x_2 ($i = 1, 2, 3, 4$). In eqn. (A1)

$$c(x) = (A_{11} + x^2 A_{13})/[x^2(A_{11}A_{33} - A_{13}^2)] \tag{A2}$$

$$d(x) = (A_{13} + x^2 A_{33})/[x(A_{11}A_{33} - A_{13}^2)] \tag{A3}$$

(2) When $x_1 = x_2$

$$\begin{aligned} Z_{13} &= [c'(x_1)/\lambda + c(x_1)z]\exp(\lambda x_1 z) \\ Z_{23} &= [d'(x_1)/\lambda + d(x_1)z]\exp(\lambda x_1 z) \\ Z_{33} &= (-x_1^{-2}/\lambda + x_1^{-1}z)\exp(\lambda x_1 z) \\ Z_{43} &= z\exp(\lambda x_1 z) \end{aligned} \tag{A4}$$

while Z_{i1} and Z_{i2} are the same as those in eqn. (A1), Z_{i4} are obtained from Z_{i3} on replacing x_1 by $-x_1$ ($i = 1, 2, 3, 4$). In eqn. (A4), the prime denotes derivative, i.e.

$$\begin{aligned} c'(x_1) &= \mathrm{d}c(x_1)/\mathrm{d}x_1 \\ d'(x_1) &= \mathrm{d}d(x_1)/\mathrm{d}x_1 \end{aligned} \tag{A5}$$

Appendix B

The elements of the layer matrix $[\mathbf{a}_k]$ in eqn. (32) are (omitting the subscript k):

(1) When $x_1 \neq x_2$

$$\begin{aligned} a_{11} = a_{33} &= [g(x_1)c(x_1)/x_1]\cosh y_1 + [g(x_2)c(x_2)/x_2]\cosh y_2 \\ a_{12} = -a_{43} &= g(x_1)c(x_1)\sinh y_1 + g(x_2)c(x_2)\sinh y_2 \\ a_{13} &= -g(x_1)c^2(x_1)\sinh y_1 - g(x_2)c^2(x_2)\sinh y_2 \\ a_{14} = -a_{23} &= -g(x_1)c(x_1)d(x_1)\cosh y_1 - g(x_2)c(x_2)d(x_2)\cosh y_2 \\ a_{21} = -a_{34} &= -[g(x_1)d(x_1)/x_1]\sinh y_1 - [g(x_2)d(x_2)/x_2]\sinh y_2 \\ a_{22} = a_{44} &= -g(x_1)d(x_1)\cosh y_1 - g(x_2)d(x_2)\cosh y_2 \\ a_{24} &= g(x_1)d^2(x_1)\sinh y_1 + g(x_2)d^2(x_2)\sinh y_2 \end{aligned} \tag{B1}$$

$$a_{31} = -\left[g(x_1)/x_1^2\right] \sinh y_1 - \left[g(x_2)/x_2^2\right] \sinh y_2$$

$$a_{32} = -a_{41} = -\left[g(x_1)/x_1\right] \cosh y_1 - \left[g(x_2)/x_2\right] \cosh y_2$$

$$a_{42} = g(x_1) \sinh y_1 + g(x_2) \sinh y_2$$

In eqn. (B1)

$$y_1 = \lambda x_1 h \tag{B2}$$

$$y_2 = \lambda x_2 h \tag{B3}$$

$$g(x) = x/\left[c(x) - xd(x)\right] \tag{B4}$$

where the definitions of functions $c(x)$ and $d(x)$ are given, respectively, in eqns. (A2) and (A3) of Appendix A.

(2) When $x_1 = x_2$

The layer matrix $[\mathbf{a}_k]$ in eqn. (32) can be expressed in the form

$$[\mathbf{a}_k] = -(x_1/c'(x_1))[\mathbf{b}_k] \tag{B5}$$

In eqn. (B5), the elements of matrix $[\mathbf{b}_k]$ are (omitting the subscript k)

$$b_{11} = b_{33} = -\left[c'(x_1)/x_1\right] \cosh y_1 - \left[\lambda hc(x_1)/x_1\right] \sinh y_1$$

$$b_{12} = -b_{43} = -\left[c(x_1)/x_1 + c'(x_1)\right] \sinh y_1 - \lambda hc(x_1) \cosh y_1$$

$$b_{13} = c(x_1)\left[c(x_1)/x_1 + 2c'(x_1)\right] \sinh y_1 + \lambda hc^2(x_1) \cosh y_1$$

$$b_{14} = -b_{23} = \lambda hc(x_1)d(x_1) \sinh y_1$$

$$b_{21} = -b_{34} = \left[d'(x_1)/x_1\right] \sinh y_1 + \left[\lambda hd(x_1)/x_1\right] \cosh y_1$$

$$b_{22} = b_{44} = \left[d(x_1)/x_1 + d'(x_1)\right] \cosh y_1 + \lambda hd(x_1) \sinh y_1$$

$$b_{24} = -d(x_1)\left[d(x_1)/x_1 + 2d'(x_1)\right] \sinh y_1 - \lambda hd^2(x_1) \cosh y_1$$

$$b_{31} = -x_1^{-3} \sinh y_1 + \lambda hx_1^{-2} \cosh y_1$$

$$b_{32} = -b_{41} = \lambda hx_1^{-1} \sinh y_1$$

$$b_{42} = -x_1^{-1} \sinh y_1 - \lambda h \cosh y_1 \tag{B6}$$

where y_1 is given in eqn. (B2), and the definitions of functions $c(x)$, $d(x)$ and their derivatives are given in Appendix A. For an isotropic layer, we have

$$x_1 = 1 \quad y_1 = \lambda h$$

$$c(x_1) = d(x_1) = (1+\nu)/E$$

$$c'(x_1) = -2(1+\nu)(1-\nu)/E$$

$$d'(x_1) = (1+\nu)(1-2\nu)/E$$

Upon the substitution of these values in (B6), we then obtain the layer matrix $[\mathbf{a}_k]$ for the isotropic case (Singh, 1970).

References

Ben-Menahem, A. and Singh, S.J., 1968. Multipolar elastic fields in a layered half-space. Bull. Seismol. Soc. Am., 58: 1519–1572.

Gilbert, F. and Backus, G., 1966. Propagator matrices in elastic wave and vibration problems. Geophysics, 31: 326–332.

Hu, H.C., 1953. On the three-dimensional problems of the theory of elasticity of a transversely isotropic body. Acta Phys. Sin., 9: 131–147 (in Chinese, with English abstract).

Hu, H.C., 1954. On the equilibrium of a transversely isotropic elastic space. Acta Phys. Sin., 10: 239–258 (in Chinese, with English abstract).

Lekhnitskii, S.G., 1963. Theory of Elasticity of an Anisotropic Elastic Body. Holden-Day, San Francisco, CA, 404 pp.

Parihar, K.S. and Sowdamini, S., 1985. Stress distribution in a two-dimensional infinite anisotropic medium with collinear cracks. J. Elasticity, 15: 193–214.

Singh, S.J., 1970. Static deformation of a multilayered half-space by internal sources. J. Geophys. Res., 75: 3257–3263.

Singh, S.J., 1986. Static deformation of a transversely isotropic multilayered half-space by surface loads. Phys. Earth Planet. Inter., 42: 263–273.

Small, J.C. and Booker, J.R., 1984. Finite layer analysis of layered elastic materials using a flexibility approach. Part 1—strip loadings. Int. J. Numer. Methods Eng., 20: 1025–1037.

Small, J.C. and Booker, J.R., 1986. Finite layer analysis of layered elastic materials using a flexibility approach. Part 2—circular and rectangular loadings. Int. J. Numer. Methods Eng., 23: 959–978.

Ulitko, A.F., 1979. Method of Special Vector Functions in Three-Dimensional Elasticity. Naukova Dumka, Kiev, 264 pp. (in Russian).

Wang, K., 1987. Analysis and calculation of stresses and displacements in layered elastic systems. Acta Mech. Sin., 3: 251–260.

Wardle, L.J., 1981. Three-dimensional solution for displacement discontinuities in cross-anisotropic elastic media. CSIRO Aust. Div. Appl. Geomech. Tech. Pap., 34: 1–32.