

Thermoelastic deformation of a transversely isotropic and layered half-space by surface loads and internal sources

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A general method is developed for the study of transient thermoelastic deformation in a transversely isotropic and layered half-space by surface loads and internal sources. A Laplace transform is first applied to the field quantities; Cartesian and cylindrical systems of vector functions are then introduced for reducing the basic equations to three sets of simultaneous linear differential equations. General solutions are obtained from these sets, and propagator matrices from the solutions by a partitioned matrix method.

Source functions for a variety of sources are derived in the Cartesian and cylindrical systems, and the Laplace transformed expressions of the field variables at the surface presented explicitly in the two systems in terms of a layer matrix. The effect of gravity is included by multiplying simply an effect matrix resulting from the modification of continuity conditions at the surface and the layer interfaces.

It should be noted that the present analytical method has great advantages over either the classical thin plate approach or the finite element method, and that the present result can be reduced directly to the solutions of the corresponding isotropic case.

1. Introduction

Quantitative study of the effect of heating on lithosphere deformation has received great attention in recent years. The classical thin plate approach is one of the frequently used models for this problem. Sleep and Snell (1976) proposed a thermo-mechanical model, to describe the observed subsidence of the Atlantic margin and mid-continent basins, which is spatially one-dimensional and includes viscoelastic flexure. More recently, Nunn and Sleep (1984) suggested a spatial two-dimensional model of finite extent to investigate the thermal contraction and flexure of the Michigan Basin. While Mareschal (1981) examined the vertical lithosphere expansion and the surface uplift of an elastic slab in response to conductive heating from below, Bills (1983) pro-

duced a systematic study of the thermoelastic deformation of the lithosphere.

Recently, the finite element method was used by McMullen and Mohraz (1987) to study the axisymmetric deformation of an elastic lithosphere subjected to thermal loading. They also discussed the advantages of this method over the classical thin plate approach. For some problems, however, analytical solutions can be obtained from the thermoelastic governing equations. For example, Lanzano (1986a) derived the thermoelastic deformation field of a spherical homogeneous Earth due to an internal heat source analytically, and evaluated the temperature profile and radial deformation in it numerically (Lanzano, 1986b). Since the thermoelastic parameters within the

Earth are generally functions of depth (Brown and Shankland, 1981; Dziewonski and Anderson, 1981), a more reasonable model would be a layered system. Rundle (1982) and Small and Booker (1986) modelled Earth structure as a thermoelastic, isotropic and horizontally layered half-space, and derived the analytical solution of deformation caused by an internal heat source. The former author employed a propagator matrix method with homogeneous and inhomogeneous solutions being separated, whereas the latter authors used a finite layer approach. It is apparent, however, that these analytical methods have great advantages over numerical methods such as the above-mentioned finite element technique, since very little computer storage and data preparation time is required.

While the solution of thermoelastic problems in an isotropic medium has been considered in great detail, comparatively little work has been done on similar problems in a transversely isotropic medium because of the greater difficulty involved. An early report on this topic was given by Sharma (1958), who developed a displacement potential method for solving the steady-state thermal-stress problem of a transversely isotropic semi-infinite elastic solid. Recently, by introducing three scalar functions, Noda et al. (1985) described a general solution method for the three-dimensional transient thermal-stress problems in transversely isotropic bodies owing to an asymmetric temperature distribution.

In two recent papers (Pan, 1989a,b), henceforth referred to as Papers I and II, respectively, the static response of a transversely isotropic and layered half-space owing to surface loads and dislocation sources was studied, by introducing two systems of vector functions and using the propagator matrix method. This solution method is extended, in the present work, to the corresponding transient thermoelastic problem. First, Laplace transform is used to suppress the time variable; the systems of vector functions are then introduced to reduce the basic equations to three sets of simultaneous linear differential equations. In order to derive the solution and propagator matrices for the present problem, we have employed a partitioned matrix method which enables the result in Paper I to be used directly. Source

functions for a variety of sources are derived in the cylindrical as well as the Cartesian system of vector functions, and solutions for the field quantities at the surface are obtained in the Laplace transformed domain in terms of the two systems. Finally, the effect of gravity and some particular results contained in the solution are discussed.

2. Governing equations and solution method

Suppose that the axis of symmetry of a transversely isotropic thermoelastic medium is along the z -axis. We can write the Duhamel-Neumann constitutive equations in Cartesian coordinates (x, y, z) as follows (Nowinski, 1978):

$$\begin{aligned}\sigma_{xx} &= A_{11}u_{x,x} + A_{12}u_{y,y} + A_{13}u_{z,z} - \beta_1\phi \\ \sigma_{yy} &= A_{12}u_{x,x} + A_{11}u_{y,y} + A_{13}u_{z,z} - \beta_1\phi \\ \sigma_{zz} &= A_{13}u_{x,x} + A_{13}u_{y,y} + A_{33}u_{z,z} - \beta_3\phi \\ \sigma_{yz} &= A_{44}(u_{y,z} + u_{z,y}) \\ \sigma_{xz} &= A_{44}(u_{x,z} + u_{z,x}) \\ \sigma_{xy} &= (A_{11} - A_{12})(u_{x,y} + u_{y,x})/2\end{aligned}\quad (1)$$

In eqn. (1), partial differentiation with respect to x , y and z is indicated with a comma followed by the variables; σ_{xx} , σ_{yy} etc. are the components of stress and (u_x, u_y, u_z) the displacement components; A_{ij} are elastic moduli; ϕ is the increment of absolute temperature over a uniform reference temperature; β_1 and β_3 are thermal moduli which are related to α_1 and α_3 by

$$\begin{aligned}\beta_1 &= (A_{11} + A_{12})\alpha_1 + A_{13}\alpha_3 \\ \beta_3 &= 2A_{13}\alpha_1 + A_{33}\alpha_3\end{aligned}$$

where α_1 and α_3 are the coefficients of linear thermal expansion in horizontal (x or y) and vertical (z) directions, respectively.

The equations of equilibrium are

$$\begin{aligned}\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + f_x &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + \sigma_{yz,z} + f_y &= 0 \\ \sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + f_z &= 0\end{aligned}\quad (2)$$

where $\mathbf{f} = (f_x, f_y, f_z)$ is the body force vector.

The third set is the heat conduction equation which governs the temperature field ϕ . If we as-

sume that the effects of thermoelastic coupling are negligible, this equation can be simplified to (Nowinski, 1978)

$$\phi_{,xx} + \phi_{,yy} + k^2\phi_{,zz} - \phi_{,t}/\kappa + \rho H/k_1 = 0 \quad (3)$$

Heat fluxes q_i are related to temperature gradients by the generalized Fourier law

$$q_x = -k_1\phi_{,x} \quad q_y = -k_1\phi_{,y} \quad q_z = -k_3\phi_{,z} \quad (4)$$

In eqns. (3) and (4),

$$k^2 = k_3/k_1 \quad \kappa = k_1/(\rho c)$$

Parameters k_1 and k_3 are the coefficients of thermal conductivity in a horizontal and vertical direction, respectively; ρ and c are the density and the specific heat of the material, respectively; κ is the diffusivity of the material, and finally H the heat produced by internal heat sources per unit time and unit mass.

The corresponding governing equations in cylindrical coordinates (r, θ, z) can also be written down easily and are similar to those presented above.

We will solve eqns. (1)–(4) under appropriate initial and boundary conditions in terms of two systems of vector functions (Paper I), i.e. the Cartesian and cylindrical systems of vector functions, and proceed in the Cartesian system only for illustration. However, one should keep in mind that the following expressions of expansion coefficients hold in the two systems.

First, since the problem is transient, we employ the Laplace transform

$$f(x, y, z; s) = \int_0^{+\infty} f(x, y, z; t) \exp(-st) dt \quad (5)$$

to suppress the variable t for functions depending upon time. It should be noted that we have used the same symbols for the functions before and after the Laplace transform, and that they are distinguished by using the Laplace variable s for the transformed one in the place of t before the transform. Further, the initial values for all field quantities are assumed to be zero.

Next, we expand the unknown displacement and 'surface' stress vectors, temperature, and heat

flux in the z -direction as follows

$$\begin{aligned} \mathbf{u}(x, y, z; s) &= \iint_{-\infty}^{+\infty} [U_L(z)\mathbf{L}(x, y) + U_M(z)\mathbf{M}(x, y) \\ &\quad + U_N(z)\mathbf{N}(x, y)] d\alpha d\beta \end{aligned} \quad (6)$$

$$\begin{aligned} \mathbf{T}(x, y, z; s) &\equiv \sigma_{xz}\mathbf{i}_x + \sigma_{yz}\mathbf{i}_y + \sigma_{zz}\mathbf{i}_z \\ &= \iint_{-\infty}^{+\infty} [T_L(z)\mathbf{L}(x, y) + T_M(z)\mathbf{M}(x, y) \\ &\quad + T_N(z)\mathbf{N}(x, y)] d\alpha d\beta \end{aligned} \quad (7)$$

$$\phi(x, y, z; s) = \iint_{-\infty}^{+\infty} \Phi(z)S(x, y) d\alpha d\beta \quad (8)$$

$$\begin{aligned} q_z(x, y, z; s) &\equiv -k_3\phi_{,z} \\ &= \iint_{-\infty}^{+\infty} Q(z)S(x, y) d\alpha d\beta \end{aligned} \quad (9)$$

In eqns. (6)–(9), scalar function S and vector functions \mathbf{L} , \mathbf{M} and \mathbf{N} were defined in Paper I; the dependence of expansion coefficients U_L , U_M , U_N , T_L , T_M , T_N , Φ and Q on the variable s and on the parameters α and β has been dropped for brevity.

Finally, by assuming that the thermoelastic system is free of body force and heat source, and proceeding as in Paper I, we find that the above expansion coefficients satisfy the following three sets of linear differential equations

$$\left. \begin{aligned} U_{L,z} &= \lambda^2 U_M A_{13}/A_{33} + T_L/A_{33} + \beta_3 \Phi/A_{33} \\ U_{M,z} &= -U_L + T_M/A_{44} \\ T_{L,z} &= \lambda^2 T_M \\ T_{M,z} &= \lambda^2 U_M (A_{11}A_{33} - A_{13}^2)/A_{33} - A_{13}T_L \\ &\quad /A_{33} + (\beta_1 - A_{13}\beta_3/A_{33})\Phi \end{aligned} \right\} \quad (10)$$

$$\left. \begin{aligned} \Phi_{,z} &= -Q/k_3 \\ Q_{,z} &= -k_1(\lambda^2 + s/\kappa)\Phi \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned} U_{N,z} &= T_N/A_{44} \\ T_{N,z} &= \lambda^2 (A_{11} - A_{12})U_N/2 \end{aligned} \right\} \quad (12)$$

In eqns. (10)–(12)

$$\lambda^2 = \alpha^2 + \beta^2$$

In obtaining eqns. (10)–(12), we have assumed that the thermoelastic parameters involved are in-

dependent of the horizontal variables x and y , but they may be any functions of the vertical variable z . The three sets of equations can therefore be used to analyse the transient thermoelastic problem in a vertically inhomogeneous half-space. A numerical propagator matrix method was developed by Pan et al. (1986) for solving this type of equation. In the following, however, we assume that the medium is vertically piecewise homogeneous, which is in accordance with the Earth's structure.

It should be noted that the deformation of type II (corresponding to the vector \mathbf{N}) is free of the thermal effect, and thus is exactly the same as the purely elastic case. Its solution to surface loads and dislocation sources was given in Papers I and II, respectively.

3. General solutions and layer matrices

If a medium is homogeneous, we find, with some algebraic manipulation, that the general solutions of eqns. (10) and (11) can be cast into

$$[\mathbf{E}(z)] = [\mathbf{F}(z)][\mathbf{K}] \quad (13)$$

where

$$[\mathbf{E}(z)] = [U_L(z), \lambda U_M(z), T_L(z)/\lambda, T_M(z), \Phi(z), Q(z)]^T$$

$$[\mathbf{K}] = [c_1, c_2, c_3, c_4, c_5, c_6]^T$$

superscript T indicates the transpose of a matrix and c_i are constants to be determined; $[\mathbf{F}(z)]$ is the general solution matrix and can be expressed as a partitioned form

$$[\mathbf{F}(z)] = \begin{bmatrix} \mathbf{Z}(z) & \mathbf{X}(z) \\ \mathbf{0} & \mathbf{Y}(z) \end{bmatrix} \quad (14)$$

In eqn. (14), $[\mathbf{0}]_{(2 \times 4)}$ is a zero submatrix, and $[\mathbf{Z}(z)]_{(4 \times 4)}$ the general solution submatrix for the purely elastic body with elements being given in Paper I; $[\mathbf{Y}(z)]_{(2 \times 2)}$ is the one for the pure heat conduction in a thermally transversely isotropic material, and its elements are found to be

$$\begin{aligned} Y_{11} &= \exp(x_3 z) & Y_{12} &= \exp(-x_3 z) \\ Y_{21} &= -k_3 x_3 \exp(x_3 z) & Y_{22} &= k_3 x_3 \exp(-x_3 z) \end{aligned} \quad (15)$$

where

$$x_3 = (\lambda^2 + s/\kappa)^{1/2}/k$$

The submatrix $[\mathbf{X}(z)]_{(4 \times 2)}$ in eqn. (14) represents the effect of a temperature field on displacements and stresses, and has the following non-zero elements

$$\begin{aligned} X_{11} &= \beta_4 \exp(x_3 z) & X_{12} &= \beta_4 \exp(-x_3 z) \\ X_{41} &= \beta_5 \exp(x_3 z) & X_{42} &= \beta_5 \exp(-x_3 z) \end{aligned} \quad (16)$$

where

$$\beta_4 = \beta_3/A_{33}$$

$$\beta_5 = \beta_1 - \beta_4 A_{13}$$

The inverse of $[\mathbf{F}(z)]$ is derived by a partitioned matrix method from Horn and Johnson (1985). Thus a propagator matrix and a propagating relation which relates the expansion coefficients at different depths of an homogeneous layer, can be obtained

$$[\mathbf{E}(z_1)] = [\mathbf{P}(h)][\mathbf{E}(z_2)] \quad (17)$$

where

$$[\mathbf{P}] = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{0} & \mathbf{c} \end{bmatrix} \quad (18)$$

is the propagator matrix, and $h = z_2 - z_1$ the depth difference. In eqn. (18), $[\mathbf{a}]_{(4 \times 4)}$ is the propagator submatrix for the purely elastic medium with elements being given in Paper I; $[\mathbf{c}]_{(2 \times 2)}$ is the one for the pure heat conduction and its elements are found as

$$\begin{aligned} c_{11} &= \cosh(x_3 h) & c_{12} &= \sinh(x_3 h)/(k_3 x_3) \\ c_{21} &= k_3 x_3 \sinh(x_3 h) & c_{22} &= \cosh(x_3 h) \end{aligned} \quad (19)$$

The submatrix $[\mathbf{b}]_{(4 \times 2)}$ in eqn. (18) can be expressed as

$$\begin{aligned} [\mathbf{b}] &= -[\mathbf{Z}(z_1)][\mathbf{Z}(z_2)]^{-1}[\mathbf{X}(z_2)][\mathbf{Y}(z_2)]^{-1} \\ &\quad + [\mathbf{X}(z_1)][\mathbf{Y}(z_2)]^{-1} \end{aligned} \quad (20)$$

where superscript -1 denotes the inverse of a matrix. After some algebra we find, in the case of

$x_1 \neq x_2$ the following non-zero elements of [b]

$$\left. \begin{aligned} b_{11} &= f(x_1)c(x_1) \cosh(\lambda x_1 h) + f(x_2)c(x_2) \\ &\quad \times \cosh(\lambda x_2 h) + \beta_4 \cosh(x_3 h) \\ b_{21} &= -f(x_1)d(x_1) \sinh(\lambda x_1 h) \\ &\quad - f(x_2)d(x_2) \sinh(\lambda x_2 h) \\ b_{31} &= -f(x_1) \sinh(\lambda x_1 h)/x_1 \\ &\quad - f(x_2) \sinh(\lambda x_2 h)/x_2 \\ b_{41} &= f(x_1) \cosh(\lambda x_1 h) + f(x_2) \cosh(\lambda x_2 h) \\ &\quad + \beta_5 \cosh(x_3 h) \\ b_{12} &= \beta_4 \sinh(x_3 h)/(k_3 x_3) \\ b_{42} &= \beta_5 \sinh(x_3 h)/(k_3 x_3) \end{aligned} \right\} \quad (21)$$

where

$$f(x) = g(x)[\beta_5 d(x) - \beta_4/x]$$

$c(x)$, $d(x)$ and $g(x)$ are the functions defined in Paper I, and x_1^2 and x_2^2 are two distinct roots of the characteristic equation

$$(A_{44}x^2 - A_{11})(A_{33}x^2 - A_{44}) + (A_{13} + A_{44})^2 x^2 = 0$$

4. Deformation of a layered thermoelastic system

If the thermoelastic parameters of a medium are vertically piecewise homogeneous, we can model it with $p-1$ parallel and homogeneous layers lying over an homogeneous half-space. The layers are numbered serially, the layer at the top being layer 1 and the half-space, layer p . We place the origins of Cartesian and cylindrical coordinates at the surface, and the z -axis is drawn into the medium. The k th layer is of thickness h_k and is bounded by the interfaces $z = z_{k-1}, z_k$. Obviously, $z_0 = 0$ and $z_{p-1} = HP$, where HP is the depth of the last interface. We further assume that suitable boundary conditions which make a problem definite are applied to the surface $z = 0$, and that a point source is situated on the z -axis at a depth, d , below the surface. Let the source layer be designated as layer s with boundaries $z = z_{s-1}, z_s$. We divide the source layer into two sub-layers, $s1$ and $s2$, of identical properties. The first

sub-layer is bounded by the planes $z = z_{s-1}$ and d and the second by $z = d$, z_s . Displacement and 'surface' stress vectors, temperature, and the heat flux in the z -direction are assumed to be continuous across any interfaces of layers except at $z = d$, which is to be discussed in the following.

4.1. Source representation

4.1.1. Point force source

The point force source in Cartesian coordinates is expressed as

$$f_i(x, y, z; t) = p(t)n_i\delta(x)\delta(y)\delta(z-d) \quad (22)$$

$$i = x, y, z$$

where (n_x, n_y, n_z) are the direction cosines of the point force vector in Cartesian coordinates, and $p(t)$ is the time-dependent factor of the source. The Laplace transform of this equation is

$$f_i(x, y, z; s) = p(s)n_i\delta(x)\delta(y)\delta(z-d) \quad (23)$$

$$i = x, y, z$$

Expanding it in the Cartesian system of vector functions, we obtain

$$\mathbf{f}(x, y, z; s) = \iint_{-\infty}^{+\infty} [F_L(z)\mathbf{L}(x, y) + F_M(z)\mathbf{M}(x, y) + F_N(z)\mathbf{N}(x, y)] d\alpha d\beta \quad (24)$$

where the expansion coefficients are

$$F_L(z) = p(s)\delta(z-d)n_z/(2\pi)$$

$$F_M(z) = ip(s)\delta(z-d)(n_x\alpha + n_y\beta)/(2\pi\lambda^2)$$

$$F_N(z) = ip(s)\delta(z-d)(n_x\beta - n_y\alpha)/(2\pi\lambda^2) \quad (25)$$

Therefore the discontinuities of $[\mathbf{E}(z)]$ and $[\mathbf{E}^L(z)] (= [\mathbf{U}_N(z), \mathbf{T}_N(z)/\lambda]^T)$ caused by this source are

$$\Delta T_L/\lambda = -p(s)n_z/(2\pi\lambda)$$

$$\Delta T_M = -ip(s)(n_x\alpha + n_y\beta)/(2\pi\lambda^2) \quad (26)$$

$$\Delta T_N/\lambda = -ip(s)(n_x\beta - n_y\alpha)/(2\pi\lambda^3)$$

A similar result in the cylindrical system of vector functions was found as (Pan, 1989c)

$$\left. \begin{aligned} \Delta T_L/\lambda &= -p(s)n_z/[\lambda(2\pi)^{1/2}] \\ m &= 0 \\ \Delta T_M &= p(s)(\pm n_x + in_y)/[2\lambda(2\pi)^{1/2}] \\ m &= \pm 1 \\ \Delta T_N/\lambda &= p(s)(in_x \pm n_y)/[2\lambda^2(2\pi)^{1/2}] \\ m &= \pm 1 \end{aligned} \right\} \quad (27)$$

Other quantities in the column matrices $[\mathbf{E}(z)]$ and $[\mathbf{E}^L(z)]$ are continuous across $z = d$.

4.1.2. Point dislocation source

A full discussion of this problem was given in Paper II, and the discontinuities caused by this type of source were obtained in the Cartesian and cylindrical systems of vector functions (Paper II). If the source depends upon time t , a factor $R(s)$, which is the Laplace transform of the time-dependent function of the dislocation source, should multiply every term of the discontinuities.

4.1.3. Point heat source

When a heat source is applied to a thermoelastic system, eqn. (11) becomes inhomogeneous

$$\begin{aligned} \Phi_{,z} &= -Q/k_3 \\ Q_{,z} &= -k_1(\lambda^2 + s/\kappa)\Phi + W \end{aligned} \quad (28)$$

where W is the expansion coefficient of body heat, i.e.

$$\rho H = \iint_{-\infty}^{+\infty} W(z)S(x, y) \, d\alpha \, d\beta \quad (29)$$

If we assume that the body heat is of a point source

$$\rho H = D(s)\delta(x)\delta(y)\delta(z-d) \quad (30)$$

with $D(s)$ being the Laplace transform of the time-dependent factor of the heat source, we then find that

$$W(z) = D(s)\delta(z-d)/(2\pi) \quad (31)$$

This source causes a discontinuity of Q with magnitude

$$\Delta Q = D(s)/(2\pi) \quad (32)$$

A similar result in the cylindrical system of vector functions is found to be

$$\Delta Q = D(s)/(2\pi)^{1/2} \quad m = 0 \quad (33)$$

4.2. Layered half-space

In order to solve problems in this system, we need to know the surface boundary condition at $z = 0$, the continuity condition at the layer interfaces, and the discontinuity caused by sources at $z = d$. First, we assume, as an example, that the 'surface' stress vector \mathbf{T} and the temperature ϕ at $z = 0$ are known, i.e. in expansion forms, we have $\mathbf{T}(x, y, 0; s)$

$$\begin{aligned} &= \iint_{-\infty}^{+\infty} [T_L(0)\mathbf{L}(x, y) + T_M(0)\mathbf{M}(x, y) \\ &\quad + T_N(0)\mathbf{N}(x, y)] \, d\alpha \, d\beta \end{aligned} \quad (34)$$

$$\phi(x, y, 0; s) = \iint_{-\infty}^{+\infty} \Phi(0)S(x, y) \, d\alpha \, d\beta \quad (35)$$

Next, it is easy to show that the continuities of displacement and 'surface' stress vectors, temperature, and heat flux in the z -direction at the layer interfaces are equivalent to those of the column matrices $[\mathbf{E}(z)]$ and $[\mathbf{E}^L(z)]$; finally, the discontinuities of $[\mathbf{E}(z)]$ and $[\mathbf{E}^L(z)]$ caused by point force, dislocation and heat sources have just been derived, and we will use

$$[\Delta \mathbf{E}] = [\Delta U_L, \lambda \Delta U_M, \Delta T_L/\lambda, \Delta T_M, \Delta \Phi, \Delta Q]^T$$

and

$$[\Delta \mathbf{E}^L] = [\Delta U_N, \Delta T_N/\lambda]^T$$

to indicate them.

Extending Singh's technique (Singh, 1970) to the present case, one may find the expressions of the field quantities at any point of the medium. In particular, we can obtain the expansion coefficients of the surface displacements and heat flux

$$Q(0) = (G_{66}/G_{56})(B_5 + \Phi(0)) - B_6 \quad (36)$$

$$\begin{aligned} U_L(0) &= \left\{ [(B_3 + T_L(0)/\lambda) \right. \\ &\quad \left. - (G_{36}/G_{56})(B_5 + \Phi(0))] G|_{24}^{14} \right. \\ &\quad \left. + [(B_4 + T_M(0)) - (G_{46}/G_{56}) \right. \\ &\quad \left. \times (B_5 + \Phi(0))] G|_{42}^{13} \right\} / G|_{24}^{34} \\ &\quad + (G_{16}/G_{56})(B_5 + \Phi(0)) - B_1 \end{aligned} \quad (37)$$

$$\begin{aligned} \lambda U_M(0) = & \left\{ [(B_3 + T_L(0)/\lambda) \right. \\ & - (G_{36}/G_{56})(B_5 + \Phi(0))] G_{24}^{24} \\ & + [(B_4 + T_M(0)) - (G_{46}/G_{56}) \\ & \times (B_5 + \Phi(0))] G_{42}^{23} / G_{24}^{34} \\ & \left. + (G_{26}/G_{56})(B_5 + \Phi(0)) - B_2 \right\} \quad (38) \end{aligned}$$

$$U_N(0) = G_{12}^L (B_2^L + T_N(0)/\lambda) / G_{22}^L - B_1^L \quad (39)$$

In eqns. (36)–(39)

$$[\mathbf{G}] = [\mathbf{P}_1][\mathbf{P}_2] - - [\mathbf{P}_{p-1}][\mathbf{F}_p(HP)] \quad (40a)$$

$$[\mathbf{G}^L] = [\mathbf{a}_1^L][\mathbf{a}_2^L] - - [\mathbf{a}_{p-1}^L][\mathbf{Z}_p^L(HP)] \quad (40b)$$

$$[\mathbf{B}] = [\mathbf{P}_1][\mathbf{P}_2] - - [\mathbf{P}_{s-1}][\mathbf{P}_{s1}][\Delta \mathbf{E}] \quad (41a)$$

$$[\mathbf{B}^L] = [\mathbf{a}_1^L][\mathbf{a}_2^L] - - [\mathbf{a}_{s-1}^L][\mathbf{a}_{s1}^L][\Delta \mathbf{E}^L] \quad (41b)$$

$$G_{kl}^{ij} = G_{ik}G_{jl} - G_{il}G_{jk}$$

where the solution matrix $[\mathbf{Z}_p^L(HP)]$ and the propagator matrix $[\mathbf{a}_p^L]$ were given in Paper I, and subscript p is attached to indicate that the quantity belongs to the p th layer.

It should be noted that the solution for the pure surface load and internal source problems can be obtained, respectively, by setting $[\mathbf{B}] = [\mathbf{B}^L] = [\mathbf{0}]$ and $T_L(0) = T_M(0) = T_N(0) = \Phi(0) = 0$ in eqns. (36)–(39).

Formulation for obtaining other field variables is simple, and similar to that given in Paper I. However, the quantity $-\beta_1\phi$ should be added to the expressions for the normal stress components σ_{xx} , σ_{yy} or σ_{rr} , $\sigma_{\theta\theta}$, as one could observe from eqn. (1). Additionally, the heat flux components in horizontal directions are given by

$$q_x = -k_1 \iint_{-\infty}^{+\infty} \Phi S_{,x} \, d\alpha \, d\beta \quad (42a)$$

$$q_y = -k_1 \iint_{-\infty}^{+\infty} \Phi S_{,y} \, d\alpha \, d\beta \quad (42b)$$

in the Cartesian system, and

$$q_r = -k_1 \sum_m \int_0^{+\infty} \Phi S_{,r} \, dr \, d\theta \quad (43a)$$

$$q_\theta = -k_1 \sum_m \int_0^{+\infty} \Phi S_{,\theta} \, dr \, d\theta \quad (43b)$$

in the cylindrical system.

4.3. Layered plate

By removing the homogeneous half-space, we thus get a layered plate consisting of $p - 1$ homogeneous layers. In this case, suitable boundary conditions should be applied to both surfaces $z = 0$ and $z = HP$. As an illustration we assume that at $z = 0$, we know \mathbf{T} and ϕ , as given by eqns. (34) and (35), and that at the bottom surface $z = HP$, we have

$$\begin{aligned} \mathbf{u}(x, y, HP; s) \\ = \iint_{-\infty}^{+\infty} [U_L(HP)\mathbf{L}(x, y) \\ + U_M(HP)\mathbf{M}(x, y) \\ + U_N(HP)\mathbf{N}(x, y)] \, d\alpha \, d\beta \quad (44) \end{aligned}$$

$$q_z(x, y, HP; s) = \iint_{-\infty}^{+\infty} Q(HP)S(x, y) \, d\alpha \, d\beta \quad (45)$$

Following the same procedure as above, we obtain the unknown quantities at both surfaces:

$$\left. \begin{aligned} \Phi(HP) &= (B_5 + \Phi(0) - G_{56}Q(HP))/G_{55} \\ T_L(HP)/\lambda &= \{G_{44}[W_1] - G_{34}[W_2]\}/G_{34}^{34} \\ T_M(HP) &= \{G_{33}[W_2] - G_{43}[W_1]\}/G_{34}^{34} \\ T_N(HP)/\lambda &= [B_2^L + T_N(0)/\lambda \\ &\quad - G_{21}^L U_N(HP)]/G_{22}^L \end{aligned} \right\} \quad (46)$$

$$\left. \begin{aligned} Q(0) &= G_{65}(B_5 + \Phi(0) - G_{56}Q(HP))/G_{55} \\ &\quad + G_{66}Q(HP) - B_6 \\ U_L(0) &= \{[W_1]G_{34}^{14} + [W_2]G_{43}^{13}\}/G_{34}^{34} \\ &\quad + G_{11}U_L(HP) + G_{12}\lambda U_M(HP) \\ &\quad + G_{15}\Phi(HP) + G_{16}Q(HP) - B_1 \\ \lambda U_M(0) &= \{[W_1]G_{34}^{24} + [W_2]G_{43}^{23}\}/G_{34}^{34} \\ &\quad + G_{21}U_L(HP) + G_{22}\lambda U_M(HP) \\ &\quad + G_{25}\Phi(HP) + G_{26}Q(HP) - B_2 \\ U_N(0) &= G_{11}^L U_N(HP) + G_{12}^L T_N(HP)/\lambda - B_1^L \end{aligned} \right\} \quad (47)$$

where

$$\begin{aligned} [W_1] &= B_3 + T_L(0)/\lambda - G_{35}\Phi(HP) - G_{36}Q(HP) \\ &\quad - G_{31}U_L(HP) - G_{32}\lambda U_M(HP) \\ [W_2] &= B_4 + T_M(0) - G_{45}\Phi(HP) - G_{46}Q(HP) \\ &\quad - G_{41}U_L(HP) - G_{42}\lambda U_M(HP) \end{aligned}$$

While the column matrices $[\mathbf{B}]$ and $[\mathbf{B}^L]$ are given by eqns. (41a,b), matrices $[\mathbf{G}]$ and $[\mathbf{G}^L]$ are obtained from eqns. (40a,b) by removing the solution matrices $[\mathbf{F}]$ and $[\mathbf{Z}^L]$, respectively, from its expression. It is noted again that the solution for the pure surface load and internal source problems can be obtained, respectively, by setting $[\mathbf{B}] = [\mathbf{B}^L] = [\mathbf{0}]$ and $T_L(0) = T_M(0) = T_N(0) = \Phi(0) = 0$ in eqns. (46) and (47).

5. Isostatic response

We have just solved the transient thermoelastic deformation problem of a transversely isotropic and layered half-space by surface loads and internal sources. When dealing with some deformation problems of the Earth medium, however, the effect of gravity must be included (Cathles, 1975; Turcotte and Schubert, 1982). The direct way, of course, is to solve the original elastic-gravitational equations, as presented by Rundle (1980, 1981). But for a transversely isotropic medium, the advantages of using this method are not as apparent as for the isotropic case, since an exact solution to the former is quite difficult. Another approach we, thus, adopt here is to solve the non-gravitating equations, which we have just done, and to modify the continuity conditions at the layer interfaces as well as at the surface for the inclusion of the gravity effect (McConnell, 1965; Iwasaki and Matsu'ura, 1982). Following Iwasaki and Matsu'ura (1982), modification is required for the normal stress component σ_{zz} only, which now becomes

$$\sigma_{zz}^k - \rho_k g_k u_z^k = \sigma_{zz}^{k+1} - \rho_{k+1} g_{k+1} u_z^k \quad (48)$$

at the k th interface, and

$$\sigma_{zz} - \rho_1 g_0 u_z = p_{zz} - \rho_0 g_0 u_z \quad (49)$$

at the surface $z=0$. In eqns. (48) and (49), ρ_k , g_k , u_z^k and σ_{zz}^k are the density, gravity acceleration, displacement and stress component, in the k th layer, respectively; p_{zz} is the surface load in vertical direction; g_0 is the surface acceleration of gravity and ρ_0 the density of the material filling the vertical lithosphere deflection (McMullen and Mohraz, 1987). It is obvious that gravity has no effect on the deformation of type II related to the vector \mathbf{N} .

Using the expansion coefficients, eqns. (48) and (49) can be equivalently expressed as

$$T_L^k - \rho_k g_k U_L^k = T_L^{k+1} - \rho_{k+1} g_{k+1} U_L^k \quad (50)$$

and

$$T_L - \rho_1 g_0 U_L = P_L - \rho_0 g_0 U_L \quad (51)$$

where P_L is the expansion coefficient of the vertical surface load p_{zz} in the Cartesian or cylindrical system of vector functions.

If we introduce a new column matrix

$$\begin{aligned} [\mathbf{A}(z)] &= [U_L(z), \lambda U_M(z), T_L(z) \\ &\quad - \rho g U_L(z), T_M(z), \Phi(z), Q(z)]^T \end{aligned} \quad (52)$$

the general solution (13) is then modified into

$$[\mathbf{A}(z)] = [\mathbf{M}][\mathbf{F}(z)][\mathbf{K}] \quad (53)$$

where $[\mathbf{M}]$ is an effect matrix with non-zero elements as below

$$\begin{aligned} (1, 1) &= (2, 2) = (4, 4) = (5, 5) = (6, 6) = 1 \\ (3, 3) &= \lambda \\ (3, 1) &= -\rho g \end{aligned} \quad (54)$$

The propagating relation for the new column matrix becomes

$$[\mathbf{A}(z_{k-1})] = [\mathbf{M}_{k-1}][\mathbf{P}(h)][\mathbf{M}_k]^{-1}[\mathbf{A}(z_k)] \quad (55)$$

where $h = z_k - z_{k-1}$. Finally, the discontinuity $[\Delta \mathbf{E}]$ caused by internal sources is also modified into

$$[\Delta \mathbf{A}] = [\mathbf{M}][\Delta \mathbf{E}] \quad (56)$$

Using these new relations, we can proceed in the same way as in the above section to study the

isostatic response. It should be noted, however, that the actual effect of gravity results in the modification of the surface boundary condition only, as one could easily observe from the propagating relation (55). Therefore, the solution to the gravitating case is nearly as simple as that to the non-gravitating case. This conclusion is coincident with McConnell (1965), and is obtained without additional restrictions on the properties of the layered structure (Iwasaki and Matsu'ura, 1982).

6. Particular cases

We first point out that since the formulation given above holds in the cylindrical system as well as in the Cartesian system of vector functions, two types of particular solution—two-dimensional and axially symmetric deformations—can be obtained directly, as we have shown in Papers I and II for the purely elastic case.

Solution for the transient, thermoelastic problem in an isotropic and layered medium can also be derived easily. In the case of a thermoelastic isotropic material, we have $k_1 = k_3$, $\alpha_1 = \alpha_3$ and the eqn. (3) in Paper I, so that

$$\begin{aligned} k &= 1 \\ x_3 &= (\lambda^2 + s/\kappa)^{1/2} \\ \beta_1 &= \beta_3 = E\alpha_1/(1 - 2\nu) \\ \beta_4 &= \alpha_1(1 + \nu)/(1 - \nu) \\ \beta_5 &= E\alpha_1/(1 - \nu) \end{aligned} \quad (57)$$

where E and ν are the Young's modulus and Poisson ratio, respectively. While the elements of the submatrices $[\mathbf{Z}]$ in eqn. (14) and $[\mathbf{a}]$ in eqn. (18) for the isotropic case have been given in Paper I, those of the submatrices $[\mathbf{Y}]$ and $[\mathbf{X}]$ in eqn. (14) and $[\mathbf{c}]$ in eqn. (18) are obtained from eqns. (15), (16) and (19), respectively, by substituting eqn. (57) into them. The elements of the submatrix $[\mathbf{b}]$ in eqn. (18) for the isotropic case are given by the following equation with $x_1 = 1$.

$$\left. \begin{aligned} b_{11} &= x_1 \left\{ -\beta_4 [c'(x_1)x_1^{-1} \cosh y_1 \right. \\ &\quad \left. + \lambda hc(x_1)x_1^{-1} \sinh y_1] \right. \\ &\quad \left. + \beta_4 c'(x_1)x_1^{-1} \cosh(x_3 h) \right. \\ &\quad \left. + \beta_5 \lambda hc(x_1)d(x_1) \sinh y_1 \right\} / c'(x_1) \\ b_{21} &= x_1 \left\{ \beta_4 [d'(x_1)x_1^{-1} \sinh y_1 \right. \\ &\quad \left. + \lambda hd(x_1)x_1^{-1} \cosh y_1] \right. \\ &\quad \left. - \beta_5 [d(x_1)(d(x_1)x_1^{-1} + 2d'(x_1)) \right. \\ &\quad \left. \times \sinh y_1 + \lambda hd^2(x_1) \cosh y_1] \right\} \\ &\quad / c'(x_1) \\ b_{31} &= x_1 \left\{ \beta_4 [-x_1^{-3} \sinh y_1 + \lambda hx_1^{-2} \cosh y_1] \right. \\ &\quad \left. - \beta_5 [d'(x_1)x_1^{-1} \sinh y_1 \right. \\ &\quad \left. + \lambda hd(x_1)x_1^{-1} \cosh y_1] \right\} / c'(x_1) \\ b_{41} &= x_1 \left\{ -\beta_4 \lambda hx_1^{-1} \sinh y_1 \right. \\ &\quad \left. + \beta_5 c'(x_1)x_1^{-1} \cosh(x_3 h) \right. \\ &\quad \left. + \beta_5 [(d(x_1)x_1^{-1} + d'(x_1)) \cosh y_1 \right. \\ &\quad \left. + \lambda hd(x_1) \sinh y_1] \right\} / c'(x_1) \\ b_{12} &= \beta_4 \sinh(x_3 h) / (k_1 x_3) \\ b_{22} &= b_{32} = 0 \\ b_{42} &= \beta_5 \sinh(x_3 h) / (k_1 x_3) \end{aligned} \right\} \quad (58)$$

in which $y_1 = \lambda x_1 h$; β_4 , β_5 and x_3 are given in eqn. (57); the definitions of functions $c(x)$, $d(x)$ and their derivatives were given in Paper I.

Finally, a solution for the steady-state thermoelastic problem in a transversely isotropic and layered half-space can also be obtained directly from the formulation above. Since in this case all quantities are time-independent, time factors in the expressions of surface boundary conditions and of internal sources should be removed. In addition, the expression of the third root x_3 should be replaced by $x_3 = \lambda/k$ for a similar reason. The solution is therefore reduced to a simple form which contains no Laplace transform.

7. Conclusion and discussion

A general solution method is developed for the study of transient thermoelastic deformation by

surface loads and internal sources in a transversely isotropic and layered half-space. The approaches proposed in Papers I and II are employed, in conjunction with a partitioned matrix method, to obtain the solution analytically. Source functions for a variety of sources are derived in the Cartesian and cylindrical systems of vector functions. The effect of gravity is included by multiplying simply an effect matrix resulting from the modification of continuity conditions at the surface and the layer interfaces. It is noted that the present solution can be reduced directly to that of the corresponding two-dimensional and axially symmetric deformations and of the corresponding isotropic case.

Since the present solution is obtained directly from the three-dimensional transient thermoelastic equations and expressed in the Laplace transformed domain in terms of two systems of vector functions, its advantages over the classical thin plate approach and the finite element method are obvious. While multiplication of the partitioned propagator matrices is required for three submatrices only since all the elements of one submatrix are always zero, we eventually must resort to some quadrature methods to evaluate the inversion of the Laplace transforms and the infinite integrals involved. Fortunately, much work on this topic has appeared in the literature. We can thus employ suitable quadrature methods to investigate quantitatively various problems of the Earth's thermoelastic deformation related to a transversely isotropic and layered model.

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