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An exact solution for transversely isotropic, simply supported and layered rectangular plates

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Abstract. Levinson's solution for the problem of a simply supported rectangular plate of arbitrary thickness by normal surface loads is extended to the transversely isotropic and layered case. The exact closed form solution is obtained by using the propagator matrix method in a system of vector functions. As a special case of the layered medium, the normal displacement or deflection of a homogeneous plate of arbitrary thickness by normal surface loads is also given. It is shown that it approaches the classical solution for the transversely isotropic thin plate as the thickness approaches zero on the one hand, and on the other hand reduces to the thick plate expression as given by Levinson when the medium is isotropic.

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List of symbols

$x = x_1, y = x_2, z = x_3$ = Cartesian coordinates; In Section 5 and Appendices A and C, x_1 and x_2 are the characteristic roots.
σ_{ij}, e_{ij} = stress and strain components respectively.
$A_{11}, A_{12}, A_{13}, A_{33}, A_{44}$ = elastic constants.
$A_{66} = (A_{11} - A_{12})/2$
E = Young's modulus.
ν = Poisson's ratio.
$(\ )_{,i} = \partial(\ )/\partial x_i$
$u_i; u_x, u_y, u_z$ = displacements.
$S(x, y; m, n)$ = scalar function.
m, n = integers.
a, b, H = length, width and thickness of the layered rectangular plate respectively.
$\nabla^2(\ ) = \partial^2(\ )/\partial x^2 + \partial^2(\ )/\partial y^2$ = Laplacian operator.
$\lambda^2 = \alpha^2 + \beta^2$
$\alpha^2 = (m\pi)^2/a^2$
$\beta^2 = (n\pi)^2/b^2$
i_x, i_y, i_z = unit vectors.
$\mathbf{L}, \mathbf{M}, \mathbf{N}$ = vector functions.
δ_{ij} = Kronecker delta.
$h_k = z_k - z_{k-1}$ = thickness of layer k.
z_{k-1}, z_k = coordinates of upper and lower interfaces of layer k.
$P_{zz} = P_{mn}S(x, y; m, n)$ = normal loading applied to the uppermost surface.

$u_i^{(k)}, \sigma_{ij}^{(k)}$ = displacements and stresses respectively in layer k.
$U_L(z), U_M(z), U_N(z)$ = expansion coefficients of displacement vector u.
$T_L(z), T_M(z), T_N(z)$ = expansion coefficients of "surface" stress vector T.
$f(z), g(z)$ = shape functions used by Levinson.
W_{mn} = deflection coefficients used by Levinson.
$[A(z)], [A^L(z)], [Z(z)], [Z^L(z)]$ = matrix functions.
$[C] = [A, B, C, D]^T, [C^L] = [A^L, B^L]^T$ = column matrices.
$[--]^T$ = transpose of matrix $[--]$.
$[--]^{-1}$ = inverse of matrix $[--]$.
$[a_k], [a_k^L]$ = layer matrices.
$[A_k(z)]$ = column matrix function of layer k.
$\Delta = F_{31}F_{42} - F_{41}F_{32}$
$[F] = [a_1][a_2]--[a_p]$
F_{ij} = elements of matrix $[F]$.
$y_1 = x_1\lambda H$
$y_2 = x_2\lambda H$
ch, sh = hyperbolic cosine and sine respectively.
$c(x), d(x)$ = functions defined in Appendix A.
$c'(x) = \mathrm{d}c(x)/\mathrm{d}x$
$d'(x) = \mathrm{d}d(x)/\mathrm{d}x$
D = bending stiffness
$s = (A_{66}/A_{44})^{1/2}$
$\bar{s} = sA_{44}$
$g(x)$ = function defined and used in Appendix C.
$[b]$ = matrix used in Appendix C.

1. Introduction

It is well-known that the classical thin plate theory places undue restrictions on certain types of plate problems and thus cannot provide satisfactory results [1]. Therefore, various "thick plate theories" based on more appropriate and rational assumptions have been developed [2]. Recently, based on his kinematic assumptions, Levinson proposed a novel thick plate theory [3] in which normal stress and strain effects as well as shear effects are incorporated. Furthermore, by making use of these assumptions, he solved the Navier equations of elasticity and obtained an exact three dimensional solution for a simply supported rectangular plate of arbitrary thickness by normal surface loads [4].

The solution of a transversely isotropic and layered elastic medium is of great interest in the fields of soil mechanics [5, 6] and of composite materials [7]. Tsai [8] solved the symmetrical problem of the contact between a spherical indenter and a thick transversely isotropic plate by using the technique of the Hankel transform; Small and Booker [6] solved the three-dimensional problem of a transversely isotropic and layered elastic medium under surface loadings using double Fourier transformations and the so-called finite layer approach. But that method requires one to solve a system of simultaneous linear equations with order proportional to the number of layers and to introduce auxiliary variables and coordinate transformations. Recently, a considerably simpler and uniform approach has been proposed by the present author to solve this problem [9]. As the method used in [9] is based on two systems of vector functions in association with the propagator matrix method [10], the formulation has avoided the complicated nature of the problem.

Based on our previous work [9], in this paper we have extended Levinson's solution [4] for the problem of a simply supported rectangular plate of arbitrary thickness by normal surface loads to the corresponding transversely isotropic and layered case. The exact closed form solution is obtained using the propagator matrix method in a system of vector functions constructed from a double Fourier sine series. For a homogeneous thick plate, the solution can be expressed in a very simple form, which is useful for investigating the effects of the thickness, elastic constants and characteristic roots on the deflection. It is found that the classical solution of a transversely isotropic and simply supported rectangular plate and the solution of an isotropic thick plate given in [4] are all the particular cases of our results.

2. Basic equations and a system of vector functions

We take the axis of symmetry of a homogeneous and transversely isotropic elastic medium as the z-axis. The generalized Hooke's law in Cartesian coordinates ($x = x_1, y = x_2, z = x_3$) can then be written as [11]

$$\begin{aligned} \sigma_{xx} &= A_{11}e_{xx} + A_{12}e_{yy} + A_{13}e_{zz} \qquad & \sigma_{yz} &= 2A_{44}e_{yz}, \\ \sigma_{yy} &= A_{12}e_{xx} + A_{11}e_{yy} + A_{13}e_{zz} \qquad & \sigma_{xz} &= 2A_{44}e_{xz}, \\ \sigma_{zz} &= A_{13}e_{xx} + A_{13}e_{yy} + A_{33}e_{zz} \qquad & \sigma_{xy} &= 2A_{66}e_{xy}, \end{aligned} \tag{2.1}$$

where

$$A_{66} = (A_{11} - A_{12})/2. \tag{2.2}$$

In (2.1), σ_{xx}, σ_{yy}, etc. are the components of stress and e_{xx}, e_{yy}, etc. are the components of strain. A_{11}, A_{12}, A_{13}, A_{33} and A_{44} are the five elastic constants of the medium. In the case of an isotropic medium

$$\begin{aligned} A_{11} &= A_{33} = E(1-\nu)/[(1+\nu)(1-2\nu)], \\ A_{12} &= A_{13} = E\nu/[(1+\nu)(1-2\nu)], \\ A_{44} &= E/[2(1+\nu)], \end{aligned} \tag{2.3}$$

where E is Young's modulus and ν is Poisson's ratio.

When the body forces are absent, the equations of equilibrium in tensor notation are

$$\sigma_{ij,j} = 0, \tag{2.4}$$

where the summation convention is implied. The subscript ",j" denotes the partial derivative with respect to x_j and $i, j = 1, 2, 3$.

The final basic equations are the well-known strain-displacement relations, i.e.

$$e_{ij} = (u_{i,j} + u_{j,i})/2, \tag{2.5}$$

where u_i is the x_i component of displacement vector.

We now introduce a double Fourier sine series and a system of vector functions constructed from it. The series

$$S(x, y; m, n) = \sin(m\pi x/a)\,\sin(n\pi y/b) \tag{2.6}$$

is assumed after examining the "edge" conditions of a simply supported plate occupying the region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and $0 \leqslant z \leqslant H$. The constants m and n are taken to be integers. It is apparent that the function $S(x, y; m, n)$ satisfies Helmholtz equation

$$\nabla^2 S + \lambda^2 S = 0, \tag{2.7}$$

where ∇^2 is the Laplacian operator,

$$\begin{aligned} \lambda^2 &= \alpha^2 + \beta^2, \\ \alpha^2 &= (m\pi)^2/a^2, \\ \beta^2 &= (n\pi)^2/b^2. \end{aligned} \tag{2.8}$$

The method of a system of vector functions has been demonstrated to be a very efficient means in theoretical physics [12] and especially in elasticity [13]. Recently, the author has tackled some layered elasticity problems [9, 14, 15] by making extensive use of this method in association with the propagator matrix method. The system used to solve the present problem is

$$\begin{aligned} \mathbf{L}(x, y; m, n) &= i_z S(x, y; m, n), \\ \mathbf{M}(x, y; m, n) &= \operatorname{grad} S = i_x\, \partial S/\partial x + i_y\, \partial S/\partial y, \\ \mathbf{N}(x, y; m, n) &= \operatorname{curl}(i_z S) = i_x\, \partial S/\partial y - i_y\, \partial S/\partial x, \end{aligned} \tag{2.9}$$

where (i_x, i_y, i_z) are the unit vectors in the (x, y, z) directions and $S(x, y; m, n)$ is given by (2.6). It is easy to show that these three vectors are mutually orthogonal and have the following normalization factors

$$\int_0^b \mathrm{d}y \int_0^a \mathrm{d}x \mathbf{L}(x, y; m, n) \cdot \mathbf{L}(x, y; m', n') = ab\delta_{mm'}\delta_{nn'}/4,$$
$$\int_0^b \mathrm{d}y \int_0^a \mathrm{d}x \mathbf{M}(x, y; m, n) \cdot \mathbf{M}(x, y; m', n') = ab\lambda^2\delta_{mm'}\delta_{nn'}/4, \tag{2.10}$$
$$\int_0^b \mathrm{d}y \int_0^a \mathrm{d}x \mathbf{N}(x, y; m, n) \cdot \mathbf{N}(x, y; m', n') = ab\lambda^2\delta_{mm'}\delta_{nn'}/4,$$

where δ_{ij} is Kronecker's delta.

3. General solutions and layer matrices

In this Section, we will derive, for every layer, the general solutions and layer matrices (also called propagator matrices) in the system (2.9). Let us first assume that the thick rectangular plate is simply supported on the "edges" $x = 0$, $x = a$, $y = 0$ and $y = b$, and is composed of p parallel, homogeneous and transversely isotropic layers. The layers are serially numbered with the layer at the top being layer 1. We place the origin of the Cartesian system (x, y, z) at the left bottom corner of the uppermost surface, and the z-axis is drawn into the plate. The layer k is of thickness h_k and is bounded by the interfaces $z = z_{k-1}, z_k$, and for simplicity, the layer interfaces are assumed to be in welded contact. Normal loading P_{zz} is applied to the uppermost surface. If we assume that this loading can be expanded in terms of $S(x, y; m, n)$, that is (when the function $S(x, y; m, n)$ occurs in the following, summation with respect to m and n is implied)

$$P_{zz} = P_{mn}S(x, y; m, n), \tag{3.1}$$

we then have the complete "boundary" conditions [4]

$$\sigma_{xz}(x, y, H) = \sigma_{yz}(x, y, H) = 0, \tag{3.2}$$

$$\sigma_{xz}(x, y, 0) = \sigma_{yz}(x, y, 0) = 0, \tag{3.3}$$

$$\sigma_{zz}(x, y, H) = 0, \tag{3.4}$$

$$\sigma_{zz}(x, y, 0) = P_{mn}S(x, y; m, n), \tag{3.5}$$

$$\begin{cases} u_z(0, y, z) = u_z(a, y, z) = 0, \\ u_y(0, y, z) = u_y(a, y, z) = 0, \\ \sigma_{xx}(0, y, z) = \sigma_{xx}(a, y, z) = 0, \end{cases} \tag{3.6}$$

$$\begin{cases} u_z(x, 0, z) = u_z(x, b, z) = 0, \\ u_x(x, 0, z) = u_x(x, b, z) = 0, \\ \sigma_{yy}(x, 0, z) = \sigma_{yy}(x, b, z) = 0, \end{cases} \tag{3.7}$$

and the continuity conditions at the layer interface $z = z_k$

$$\begin{aligned} &u_i^{k+1}(x, y, z_k) = u_i^k(x, y, z_k), \\ &\sigma_{iz}^{k+1}(x, y, z_k) = \sigma_{iz}^k(x, y, z_k) \quad i = x, y, z, \end{aligned} \tag{3.8}$$

where superscript k $(k + 1)$ is attached to denote that the corresponding quantity is in layer k $(k + 1)$.

For any layer k, we will seek the solution of equations (2.4) under the above conditions in the form [13]

$$\begin{aligned} \mathbf{u}(x, y, z) &= U_L(z)\mathbf{L}(x, y) + U_M(z)\mathbf{M}(x, y) + U_N(z)\mathbf{N}(x, y), \\ \mathbf{T}(x, y, z) &\equiv \sigma_{xz} i_{\mathbf{x}} + \sigma_{yz} i_{\mathbf{y}} + \sigma_{zz} i_{\mathbf{z}} \\ &= T_L(z)\mathbf{L}(x, y) + T_M(z)\mathbf{M}(x, y) + T_N(z)\mathbf{N}(x, y), \end{aligned} \tag{3.9}$$

where the parameters m and n in **L**, **M** and **N** have been omitted for simplicity.

On substituting (3.9) into (2.1), (2.4) and (2.5), the equations of equilibrium can finally be reduced to two systems of simultaneous linear differential equations [9]. They are called type I and type II respectively. For type I

$$\begin{aligned} &\mathrm{d}U_L/\mathrm{dz} = \lambda^2 U_M A_{13}/A_{33} + T_L/A_{33}, \\ &\mathrm{d}U_M/\mathrm{dz} = -U_L + T_M/A_{44}, \\ &\mathrm{d}T_L/\mathrm{dz} = \lambda^2 T_M, \\ &\mathrm{d}T_M/\mathrm{dz} = \lambda^2 U_M(A_{11}A_{33} - A_{13}^2)/A_{33} - A_{13}T_L/A_{33}, \end{aligned} \tag{3.10}$$

and for type II

$$\begin{aligned} &\mathrm{d}U_N/\mathrm{dz} = T_N/A_{44}, \\ &\mathrm{d}T_N/\mathrm{dz} = \lambda^2 A_{66} U_N. \end{aligned} \tag{3.11}$$

Before continuing our derivation, we shall mention here two facts concerning (3.10) and (3.11). First, it is evident that the original problem breaks up into two independent problems, which is the natural result of the orthogonality of the system (2.9). Thus we can solve them separately and then use equation (3.9) to get the total solution of the problem. Second, the coefficients of the displacement components in the system of vector functions for type I correspond to the shape functions in [4]. Actually, the relations between them are

$$U_L(z) = f(z)W_{mn},$$
$$U_M(z) = -g(z)W_{mn}. \tag{3.12}$$

By the standard method, the general solutions of type I and type II are derived as [9]

$$[A(z)] = [Z(z)][C], \tag{3.13}$$

$$[A^L(z)] = [Z^L(z)][C^L]. \tag{3.14}$$

In equations (3.13) and (3.14), the elements of solution matrices $[Z(z)]$ and $[Z^L(z)]$ are given in Appendices A and B respectively, and superscript L is attached to denote that the corresponding quantities belong to the type II. The column matrices are defined by

$$[A(z)] = [U_L(z), \lambda U_M(z), T_L(z)/\lambda, T_M(z)]^T, \tag{3.15}$$

$$[A^L(z)] = [U_N(z), T_N(z)/\lambda]^T, \tag{3.16}$$

$$[C] = [A, B, C, D]^T, \tag{3.17}$$

$$[C^L] = [A^L, B^L]^T, \tag{3.18}$$

where A, B, C, D, A^L and B^L are arbitrary functions of m and n, and $[--]^T$ denotes the transpose of the matrix $[--]$.

From (3.13) and (3.14), we can get the following important relations

$$[A(z_{k-1})] = [a_k][A(z_k)], \tag{3.19}$$

$$[A^L(z_{k-1})] = [a_k^L][A^L(z_k)], \tag{3.20}$$

where

$$[a_k] = [Z(z_{k-1})][Z(z_k)]^{-1} \tag{3.21}$$

and

$$[a_k^L] = [Z^L(z_{k-1})][Z^L(z_k)]^{-1} \tag{3.22}$$

are the layer matrices or the propagator matrices of layer k, and are the essentials of the propagator matrix method [10]. Their elements have been obtained analytically in [9] and are listed in Appendices C and B for the purpose of completeness.

4. Particular solutions

In this Section, we will find the solutions of (3.19) and (3.20) which satisfy the given conditions (3.2)–(3.8). It is evident that the type II system has only the trivial solution for the given conditions. Our task is then to investigate the type I system. From (3.9) and (4.9) below, we notice that the "edge" conditions (3.6) and (3.7) have been automatically satisfied because of our selection of the system of vector functions. The remaining requirements are that the coefficients in the system satisfy the traction boundary conditions (3.2)–(3.5) at $z = 0$ and $z = H$, and the continuity conditions (3.8) at the layer interfaces. With regard to (3.9), the conditions (3.8) yield

$$[A_{k-1}(z_{k-1})] = [A_k(z_{k-1})], \tag{4.1}$$

where subscript k ($k-1$) is attached to the column matrix $[A]$ to denote that it is in layer k ($k-1$). Using relation (3.19), we get

$$[A_{k-1}(z_{k-1})] = [a_k][A_k(z_k)]. \tag{4.2}$$

By making repeated use of (4.2), we finally arrive at

$$[A_1(0)] = [a_1][a_2] - [a_p][A_p(H)]. \tag{4.3}$$

Using the remaining boundary conditions (3.2)–(3.5) which give

$$\begin{aligned} &T_L(0) = P_{mn}, \quad T_M(0) = 0, \\ &T_L(H) = 0, \quad T_M(H) = 0, \end{aligned} \tag{4.4}$$

we thus obtain the following expressions for the unknown quantities on both sides of (4.3)

$$\begin{cases} U_L(H) = F_{42}P_{mn}/(\lambda\Delta), \\ \lambda U_M(H) = -F_{41}P_{mn}/(\lambda\Delta), \end{cases} \tag{4.5}$$

$$\begin{cases} U_L(0) = (F_{11}F_{42} - F_{12}F_{41})P_{mn}/(\lambda\Delta), \\ \lambda U_M(0) = (F_{21}F_{42} - F_{22}F_{41})P_{mn}/(\lambda\Delta), \end{cases} \tag{4.6}$$

where

$$\Delta = F_{31}F_{42} - F_{41}F_{32},$$

$$[F] = [a_1][a_2]--[a_p].$$

Knowing $U_L(H)$ and $U_M(H)$ from (4.5), we can get the coefficients of the displacement and "surface" stress vectors at any point z of the medium, say $z_{k-1} \leqslant z \leqslant z_k$, from the right-hand side of (4.3) by multiplications of matrices

$$[A_k(z)] = [a_k(z_k - z)][a_{k+1}]--[a_p][A_p(H)], \tag{4.7}$$

where $[a_k(z_k - z)]$ is obtained from $[a_k]$ in Appendix C on replacing h_k by $z_k - z$.

So far, we have derived the displacement and "surface" stress vectors at any point of the medium caused by the normal loading (3.5) at its uppermost surface $z = 0$. They are

$$\begin{aligned} \mathbf{u}(x, y, z) &= U_L(z)\mathbf{L}(x, y) + U_M(z)\mathbf{M}(x, y), \\ \mathbf{T}(x, y, z) &= T_L(z)\mathbf{L}(x, y) + T_M(z)\mathbf{M}(x, y), \end{aligned} \tag{4.8}$$

where the coefficients are given by (4.7).

The final task is to get the remaining stress components σ_{xx}, σ_{xy} and σ_{yy} at any point of the medium. It is shown in [9] that with the generalized Hooke's law (2.1), the strain-displacement relations (2.5) and the reduced system of linear differential equations (3.10), they can be expressed as linear combinations of the known elements in (4.7); that is

$$\begin{aligned} \sigma_{xx}(x, y, z) &= [T_L A_{13}/A_{33} + (\lambda^2 A_{13}^2/A_{33} - \alpha^2 A_{11} - \beta^2 A_{12})U_M]S(x, y; m, n), \\ \sigma_{xy}(x, y, z) &= 2A_{66}U_M \frac{\partial^2}{\partial x\, \partial y} S(x, y; m, n), \\ \sigma_{yy}(x, y, z) &= [T_L A_{13}/A_{33} + (\lambda^2 A_{13}^2/A_{33} - \alpha^2 A_{12} - \beta^2 A_{11})U_M]S(x, y; m, n), \end{aligned} \tag{4.9}$$

where the elastic constants are understood as those of layer k.

5. Particular cases

5.1. Transversely isotropic and homogeneous thick plate

When the medium between z_{k-1} and z_{k+1} is homogeneous, one can get the following relation from (3.21)

$$[a(z_k - z_{k-1})][a(z_{k+1} - z_k)] = [a(z_{k+1} - z_{k-1})]. \tag{5.1}$$

For a homogeneous plate with thickness H, we therefore find that

$$[F] = [a(z_p - z_0)] = [a(H)]. \tag{5.2}$$

In this case, the solution can be expressed in a very simple form. For example, using the equation (5.2) and the elements of matrix $[a]$ in Appendix C, we can get the normal displacement or deflection at the upper surface from the first formula of (4.6):

$$\begin{aligned} u_z(x, y, 0) = {} & (P_{mn}/\lambda)\{[c(x_2) - c(x_1)] \\ & \times(\mathrm{sh}\, y_1\, \mathrm{ch}\, y_2/x_2 - \mathrm{ch}\, y_1\, \mathrm{sh}\, y_2/x_1)S(x, y; m, n)\} \\ & /\{-2/(x_1x_2) + 2\, \mathrm{ch}\, y_1\, \mathrm{ch}\, y_2/(x_1x_2) \\ & -(x_1^{-2} + x_2^{-2})\, \mathrm{sh}\, y_1\, \mathrm{sh}\, y_2\} \quad x_1 \neq x_2, \end{aligned} \tag{5.3}$$

$$\begin{aligned} u_z(x, y, 0) = {} & (P_{mn}/\lambda)x_1 c'(x_1)(\lambda H + \mathrm{ch}\, y_1\, \mathrm{sh}\, y_1/x_1)S(x, y; m, n) \\ & /[\mathrm{sh}^2\, y_1/x_1^2 - (\lambda H)^2], \quad x_1 = x_2 \end{aligned} \tag{5.4}$$

where $y_1 = x_1\lambda H$ and $y_2 = x_2\lambda H$; x_1 and x_2 are the characteristic roots determined by the elastic constants of the plate; ch and sh stand for hyperbolic cosine and sine respectively. The function $c(x)$ and its derivative $c'(x)$ are also dependent on the elastic constants of the plate and their definitions are given in Appendix A. It is of interest to note that for the transversely isotropic, simply supported and homogeneous thick plate, the deflection caused by the normal surface loading is dependent on the characteristic roots no matter whether they are equal or not.

With some algebraic manipulations it is also found that the formula (5.3) will reduce to (5.4) as $x_2 \to x_1$, and they all reduce to the following expression as $x_1 \to 1$:

$$u_z(x, y, 0) = [-2(1-\nu^2)/E](P_{mn}/\lambda)[(\lambda H + \mathrm{ch}\, y_1 \,\mathrm{sh}\, y_1)S(x, y; m, n)]$$

$$/[\mathrm{sh}^2\, y_1 - (\lambda H)^2]. \tag{5.5}$$

Noticing that $y_1 = \lambda H$ for the isotropic case, and $\lambda^2 = \pi^2(m^2/a^2 + n^2/b^2)$, we see that this expression is then exactly the same as the formula (30) in reference [4]. That is, the solution in [4] may be considered to be a special case of the present result. It has been shown in [4] that (5.5) will approach the classical Navier solution [16] for the simply supported rectangular plate as H approaches zero.

5.2. Transversely isotropic thin plate

We now examine the thin plate limit of (5.3) and (5.4). By expanding all functions on the right-hand sides of these equations in power series of λH and retaining only the lowest order terms in both the numerator and denominator, it is found that they all approach the following result

$$u_z(x, y, 0) = -12P_{mn}S(x, y; m, n)/[\lambda^4 H^3(A_{11} - A_{13}^2/A_{33})], \tag{5.6}$$

which is independent of the characteristic roots and depends on only three elastic constants. On using the definition of bending stiffness for a transversely isotropic thin plate [7]

$$D = (A_{11} - A_{13}^2/A_{33})H^3/12 \tag{5.7}$$

our result is then exactly the same as the classical solution for the corresponding thin plate theory [7]. It is obvious that (5.6) can be directly reduced to the classical Navier solution by using the expressions of A_{11}, A_{13} and A_{33} in (2.3) for the isotropic medium.

6. Discussion and conclusion

Though the solutions of a simply supported, orthotropic and layered rectangular plate have been reported by some previous researchers [17, 18] within the three-dimensional theory of linear elasticity, all of them require the solution of a system of simultaneous linear equations with order proportional to the number of layers, and no explicit solution has ever been obtained for the corresponding homogeneous rectangular plate. Levinson [4] is probably the first who provided an exact closed form solution for the problem of a simply supported and isotropic rectangular plate of arbitrary thickness by making use of his kinematic assumption [3].

The present solution is an extension of Levinson's result [4] to the transversely isotropic and layered case. As the solution is based on a system of vector functions in association with the propagator matrix method, the result is presented in a simple and exact closed form, which is of interest to laminated plate theory as well as to various thick plate theories. It has been shown that the present solution contains Levinson's thick plate result [4] as a special case and the classical solution for the transversely isotropic thin plate as a limiting case. It is noted that for the isotropic case, the dependence of deflection on the elastic constants is the same no matter whether the plate is thick or thin; for the transversely isotropic case, however, this dependence is more complicated for the thick plate than for the thin plate.

Finally we point out that, with slight modifications, the present result can be used to solve the corresponding shear surface loading problem, which may be of interest to foundation theory because the frictional forces on the plate-foundation interface need to be considered by more appropriate methods [19, 20].

Appendix A

The elements of the solution matrix $[Z(z)]$ in (3.13) are given below.

1. When $x_1 \neq x_2$,

$$\begin{aligned}
Z_{11} &= c(x_1)\exp(\lambda x_1 z), & Z_{12} &= c(x_1)\exp(-\lambda x_1 z),\\
Z_{21} &= d(x_1)\exp(\lambda x_1 z), & Z_{22} &= -d(x_1)\exp(-\lambda x_1 z),\\
Z_{31} &= x_1^{-1}\exp(\lambda x_1 z), & Z_{32} &= -x_1^{-1}\exp(-\lambda x_1 z),\\
Z_{41} &= \exp(\lambda x_1 z), & Z_{42} &= \exp(-\lambda x_1 z),
\end{aligned} \tag{A.1}$$

where x_1 and x_2 are the characteristic roots of the equation

$$(A_{44}x^2 - A_{11})(A_{33}x^2 - A_{44}) + (A_{13} + A_{44})^2 x^2 = 0.$$

Z_{i3} and Z_{i4} are obtained from Z_{i1} and Z_{i2} respectively on replacing x_1 by x_2

$(i = 1, 2, 3, 4)$. In equation (A.1),

$$c(x) = (A_{11} + x^2 A_{13})/[x^2(A_{11}A_{33} - A_{13}^2)], \tag{A.2}$$

$$d(x) = (A_{13} + x^2 A_{33})/[x(A_{11}A_{33} - A_{13}^2)]. \tag{A.3}$$

2. When $x_1 = x_2$,

$$\begin{cases} Z_{13} = [c'(x_1)/\lambda + c(x_1)z] \exp(\lambda x_1 z), \\ Z_{23} = [d'(x_1)/\lambda + d(x_1)z] \exp(\lambda x_1 z), \\ Z_{33} = (-x_1^{-2}/\lambda + x_1^{-1}z] \exp(\lambda x_1 z), \\ Z_{43} = z \exp(\lambda x_1 z); \end{cases} \tag{A.4}$$

while Z_{i1} and Z_{i2} are the same as in equation (A.1), the Z_{i4} are obtained from Z_{i3} on replacing x_1 by $-x_1$ $(i = 1, 2, 3, 4)$. In equation (A.4), the prime denotes differentiation, i.e.

$$c'(x_1) = \mathrm{d}c(x_1)/\mathrm{d}x_1, \quad d'(x_1) = \mathrm{d}d(x_1)/\mathrm{d}x_1. \tag{A.5}$$

Appendix B

The elements of the solution matrix $[Z^L(z)]$ in (3.14) are

$$\begin{aligned} &Z_{11}^L = \exp(-\lambda s z), \qquad Z_{12}^L = \exp(\lambda s z), \\ &Z_{21}^L = -\bar{s} \exp(-\lambda s z), \; Z_{22}^L = \bar{s} \exp(\lambda s z), \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} &s = (A_{66}/A_{44})^{1/2}, \\ &\bar{s} = sA_{44} = (A_{44}A_{66})^{1/2}. \end{aligned} \tag{B.2}$$

The elements of the layer matrix $[a_k^L]$ in (3.22) are (omitting the subscript k)

$$\begin{aligned} &a_{11}^L = \cosh(s\lambda h), \qquad a_{12}^L = -\bar{s}^{-1} \sinh(s\lambda h), \\ &a_{21}^L = -\bar{s} \sinh(s\lambda h), \quad a_{22}^L = \cosh(s\lambda h). \end{aligned} \tag{B.3}$$

For the isotropic medium, we have

$$s = 1,$$

$$\bar{s} = E/[2(1+\nu)];$$

the elements of $[a_k^L]$ are then reduced to those of the isotropic layer matrix [14].

Appendix C

The elements of the layer matrix $[a_k]$ in (3.21) are (omitting the subscript k) given below.

1. When $x_1 \neq x_2$,

$$\begin{aligned}
a_{11} &= a_{33} = [g(x_1)c(x_1)/x_1] \cosh y_1 + [g(x_2)c(x_2)/x_2] \cosh y_2, \\
a_{12} &= -a_{43} = g(x_1)c(x_1) \sinh y_1 + g(x_2)c(x_2) \sinh y_2, \\
a_{13} &= -g(x_1)c^2(x_1) \sinh y_1 - g(x_2)c^2(x_2) \sinh y_2, \\
a_{14} &= -a_{23} = -g(x_1)c(x_1)d(x_1) \cosh y_1 - g(x_2)c(x_2)d(x_2) \cosh y_2, \\
a_{21} &= -a_{34} = -[g(x_1)d(x_1)/x_1] \sinh y_1 - [g(x_2)d(x_2)/x_2] \sinh y_2, \\
a_{22} &= a_{44} = -g(x_1)d(x_1) \cosh y_1 - g(x_2)d(x_2) \cosh y_2, \\
a_{24} &= g(x_1)d^2(x_1) \sinh y_1 + g(x_2)d^2(x_2) \sinh y_2, \\
a_{31} &= -[g(x_1)/x_1^2] \sinh y_1 - [g(x_2)/x_2^2] \sinh y_2, \\
a_{32} &= -a_{41} = -[g(x_1)/x_1] \cosh y_1 - [g(x_2)/x_2] \cosh y_2, \\
a_{42} &= g(x_1) \sinh y_1 + g(x_2) \sinh y_2.
\end{aligned} \tag{C.1}$$

In equation (C.1),

$$y_1 = \lambda x_1 h, \tag{C.2}$$

$$y_2 = \lambda x_2 h, \tag{C.3}$$

$$g(x) = x/[c(x) - xd(x)], \tag{C.4}$$

where the definitions of functions $c(x)$ and $d(x)$ are given respectively in equations (A.2) and (A.3) of Appendix A.

2. When $x_1 = x_2$,
The layer matrix $[a_k]$ in (3.21) can be expressed in the form

$$[a_k] = -(x_1/c'(x_1))[b_k]. \tag{C.5}$$

In equation (C.5), the elements of matrix $[b_k]$ are (omitting the subscript k)

$$\begin{aligned}
b_{11} &= b_{33} = -[c'(x_1)/x_1] \cosh y_1 - [\lambda h c(x_1)/x_1] \sinh y_1, \\
b_{12} &= -b_{43} = -[c(x_1)/x_1 + c'(x_1)] \sinh y_1 - \lambda h c(x_1) \cosh y_1, \\
b_{13} &= c(x_1)[c(x_1)/x_1 + 2c'(x_1)] \sinh y_1 + \lambda h c^2(x_1) \cosh y_1, \\
b_{14} &= -b_{23} = \lambda h c(x_1) d(x_1) \sinh y_1, \\
b_{21} &= -b_{34} = [d'(x_1)/x_1] \sinh y_1 + [\lambda h d(x_1)/x_1] \cosh y_1, \\
b_{22} &= b_{44} = [d(x_1)/x_1 + d'(x_1)] \cosh y_1 + \lambda h d(x_1) \sinh y_1, \\
b_{24} &= -d(x_1)[d(x_1)/x_1 + 2d'(x_1)] \sinh y_1 - \lambda h d^2(x_1) \cosh y_1, \\
b_{31} &= -x_1^{-3} \sinh y_1 + \lambda h x_1^{-2} \cosh y_1, \\
b_{32} &= -b_{41} = \lambda h x_1^{-1} \sinh y_1, \\
b_{42} &= -x_1^{-1} \sinh y_1 - \lambda h \cosh y_1,
\end{aligned} \tag{C.6}$$

where y_1 is given in (C.2), and the definitions of the functions $c(x)$, $d(x)$ and their derivatives are given in Appendix A. For the isotropic medium, we have

$$\begin{aligned}
&x_1 = 1, \quad y_1 = \lambda h, \\
&c(x_1) = d(x_1) = (1 + \nu)/E, \\
&c'(x_1) = -2(1 + \nu)(1 - \nu)/E, \\
&d'(x_1) = (1 + \nu)(1 - 2\nu)/E.
\end{aligned}$$

Upon the substitution of these values in (C.6), we then obtain the layer matrix $[a_k]$ for the isotropic case [14].

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