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Vibration of a transversely isotropic, simply supported and layered rectangular plate

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Abstract. The propagator matrix method is used in this paper to study the vibration of a transversely isotropic, simply supported and layered rectangular plate. A new system of vector functions is constructed to deal with general surface loading, and general solutions and layer matrices of exact closed form are obtained in this system. The particular solution for forced vibration, and the characteristic equations for free vibration of various surface conditions are then obtained by simple multiplication of layer matrices. These results are presented in such a way that the dilatational and distortional modes of vibration are separated. As a special case of the layered plate, results for the corresponding homogeneous thick plate are also derived. They are presented in a very simple form, and contain the previous results for the static transversely isotropic and the dynamic isotropic plates.

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List of symbols

$h_k = z_k - z_{k-1}$ = thickness of layer k
z_{k-1}, z_k = coordinates of upper and lower interfaces of layer k
x, y, z = Cartesian coordinates
a, b, H = length, width and thickness of the layered rectangular plate, respectively
ω = harmonic oscillation frequency as in $\exp(-i\omega t)$
ρ = mass density
u_i = displacements
σ_{ij} = stress components
$S(x, y; m, n), S_1(x, y; m, n)$ = scalar functions
m, n = integers
$P_{xz}(m, n), P_{yz}(m, n), P_{zz}(m, n)$ = loading coefficients
i_x, i_y, i_z = unit vectors
$\mathbf{L}, \mathbf{M}, \mathbf{N}$ = vector functions
$U_L(z), U_M(z), U_N(z)$ = expansion coefficients of displacement vector $\mathbf{u}$
$T_L(z), T_M(z), T_N(z)$ = expansion coefficients of traction vector $\mathbf{T}$
$A_{11}, A_{13}, A_{33}, A_{44}, A_{66}$ = elastic constants
$\lambda^2 = \alpha^2 + \beta^2, \alpha = m\pi/a, \beta = n\pi/b$
$[A(z)], [A^L(z)], [Z(z)], [Z^L(z)]$ = matrix functions
$[C], [C^L]$ = column matrices
$[--]^{-1}$ = inverse of matrix $[--]$
$[a_k], [a_k^L]$ = propagator matrices of layer k
$[A_k(z)], [A_k^L(z)]$ = column matrices of layer k

$[F]$, $[F^L]$ = matrices
$\Delta = F_{31}F_{42} - F_{32}F_{41}$
F_{ij}, F^L_{ij} = elements of matrices $[F]$ and $[F^L]$, respectively
a_{ij} = elements of matrices $[a]$
ch, sh = hyperbolic cosine and sine, respectively
$a(x_1)$, $c(x_1)$, $d(x_1)$ = functions defined in Appendix A
x_1, x_2, x_3 = characteristic roots
g = normal spring constant
G = shear modulus

1. Introduction

There are many examples in various branches of engineering in which the structure appears as a stack of parallel layers. One of the commonly studied problems related to this type of structure is wave propagation in layered elastic media of infinite extent. This is a subject applicable to many areas, such as seismology [1–4], composite materials [5, 6], etc.

The response problem of a layered elastic structure of finite extent subject to static or dynamic loading conditions is also an interesting topic in the area of composite materials, in particular, of thick plate and laminated plate theories. Although several approximate plate theories have been proposed in this field, only a few exact three dimensional solutions have been obtained, to the author's knowledge, within the linear theory of elasticity. In the static aspect, Levinson [7] obtained an exact closed form solution for an isotropic and simply supported rectangular plate, and recently, by constructing a system of vector functions and introducing the propagator matrix method, the author [8] extended Levinson's solution to the corresponding transversely isotropic and layered case. In the dynamic aspect, Lee and Reismann [9] and Srinivas et al. [10] presented exact solutions for the dynamic response of an isotropic and simply supported rectangular plate and for the vibration of the corresponding laminated plates, respectively. Srinivas and Rao [11] then derived the exact solution for the bending, vibration and buckling of simply supported orthotropic rectangular plates and laminates. Although these dynamic analyses are of interest to various plate theories, they require one to solve a system of simultaneous linear equations with order proportional to the number of layers, which is a troublesome procedure when handling multilayered plates. Further, no explicit expression of the characteristic equation has ever been obtained for the vibration of homogeneous thick plate except perhaps for the isotropic medium. Recently, Bottega [12] obtained an axisymmetric solution for the transient problem of a finite multilayered isotropic elastic disk or cylinder. The result was presented in terms of layer matrix and in the form of an eigenfunction expansion. He then discussed the orthogonality of the normal modes for multilayered elastic solids [13].

A new system of vector functions is constructed in this paper to solve the vibration of a transversely isotropic, simply supported and layered rectangular plate. The solution for the forced vibration under general surface loading is presented in an exact closed form in this new system by multiplication of layer matrices. The characteristic equations for some cases of free vibration are also obtained and several reduced results of the present solution are discussed and compared with the previous ones.

2. Problem statement

We consider a simply supported and layered rectangular plate (Fig. 1), which contains p layers of parallel, homogeneous and transversely isotropic elastic material. The kth layer is of thickness h_k and is bounded by the interfaces $z = z_{k-1}, z_k$, and the layer interfaces are assumed to be in welded contact for simplicity. The z-axis has been taken to be the axis of symmetry of this medium and is drawn into the plate.

In the absence of body force, the equation of motion for a body executing

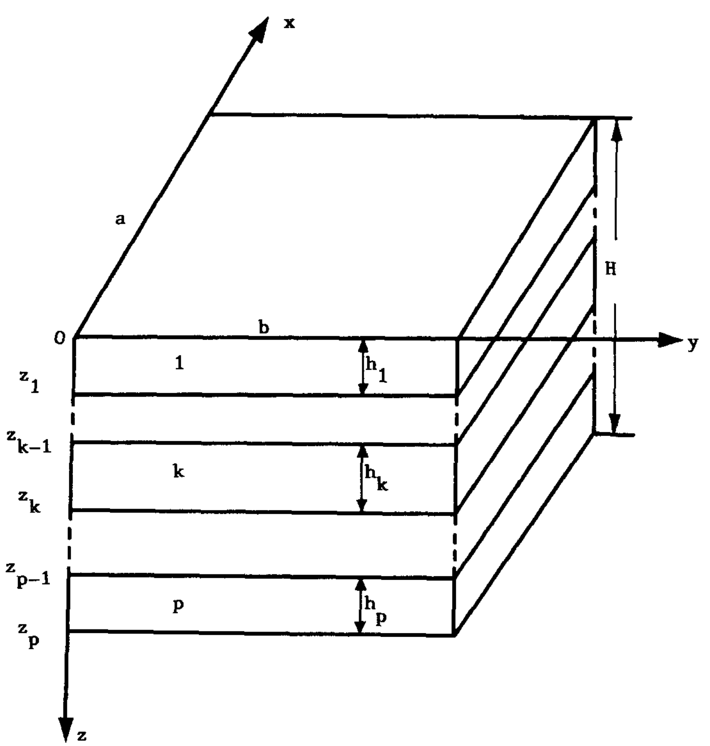

Fig. 1. Geometry of a layered plate occupying region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and $0 \leqslant z \leqslant H$.

simple harmonic oscillations with frequency ω is [in this context, the harmonic factor $\exp(-i\omega t)$ has been omitted for brevity]

$$\sigma_{ij,j} + \rho\omega^2 u_i = 0, \tag{2.1}$$

where the summation convention is implied, σ_{ij} and u_i are the components of stress tensor and displacement vector, respectively; The subscript ",j" denotes the partial derivative with respect to x, y or z as appropriate. It is noted that the static problem discussed in [8] is a special case of the present work with $\omega = 0$.

Other basic equations are the generalized Hooke's law and the well-known strain-displacement relations, cf., e.g., [8]. For a complete treatment, we need the simply-supported condition at "edges", the continuity conditions at the layer interfaces and the boundary condition at the surface. These conditions have been presented in [8]. For the third type of condition, however, we will assume a general loading, i.e., at the lowermost surface, we still have

$$\sigma_{xz}(x, y, H) = \sigma_{yz}(x, y, H) = \sigma_{zz}(x, y, H) = 0, \tag{2.2}$$

but at the uppermost surface, we have

$$\begin{aligned} \sigma_{xz}(x, y, 0) &= P_{xz}(m, n)\, \partial S/\partial x, \\ \sigma_{yz}(x, y, 0) &= P_{yz}(m, n)\, \partial S/\partial y, \\ \sigma_{zz}(x, y, 0) &= P_{zz}(m, n) S, \end{aligned} \tag{2.3}$$

where the scalar function S is defined by

$$S(x, y; m, n) = \sin(m\pi x/a)\sin(n\pi y/b). \tag{2.4}$$

It is assumed that whenever the function S occurs, summation with respect to m and n is implied, and the same convention applies to the following function S_1. In addition, the loading coefficients P_{xz}, P_{yz} and P_{zz} may be frequency dependent.

3. Particular solution for forced vibration

3.1. *Multilayered thick plates*

As we have pointed out in [8], the method of systems of vector functions can be used to handle the general loading problem. For the present "boundary" case, we construct the new system

$$\mathbf{L}(x, y; m, n) = i_z S(x, y; m, n),$$

$$\mathbf{M}(x, y; m, n) = \operatorname{grad} S = i_x \, \partial S/\partial x + i_y \, \partial S/\partial y, \tag{3.1}$$

$$\mathbf{N}(x, y; m, n) = \operatorname{curl}(i_z S_1) = i_x \, \partial S_1/\partial y - i_y \, \partial S_1/\partial x,$$

where the scalar function S_1 is defined by

$$S_1(x, y; m, n) = \cos(m\pi x/a) \cos(n\pi y/b). \tag{3.2}$$

It can be verified that this system is still mutually orthogonal and has the normalization factors given in [8]. Further, the solutions related to vector function **N** and to **L** and **M** are associated with the so-called distortional and dilatational modes, respectively.

For any layer k, we seek the solution in the form [8]

$$\mathbf{u}(x, y, z) = U_L(z)\mathbf{L}(x, y) + U_M(z)\mathbf{M}(x, y) + U_N(z)\mathbf{N}(x, y),$$

$$\mathbf{T}(x, y, z) \equiv \sigma_{xz} i_x + \sigma_{yz} i_y + \sigma_{zz} i_z \tag{3.3}$$

$$= T_L(z)\mathbf{L}(x, y) + T_M(z)\mathbf{M}(x, y) + T_N(z)\mathbf{N}(x, y).$$

Following the same procedure as in [8], we obtain two sets of simultaneous linear differential equations for determining the expansion coefficients U_L, U_M, U_N, T_L, T_M and T_N. They are, for type I,

$$\mathrm{d}U_L/\mathrm{d}z = \lambda^2 U_M A_{13}/A_{33} + T_L/A_{33},$$

$$\mathrm{d}U_M/\mathrm{d}z = -U_L + T_M/A_{44}, \tag{3.4}$$

$$\mathrm{d}T_L/\mathrm{d}z = -\rho\omega^2 U_L + \lambda^2 T_M,$$

$$\mathrm{d}T_M/\mathrm{d}z = U_M[\lambda^2(A_{11}A_{33} - A_{13}^2)/A_{33} - \rho\omega^2] - A_{13}T_L/A_{33},$$

and for type II,

$$\mathrm{d}U_N/\mathrm{d}z = T_N/A_{44}, \tag{3.5}$$

$$\mathrm{d}T_N/\mathrm{d}z = (\lambda^2 A_{66} - \rho\omega^2)U_N,$$

where A_{11}, A_{13}, A_{33}, A_{44} and A_{66} are the five elastic constants [8], and $\lambda^2 = \alpha^2 + \beta^2$, $\alpha = m\pi/a$, $\beta = n\pi/b$.

The general solutions of equations (3.4) and (3.5) are derived as

$$[A(z)] = [Z(z)][C] \tag{3.6}$$

$$[A^L(z)] = [Z^L(z)][C^L]. \tag{3.7}$$

In equations (3.6) and (3.7), the elements of the solution matrices $[Z(z)]$ and $[Z^L(z)]$ are given in Appendix A, and the superscript L is attached to denote that a quantity belongs to type II. The definitions of the column matrices $[A(z)]$, $[A^L(z)]$, $[C]$ and $[C^L]$ are the same as in [8].

From (3.6) and (3.7), we can derive the following propagating relations

$$[A(z_{k-1})] = [a_k][A(z_k)], \tag{3.8}$$

$$[A^L(z_{k-1})] = [a_k^L][A^L(z_k)], \tag{3.9}$$

where

$$[a_k] = [Z(z_{k-1})][Z(z_k)]^{-1} \tag{3.10}$$

and

$$[a_k^L] = [Z^L(z_{k-1})][Z^L(z_k)]^{-1} \tag{3.11}$$

are the propagator matrices of the kth layer. Their elements are obtained by utilizing the symmetry of equations (3.4) and (3.5) [14, 15] and are given in Appendix B.

Proceeding as in [8], we find the following relations:

$$[A_1(0)] = [a_1][a_2] \;--\; [a_p][A_p(H)], \tag{3.12}$$

$$[A_1^L(0)] = [a_1^L][a_2^L] \;--\; [a_p^L][A_p^L(H)], \tag{3.13}$$

where subscript p is attached to the column matrices $[A]$ and $[A^L]$ to denote that they are in the pth layer. Employing the boundary conditions (2.2) and (2.3), which are equivalent to

$$T_L(H) = T_M(H) = T_N(H) = 0, \tag{3.14}$$

$$T_L(0) = P_{zz}, \tag{3.15}$$

$$T_M(0) = (P_{xz}\alpha^2 + P_{yz}\beta^2)/\lambda^2, \tag{3.16}$$

$$T_N(0) = (P_{yz} - P_{xz})\alpha\beta/\lambda^2, \tag{3.17}$$

we then obtain the expressions for the unknown quantities on both sides of (3.12) and (3.13)

$$\begin{aligned} U_L(H) &= [F_{42}T_L(0)/\lambda - F_{32}T_M(0)]/\Delta, \\ \lambda U_M(H) &= -[F_{41}T_L(0)/\lambda - F_{31}T_M(0)]/\Delta, \\ U_N(H) &= T_N(0)/(\lambda F_{21}^L); \end{aligned} \tag{3.18}$$

$$\begin{aligned} U_L(0) &= [(F_{11}F_{42} - F_{12}F_{41})T_L(0)/\lambda + (F_{12}F_{31} - F_{11}F_{32})T_M(0)]/\Delta, \\ \lambda U_M(0) &= [(F_{21}F_{42} - F_{22}F_{41})T_L(0)/\lambda + (F_{22}F_{31} - F_{21}F_{32})T_M(0)]/\Delta, \\ U_N(0) &= F_{11}^L T_N(0)/(\lambda F_{21}^L), \end{aligned} \tag{3.19}$$

where

$$[F] = [a_1][a_2] -- [a_p], \tag{3.20}$$

$$[F^L] = [a_1^L][a_2^L] -- [a_p^L], \tag{3.21}$$

$$\Delta = F_{31}F_{42} - F_{32}F_{41}. \tag{3.22}$$

It is noted that equations (3.18) and (3.19) are extensions of equations (4.5) and (4.6) in [8], respectively. The coefficients of the displacement and traction vectors at any level of the media, say $z_{k-1} \leqslant z \leqslant z_k$, can then be expressed by

$$[A_k(z)] = [a_k(z_k - z)][a_{k+1}] -- [a_p][A_p(H)], \tag{3.23}$$

$$[A_k^L(z)] = [a_k^L(z_k - z)][a_{k+1}^L] -- [a_p^L][A_p^L(H)], \tag{3.24}$$

where $[a_k(z_k - z)]$ and $[a_k^L(z_k - z)]$ are obtained from Appendix B by replacing h_k with $z_k - z$.

So far we have derived the displacement and traction vectors at any point of the medium. They are given by (3.3) with coefficients being determined by (3.23) and (3.24). The remaining stress components for the present general case are found to be

$$\sigma_{xx}(x, y, z) = [T_L A_{13}/A_{33} + (\lambda^2 A_{13}^2/A_{33} - \alpha^2 A_{11} - \beta^2 A_{12})U_M]S$$
$$+ 2A_{66} U_N \, \partial^2 S_1/(\partial x \, \partial y),$$
$$\sigma_{xy}(x, y, z) = 2A_{66} U_M \, \partial^2 S/(\partial x \, \partial y) + A_{66}(\alpha^2 - \beta^2) U_N S_1, \tag{3.25}$$
$$\sigma_{yy}(x, y, z) = [T_L A_{13}/A_{33} + (\lambda^2 A_{13}^2/A_{33} - \alpha^2 A_{12} - \beta^2 A_{11})U_M]S$$
$$- 2A_{66} U_N \, \partial^2 S_1/(\partial x \, \partial y).$$

Again, this equation is an extension of equation (4.9) in [8].

3.2. *Homogeneous thick plate*

For a homogeneous plate with arbitrary thickness H, equations (3.20)–(3.22) reduce to

$$[F] = [a(H)], \tag{3.26}$$

$$[F^L] = [a^L(H)], \tag{3.27}$$

$$\Delta = a_{31} a_{42} - a_{32} a_{41}. \tag{3.28}$$

In this case, all results become very simple. For example, from equation (3.19), the normal displacement or deflection at the surface caused by normal surface loading only can be reduced to

$$u_z(x, y, 0) = T_L(0)(a_{11} a_{42} - a_{12} a_{41}) S/(\lambda \Delta). \tag{3.29}$$

Substituting equation (3.15) and the elements of the matrix $[a]$ in Appendix B into equation (3.29), and after performing some algebraic manipulations, we finally find that

$$u_z(x, y, 0) = P_{zz} \lambda^{-1} S\{[d(x_1)a(x_2)$$
$$- d(x_2)a(x_1)][d(x_2)c(x_1) \operatorname{ch}(x_1 H) \operatorname{sh}(x_2 H)$$
$$- d(x_1)c(x_2) \operatorname{ch}(x_2 H) \operatorname{sh}(x_1 H)]\}$$
$$/\{2c(x_1)c(x_2)d(x_1)d(x_2)[1 - \operatorname{ch}(x_1 H) \operatorname{ch}(x_2 H)]$$
$$+ [c^2(x_1)d^2(x_2) + c^2(x_2)d^2(x_1)] \operatorname{sh}(x_1 H) \operatorname{sh}(x_2 H)\}, \tag{3.30}$$

where the definitions of functions $a(x_1)$, $c(x_1)$ and $d(x_1)$ are given in Appendix A; ch and sh stand for hyperbolic cosine and sine, respectively.

4. Free vibration with stress-free surfaces

If the uppermost surface is also free of traction, we see that equations (3.12) and (3.13) then reduce to

$$[U_L(0), \lambda U_M(0), 0, 0]^T = [F][U_L(H), \lambda U_M(H), 0, 0]^T, \tag{4.1}$$

$$[U_N(0), 0]^T = [F^L][U_N(H), 0]^T, \tag{4.2}$$

where $[--]^T$ denotes the transpose of matrix $[--]$. These two systems have nontrivial solutions provided that

$$F_{31}F_{42} - F_{32}F_{41} = 0, \tag{4.3}$$

$$F_{21}^L = 0, \tag{4.4}$$

which are just the denominators in expressions (3.18) and (3.19).

Equations (4.3) and (4.4) are the characteristic equations for determining the eigenfrequencies of the free vibration of a transvesely isotropic and layered rectangular plate with stress-free surfaces, and their eigenvalues correspond to the dilatational and distortional modes, respectively. As these expressions are given in exact closed form, they may be used to measure various thick and laminated plate theories.

For a homogeneous thick plate, equations (4.3) and (4.4) can be expressed explicitly in very simple forms

$$\begin{aligned} &2c(x_1)c(x_2)d(x_1)d(x_2)[1 - \mathrm{ch}(x_1 H)\,\mathrm{ch}(x_2 H)] \\ &\quad + [c^2(x_1)d^2(x_2) + c^2(x_2)d^2(x_1)]\,\mathrm{sh}(x_1 H)\,\mathrm{sh}(x_2 H) = 0, \end{aligned} \tag{4.5}$$

$$\mathrm{sh}(x_3 H) = 0. \tag{4.6}$$

Further, by using equation (2.3) in [8], it is found that these two equations can be reduced to the characteristic equation obtained by Srinivas et al. [10] for an isotropic plate. As pointed out in [10], the solution of this transcendental equation for each combination of m and n yields an infinite sequence of eigenvalues, instead of only one family by thin plate theory [16] or three by Mindlin's thick plate theory [17].

The displacement and stress components for the free vibration problem may be easily obtained by using the formulae in Section 3, and, as is well-known, can be determined in relative magnitude only.

5. Free vibration with other surface conditions

For a complete analysis, we provide here the exact closed form characteristic equations for some other surface conditions frequently discussed in the literatures, cf., e.g., [10].

5.1. *Layered plates with smooth rigid surfaces*

The surface conditions on both surfaces in this case are

$$z = 0, H: \quad u_z = \sigma_{xz} = \sigma_{yz} = 0 \tag{5.1}$$

or, equivalently,

$$z = 0, H: \quad U_L = T_M = T_N = 0. \tag{5.2}$$

The characteristic equations for dilatational and distortional modes are, respectively,

$$F_{12}F_{43} - F_{42}F_{13} = 0, \tag{5.3}$$

$$F_{21}^{L} = 0. \tag{5.4}$$

It is noted that the second type of modes in this case is the same as that for the case of stress-free surfaces. For a homogeneous thick plate, equations (5.3) and (5.4) can be reduced to

$$\mathrm{sh}(x_1 H)\,\mathrm{sh}(x_2 H) = 0, \tag{5.5}$$

$$\mathrm{sh}(x_3 H) = 0, \tag{5.6}$$

which can be further reduced to the corresponding isotropic result as given in [10].

5.2. *Layered plates with rigid surfaces*

In this case, we have

$$z = 0, H: \quad u_x = u_y = u_z = 0 \tag{5.7}$$

or, equivalently,

$$z = 0, H: \quad U_L = U_M = U_N = 0. \tag{5.8}$$

The characteristic equations are

$$F_{13}F_{24} - F_{14}F_{23} = 0, \tag{5.9}$$

$$F_{12}^L = 0. \tag{5.10}$$

Notice that the equation for the distortional modes in this case is different from that of the two types discussed above. But for a homogeneous thick plate, they are not distinguishable, since in this special case, we have

$$2a(x_1)a(x_2)[1 - \mathrm{ch}(x_1 H)\,\mathrm{ch}(x_2 H)]$$

$$+ [a^2(x_1) + a^2(x_2)]\,\mathrm{sh}(x_1 H)\,\mathrm{sh}(x_2 H) = 0, \tag{5.11}$$

$$\mathrm{sh}(x_3 H) = 0. \tag{5.12}$$

Again, in the isotropic case, these results will reduce to those given in [10].

5.3. *Layered plates on elastic foundation*

The boundary conditions on both surfaces are assumed to be [10]

$$z = 0: \quad \sigma_{zz} - g u_z = 0, \quad \sigma_{xz} = \sigma_{yz} = 0, \tag{5.13}$$

$$z = H: \quad \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0, \tag{5.14}$$

where g is the normal spring constant. On expressing equations (5.13) and (5.14) in terms of the coefficients of the displacement and traction vectors, they are seen to be equivalent to

$$z = 0: \quad T_L - gU_L = 0, \quad T_M = T_N = 0, \tag{5.15}$$

$$z = H: \quad T_L = T_M = T_N = 0. \tag{5.16}$$

The characteristic equations in this case are

$$F_{41}(\lambda F_{32} - gF_{12}) - F_{42}(\lambda F_{31} - gF_{11}) = 0, \tag{5.17}$$

$$F_{21}^{L} = 0. \tag{5.18}$$

For the corresponding homogeneous case, they are reduced to

$$\begin{aligned} & g\lambda^{-1}[d(x_1)a(x_2) - d(x_2)a(x_1)][d(x_1)c(x_2)\,\mathrm{ch}(x_2H)\,\mathrm{sh}(x_1H) \\ & \quad - d(x_2)c(x_1)\,\mathrm{ch}(x_1H)\,\mathrm{sh}(x_2H)] \\ & \quad + 2c(x_1)c(x_2)d(x_1)d(x_2)[1 - \mathrm{ch}(x_1H)\,\mathrm{ch}(x_2H)] \\ & \quad + [c^2(x_1)d^2(x_2) + c^2(x_2)d^2(x_1)]\,\mathrm{sh}(x_1H)\,\mathrm{sh}(x_2H) = 0, \end{aligned} \tag{5.19}$$

$$\mathrm{sh}(x_3H) = 0. \tag{5.20}$$

It should be pointed out that, in the isotropic case, (5.19) will be reduced to the result given by Srinivas et al. [10] with g being replaced by g/G; this may be caused by misprint.

6. Discussion and conclusions

The propagator matrix method has been used to solve the vibration problem of a transversely isotropic, simply supported and layered rectangular plate. For dealing with the general surface loading case, a new system of vector functions has been constructed. While the expressions for displacements and stresses for forced vibration are derived in terms of this new system by multiplication of layer matrices, those of the characteristic equations for several types of free vibration are obtained by multiplication of layer matrices only. As these expressions are very simple, they are particularly suitable for laminate plates. Further, it is shown that the present result contains results of some previous studies as special cases.

Although we have discussed the vibration problem only, the solution for the general dynamic problem may be obtained by employing the eigenfunction expression technique proposed by Bottega [12], because the present case satisfies the orthogonality condition discussed in [13]. Finally, we point out that the present general solution and the layer matrix may be employed to extend Bottega's recent work [12] to the corresponding transversely isotropic case, which is currently under investigation.

Appendix A

The elements of the solution matrix $[Z(z)]$ in (3.6) are

$$\begin{aligned} &Z_{11} = a(x_1)\exp(x_1 z) \quad Z_{12} = -a(x_1)\exp(-x_1 z), \\ &Z_{21} = \exp(x_1 z) \quad Z_{22} = \exp(-x_1 z), \\ &Z_{31} = c(x_1)\exp(x_1 z) \quad Z_{32} = c(x_1)\exp(-x_1 z), \\ &Z_{41} = d(x_1)\exp(x_1 z) \quad Z_{42} = -d(x_1)\exp(-x_1 z); \end{aligned} \tag{A1}$$

Z_{i3} and Z_{i4} are obtained from Z_{i1} and Z_{i2}, respectively, by replacing x_1 with x_2 $(i = 1, 2, 3, 4)$; x_1^2 and x_2^2 are two distinct roots of the equation

$$\begin{aligned} &x^4 + [\rho\omega^2(A_{33} + A_{44}) + \lambda^2(2A_{13}A_{44} + A_{13}^2 - A_{11}A_{33})]x^2/(A_{33}A_{44}) \\ &\quad + (\lambda^2 A_{44} - \rho\omega^2)(\lambda^2 A_{11} - \rho\omega^2)/(A_{33}A_{44}) = 0. \end{aligned} \tag{A2}$$

Functions $a(x)$, $c(x)$ and $d(x)$ are defined by

$$a(x) = \lambda(A_{13} + A_{44})x/\Delta(x), \tag{A3}$$

$$c(x) = [A_{33}A_{44}x^2 - A_{13}(\rho\omega^2 - \lambda^2 A_{44})]/\Delta(x), \tag{A4}$$

$$d(x) = (\rho\omega^2 + \lambda^2 A_{13} + x^2 A_{33})A_{44}x/[\lambda\Delta(x)], \tag{A5}$$

where

$$\Delta(x) = \rho\omega^2 - \lambda^2 A_{44} + x^2 A_{33}. \tag{A6}$$

The elements of the solution matrix $[Z^L(z)]$ in (3.7) are

$$\begin{aligned} &Z_{11}^L = \exp(x_3 z) \qquad Z_{12}^L = \exp(-x_3 z), \\ &Z_{21}^L = \bar{s}\exp(x_3 z) \quad Z_{22}^L = -\bar{s}\exp(-x_3 z), \end{aligned} \tag{A7}$$

where

$$\bar{s} = A_{44}x_3/\lambda, \tag{A8}$$

$$x_3^2 = \lambda^2 A_{66}/A_{44} - \rho\omega^2/A_{44}. \tag{A9}$$

Appendix B

The elements of the layer matrix $[a_k]$ in (3.10) are (dropping the subscript k)

$$
\begin{aligned}
a_{11} &= a_{33} = g(x_1)a(x_1)c(x_1)\ \mathrm{ch}\ y_1 + g(x_2)a(x_2)c(x_2)\ \mathrm{ch}\ y_2,\\
a_{12} &= -a_{43} = g(x_1)a(x_1)d(x_1)\ \mathrm{sh}\ y_1 + g(x_2)a(x_2)d(x_2)\ \mathrm{sh}\ y_2,\\
a_{13} &= -g(x_1)a^2(x_1)\ \mathrm{sh}\ y_1 - g(x_2)a^2(x_2)\ \mathrm{sh}\ y_2,\\
a_{14} &= -a_{23} = -g(x_1)a(x_1)\ \mathrm{ch}\ y_1 - g(x_2)a(x_2)\ \mathrm{ch}\ y_2,\\
a_{21} &= -a_{34} = -g(x_1)c(x_1)\ \mathrm{sh}\ y_1 - g(x_2)c(x_2)\ \mathrm{sh}\ y_2,\\
a_{22} &= a_{44} = -g(x_1)d(x_1)\ \mathrm{ch}\ y_1 - g(x_2)d(x_2)\ \mathrm{ch}\ y_2,\\
a_{24} &= g(x_1)\ \mathrm{sh}\ y_1 + g(x_2)\ \mathrm{sh}\ y_2,\\
a_{31} &= -g(x_1)c^2(x_1)\ \mathrm{sh}\ y_1 - g(x_2)c^2(x_2)\ \mathrm{sh}\ y_2,\\
a_{32} &= -a_{41} = -g(x_1)d(x_1)c(x_1)\ \mathrm{ch}\ y_1 - g(x_2)d(x_2)c(x_2)\ \mathrm{ch}\ y_2,\\
a_{42} &= g(x_1)d^2(x_1)\ \mathrm{sh}\ y_1 + g(x_2)d^2(x_2)\ \mathrm{sh}\ y_2,
\end{aligned}
\tag{B1}
$$

where

$$y_1 = x_1 h, \tag{B2}$$

$$y_2 = x_2 h, \tag{B3}$$

$$g(x) = 1/[a(x)c(x) - d(x)]. \tag{B4}$$

The elements of the layer matrix $[a_k^L]$ in (3.11) are (dropping the subscript k)

$$
\begin{aligned}
a_{11}^L &= \mathrm{ch}\ y_3 \quad a_{12}^L = -\bar{s}^{-1}\ \mathrm{sh}\ y_3,\\
a_{21}^L &= -\bar{s}\ \mathrm{sh}\ y_3 \quad a_{22}^L = \mathrm{ch}\ y_3,
\end{aligned}
\tag{B5}
$$

where

$$y_3 = x_3 h. \tag{B6}$$

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