Gravitational Stresses in Anisotropic Rock Masses with Inclined Strata

B. AMADEI†
E. PAN†

This paper presents closed-form solutions for the stress field induced by gravity in generally anisotropic, orthotropic and transversely isotropic rock masses. These rocks are assumed to be homogeneous and linearly elastic continua with strata inclined with respect to a horizontal ground surface. It is found that the stress field is multiaxial. The vertical stress is always a principal stress and is equal to the weight of the overlying material. The horizontal stresses are strongly correlated to the rock mass fabric. The expressions for the gravity-induced horizontal stresses are different for rock masses deforming under conditions of no lateral strain and no lateral displacement (uniaxial strain). The gravity-induced horizontal stresses depend on several parameters such as the type, degree and orientation of the rock anisotropy with respect to the ground surface. It is found that depending on the value of those parameters, and constrained by the thermodynamic requirement that the strain energy of the rock must always be positive-definite, the gravity induced horizontal stresses can be larger, equal or less than the vertical stress. Furthermore, for a certain range of elastic properties of a transversely isotropic rock mass with inclined strata only, it is thermodynamically admissible for the horizontal stress parallel to the dip direction of the strata to be tensile.

INTRODUCTION

In two recent papers [1, 2], closed-form solutions were presented for the components of the stress field induced by gravitational loading of laterally restrained anisotropic rock masses with a horizontal ground surface. The rock masses were modelled as orthotropic or transversely isotropic linearly elastic materials that were either homogeneous or stratified with homogeneous layers. The solutions were limited to orthotropic and transversely isotropic rock masses with horizontal or vertical planes of symmetry.

The analytical solutions of Amadei *et al.* [1, 2] showed that for anisotropic rock masses under gravity and a condition of no lateral displacements, the horizontal stresses could not be predicted by the classical isotropic solution of Terzaghi and Richart [3]. Recall that in the isotropic solution, the horizontal stresses are equal to $v/(1-v)$ times the vertical stress where v is the rock Poisson's ratio. It was found that the magnitude of gravity-induced horizontal stresses greatly depends on the type, degree and orientation of rock anisotropy with respect to the ground surface. Also, it appeared that inclusion of anisotropy could broaden the range of permissible values of gravity-induced horizontal stresses

in rock masses. In fact, for some range of anisotropic rock properties, it was found that it is thermodynamically admissible for gravity-induced horizontal stresses to exceed the vertical stress component which is not possible with the isotropic solution.

The purpose of this paper is to consider the nature of the gravity-induced stress field in homogeneous orthotropic and transversely isotropic rock masses with strata which are now inclined with respect to a horizontal ground surface. At the outset, new closed-form solutions for the stresses derived by the second author are presented for gravitational loading of generally anisotropic, orthotropic and transversely isotropic rock masses under no lateral strain and no lateral displacement conditions. This is followed by a parametric study on the effect of rock anisotropic properties and rock strata inclination on gravity-induced stresses. Finally, it is shown how the domains of variation of gravity-induced horizontal stresses in anisotropic rocks are controlled by thermodynamic constraints on their elastic properties.

ANALYTICAL SOLUTION

General solution

Consider the equilibrium of a flat lying horizontal elastic half-space representing a rock mass of uniform density ρ under gravity alone. Let x, y, z be an arbitrary

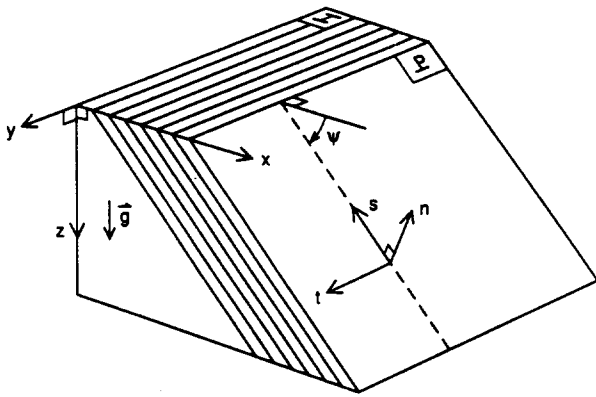†Department of Civil Engineering, University of Colorado, Boulder, CO 80309, U.S.A.

Fig. 1. Three-dimensional problem geometry showing the orientation of one of the planes of symmetry P in the x, y, z coordinate system.

coordinate system attached to the half-space such that the x- and y-axes are in the horizontal plane and the z-axis is positive downward (Fig. 1). The constitutive model for the rock mass is assumed to be described by Hooke's law which can be written in the x, y, z coordinate system as follows:

$$(\epsilon) = (A)(\sigma) \tag{1}$$

or

$$(\sigma) = (C)(\epsilon), \tag{2}$$

where (σ) and (ϵ) are, respectively, (6×1) column matrix representations of the stress and strain tensors in the x, y, z-coordinate system. (A) is a (6×6) symmetric compliance matrix with 21 independent components a_{ij} $(i, j = 1 \rightarrow 6)$ and (C) is the corresponding matrix of elastic parameters with components c_{ij} $(i, j = 1 \rightarrow 6)$ and is such that $(C) = (A)^{-1}$. The components a_{ij} and c_{ij} are assumed to be independent of x and y but are allowed to vary with the z-coordinate.

Since the upper surface of the rock mass is free of any loads, the three stress components σ_z, τ_{xz} and τ_{yz} must vanish at $z = 0$. In addition, under gravitational loading in the z-direction, at any point in the rock mass, the six stress components must satisfy the following three equations of equilibrium:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} - \rho g = 0. \tag{3}$$

In this paper, compressive stresses and contractile strains are taken as positive and the negative direction of the gravitational body force ρg is in the positive z direction (e.g. downward) where g is the acceleration due to gravity.

Because the body force of gravity, the ground surface boundary conditions and the elastic properties of the

rock mass are independent of the x- and y-coordinates, all the components of stress, strain and displacement induced by gravity can be assumed to be independent of x and y and to depend on z only. For this assumption, the horizontal strain components ϵ_x, ϵ_y and γ_{xy} vanish, that is, the rock mass deforms under a condition of no lateral strain. The three non-vanishing strain components are equal to:

$$\epsilon_z = -\frac{\partial u_z}{\partial z},$$

$$\gamma_{yz} = -\frac{\partial u_y}{\partial z},$$

$$\gamma_{xz} = -\frac{\partial u_x}{\partial z}. \tag{4}$$

Since the stress components are independent of x and y, the equations of equilibrium [equation (3)] reduce to:

$$\frac{\partial \tau_{xz}}{\partial z} = 0;$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0;$$

$$\frac{\partial \sigma_z}{\partial z} = \rho g. \tag{5}$$

Using the stress-free boundary condition at the ground surface, integration of equation (5) gives that the shear stresses τ_{xz} and τ_{yz} always vanish at any depth z and that σ_z is always equal to the weight $\rho g z$ of the material above z. The other three stress components σ_x, σ_y and τ_{xy} can be obtained by substituting the conditions $\tau_{xz} = \tau_{yz} = 0$ and $\sigma_z = \rho g z$ into equation (2). This results in the following system of three equations:

$$c_{33}\epsilon_z + c_{34}\gamma_{yz} + c_{35}\gamma_{xz} = \rho g z,$$

$$c_{34}\epsilon_z + c_{44}\gamma_{yz} + c_{45}\gamma_{xz} = 0,$$

$$c_{35}\epsilon_z + c_{45}\gamma_{yz} + c_{55}\gamma_{xz} = 0, \tag{6}$$

that can be solved for the three unknown strains ϵ_z, γ_{yz} and γ_{xz}, e.g.

$$\epsilon_z = \frac{\rho g z}{\Delta} \cdot (c_{44}c_{55} - c_{45}^2),$$

$$\gamma_{yz} = -\frac{\rho g z}{\Delta} \cdot (c_{34}c_{55} - c_{35}c_{45}),$$

$$\gamma_{xz} = \frac{\rho g z}{\Delta} \cdot (c_{34}c_{45} - c_{35}c_{44}), \tag{7}$$

where

$$\Delta = c_{33}(c_{44}c_{55} - c_{45}^2) - c_{34}(c_{34}c_{55} - c_{35}c_{45})$$

$$+ c_{35}(c_{34}c_{45} - c_{35}c_{44}).$$

Substituting those three strain components into equation (2), gives the expression for the stress components σ_x, σ_y and τ_{xy}:

$$\sigma_x = \frac{\rho g z}{\Delta} \cdot [c_{13}(c_{44}c_{55} - c_{45}^2)$$
$$- c_{14}(c_{34}c_{55} - c_{35}c_{45}) + c_{15}(c_{34}c_{45} - c_{35}c_{44})],$$

$$\sigma_y = \frac{\rho g z}{\Delta} \cdot [c_{23}(c_{44}c_{55} - c_{45}^2)$$
$$- c_{24}(c_{34}c_{55} - c_{35}c_{45}) + c_{25}(c_{34}c_{45} - c_{35}c_{44})],$$

$$\tau_{xy} = \frac{\rho g z}{\Delta} \cdot [c_{36}(c_{44}c_{55} - c_{45}^2)$$
$$- c_{46}(c_{34}c_{55} - c_{35}c_{45}) + c_{56}(c_{34}c_{45} - c_{35}c_{44})]. \quad (8)$$

In addition,

$$\sigma_z = \rho g z, \quad \tau_{yz} = 0, \quad \tau_{xz} = 0. \quad (9)$$

Equations (8) and (9) show that for a general anisotropic rock mass with 21 elastic parameters under gravity and a no lateral strain condition, the vertical stress is always a principal stress and its magnitude is independent of the rock mass elastic properties. The two horizontal principal stresses are not equal and their magnitude and orientation in the x, y plane depend on the type and degree of rock mass anisotropy.

Consider now the special case when the rock mass is *orthotropic* in a local n, s, t coordinate system attached to three orthogonal planes of symmetry. The orientation of that coordinate system with respect to the global x, y, z coordinate system is shown in Fig. 1. It is such that the t- and y-axes are parallel. Therefore, one of the three planes of symmetry (defined as P in Fig. 1) strikes parallel to the y-axis. Let ψ be the dip angle of that plane. The constitutive equation for the orthotropic rock mass in the n, s, t coordinate system is given by the following equation:

$$\begin{bmatrix} \epsilon_n \\ \epsilon_s \\ \epsilon_t \\ \gamma_{st} \\ \gamma_{nt} \\ \gamma_{ns} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_n} & -\frac{v_{sn}}{E_s} & -\frac{v_{tn}}{E_t} & 0 & 0 & 0 \\ -\frac{v_{ns}}{E_n} & \frac{1}{E_s} & -\frac{v_{ts}}{E_t} & 0 & 0 & 0 \\ -\frac{v_{nt}}{E_n} & -\frac{v_{st}}{E_s} & \frac{1}{E_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{st}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{nt}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{ns}} \end{bmatrix}$$

$$\times \begin{bmatrix} \sigma_n \\ \sigma_s \\ \sigma_t \\ \tau_{st} \\ \tau_{nt} \\ \tau_{ns} \end{bmatrix}, \quad (10)$$

or in a more compact matrix form:

$$(\epsilon)_{nst} = (H)(\sigma)_{nst} \quad (11)$$

Nine independent elastic parameters are needed to describe the deformability of the rock in the n, s, t coordinate system. E_n, E_s and E_t are the Young's moduli in the n, s and t directions, respectively. G_{ns}, G_{nt} and G_{st} are the shear moduli in planes parallel to the ns, nt and st planes, respectively. Finally, v_{ij} ($i, j = n, s, t$) are the Poisson's ratios that characterize the normal strains in the symmetry directions j when a stress is applied in the symmetry directions i. Because of symmetry of the compliance matrix (H), Poisson's ratios v_{ij} and v_{ji} are such that $v_{ij}/E_i = v_{ji}/E_j$.

Using the coordinate transformation rules for Cartesian tensors, the components a_{ij} and c_{ij} of matrices (A) and (C) in equations (1) and (2) can be expressed in terms of the nine elastic parameters of the rock in the n, s, t coordinate system and the dip angle ψ (see Appendix). For the geometry of Fig. 1, it is shown in the Appendix that several components of matrices (A) and (C) vanish, e.g. a_{45}, a_{56}, c_{45}, c_{56} and a_{i4}, a_{i6}, c_{i4}, c_{i6} for $i = 1, 2, 3$. Substituting these conditions into equation (8), the shear stress τ_{xy} vanishes and σ_x and σ_y have the following expressions:

$$\sigma_x = \rho g z \cdot \frac{c_{13}c_{55} - c_{15}c_{35}}{c_{33}c_{55} - c_{35}^2},$$

$$\sigma_y = \rho g z \cdot \frac{c_{23}c_{55} - c_{25}c_{35}}{c_{33}c_{55} - c_{35}^2}. \quad (12)$$

Equations (9) and (12) show that if one plane of symmetry of the orthotropic rock mass strikes parallel to the y-axis, the three stress components in the x, y, z coordinate system σ_x, σ_y and σ_z are always principal stresses. Because of the linear relations existing between coefficients a_{ij} and h_{ij} of matrices and (A) and (H) in equations (1) and (11), respectively, it can be shown that the stress ratios $\sigma_x/\rho g z$ and $\sigma_y/\rho g z$ defined in equation (12) depend on the dip angle ψ and eight dimensionless quantities:

$$\frac{E_s}{E_n}; \quad \frac{E_s}{E_t}; \quad v_{sn}; \quad v_{tn}; \quad v_{ts}; \quad \frac{E_s}{G_{st}}; \quad \frac{E_s}{G_{nt}}; \quad \frac{E_s}{G_{ns}}. \quad (13)$$

Equations (10–13) still apply if the rock mass is *transversely isotropic* in one of the three ns-, nt- or st-planes. In that case, only five independent elastic parameters are needed to describe the deformability of the medium in the n, s, t coordinate system. In this paper, these parameters are called E, E', v, v' and G' with the following definitions:

(i) E and E' are Young's moduli in the plane of transverse isotropy and in direction normal to it, respectively;

(ii) v and v' are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively; and

(iii) G' is the shear modulus in planes normal to the plane of transverse isotropy.

Relations exist between E, E', v, v', G and G' and the coefficients of matrix (H) in equation (11). For instance, for transverse isotropy in the st-plane:

$$\frac{1}{E_n} = \frac{1}{E'}; \quad \frac{1}{E_s} = \frac{1}{E_t} = \frac{1}{E}; \quad \frac{1}{G_{ns}} = \frac{1}{G_{nt}} = \frac{1}{G'},$$

$$\frac{v_{ns}}{E_n} = \frac{v_{nt}}{E_n} = \frac{v'}{E'};$$

$$\frac{v_{st}}{E_s} = \frac{v_{ts}}{E_t} = \frac{v}{E};$$

$$\frac{1}{G_{st}} = \frac{1}{G} = \frac{2(1+v)}{E}. \tag{14}$$

The stress ratios $\sigma_x/\rho gz$ and $\sigma_y/\rho gz$ defined in equation (12) now depend on the dip angle ψ and the following four dimensionless quantities:

$$\frac{E}{E'}; \quad v; \quad v'; \quad \frac{G}{G'}. \tag{15}$$

No lateral displacement solution

Equations (8), (9) and (12) were derived assuming that the components of stress, strain and displacement induced by gravity are independent of the x- and y-coordinates. This assumption results in a no lateral strain condition $\epsilon_x = \epsilon_y = \gamma_{xy} = 0$. Since γ_{yz} and γ_{xz} and ϵ_z do not vanish, this no lateral strain condition is not as restrictive as the classical uniaxial strain condition where all strain components except ϵ_z vanish. Furthermore, combining equations (4) and (7), the displacement components u_x, u_y and u_z are quadratic functions of the depth z whereas u_x and u_y vanish for the uniaxial strain condition.

Consider now the case where, in addition to the assumptions associated with the no lateral strain condition, the anisotropic rock mass is not allowed to deform in the horizontal plane under gravitational loading, i.e. $u_x = u_y = 0$. Then, the strain components γ_{yz} and γ_{xz} vanish along with ϵ_x, ϵ_y and γ_{xy} and the vertical strain ϵ_z is the only non-vanishing strain component (uniaxial strain condition). Substituting the condition $\gamma_{yz} = \gamma_{xz} = 0$ into equation (6) results in $\epsilon_z = \rho gz/c_{33}$ and c_{34} and c_{35} vanishing. Then, substituting $c_{34} = c_{35} = 0$ into equation (8), the stress components σ_x, σ_y and τ_{xy} can be expressed as follows:

$$\sigma_x = \rho gz \cdot \frac{c_{13}}{c_{33}},$$

$$\sigma_y = \rho gz \cdot \frac{c_{23}}{c_{33}},$$

$$\tau_{xy} = \rho gx \cdot \frac{c_{36}}{c_{33}}. \tag{16}$$

Unlike the stress components in equation (8), the stress components in equation (16) cannot be used for all rock mass anisotropy types and orientations. They can only be used for rock masses with planes of symmetry oriented such that c_{34} and c_{35} vanish. For such rock masses, the no lateral strain condition reduces to the

uniaxial strain condition. An example of rock mass anisotropy for which the two conditions coincide is presented below.

Special case when $\psi = 0$ or $90°$

When the dip angle ψ in Fig. 1 is equal to 0 or $90°$, the orthotropic or transversely isotropic rock mass has planes of symmetry normal to the x-, y- and z-axes. For these anisotropy orientations, the rock mass has nine or five independent elastic properties in the x, y, z coordinate system and many components of matrices (A) and (C) vanish, e.g. a_{45}, a_{46}, a_{56}, c_{45}, c_{46}, c_{56} and a_{i4}, a_{i5}, a_{i6}, c_{i4}, c_{i5}, c_{i6} for $i = 1, 2, 3$. In particular, since c_{34} and c_{35} vanish and c_{36} also vanishes, equation (16) gives the following expressions for the principal horizontal stresses σ_x and σ_y:

$$\sigma_x = \rho gz \cdot \frac{c_{13}}{c_{33}},$$

$$\sigma_y = \rho gz \cdot \frac{c_{23}}{c_{33}}. \tag{17}$$

Substituting the relations existing between the c_{ij} and a_{ij} components of matrices (C) and (A) and the nine or five elastic properties of the rock mass into equation (17), gives the expressions for the gravity-induced stresses proposed by Amadei et al. [1] for orthotropic and transversely isotropic rock masses. For an orthotropic rock mass, the stress ratios $\sigma_x/\rho gz$ and $\sigma_y/\rho gz$ are equal to:

$$\frac{\sigma_x}{\rho gz} = \frac{a_{12}a_{23} - a_{22}a_{13}}{a_{12}a_{22} - a_{12}^2} = \frac{v_{xz} + v_{yz}v_{xy}}{1 - v_{xy}v_{yx}},$$

$$\frac{\sigma_y}{\rho gz} = \frac{a_{12}a_{13} - a_{11}a_{23}}{a_{12}a_{22} - a_{12}^2} = \frac{v_{yz} + v_{yx}v_{xz}}{1 - v_{xy}v_{yx}}. \tag{18}$$

For a horizontally transversely isotropic rock mass, equation (17) gives:

$$\frac{\sigma_x}{\rho gz} = \frac{\sigma_y}{\rho gz} = v' \frac{E}{E'} \cdot \frac{1}{1-v}. \tag{19}$$

If the transverse isotropy is vertical and parallel to the y-axis of Fig. 1, equation (19) is replaced by the following:

$$\frac{\sigma_x}{\rho gz} = \frac{v'(1+v)}{1 - v'^2 \dfrac{E}{E'}}; \quad \frac{\sigma_y}{\rho gz} = \frac{v + v'^2 \dfrac{E}{E'}}{1 - v'^2 \dfrac{E}{E'}}. \tag{20}$$

Finally, for isotropic rock masses, equation (17) reduces to:

$$\frac{\sigma_x}{\rho gz} = \frac{\sigma_y}{\rho gz} = \frac{v}{1-v}. \tag{21}$$

Note that in equations (18–21), the stress components are all independent of the shear moduli of the rock. This is because the planes of symmetry of the rock coincide

with the x, y, z coordinate system when ψ is equal to 0 or 90°.

PARAMETRIC STUDY

In order to illustrate the analytical solution presented above, a parametric study was carried out to assess the effect of the elastic properties of rock strata and their orientation on gravitational stresses induced in *transversely isotropic* rock masses. In the parametric study, the geometry of Fig. 1 was adopted with planes of transverse isotropy parallel to the st-plane. The domains of variation selected for E/E', G/G', ν and ν' were based on literature surveys on elastic properties of intact anisotropic rocks conducted by Gerrard [4] and Amadei *et al.* [1].

As a numerical example, Figs 2a–c show, respectively, the variations of $\sigma_x/\rho g z$, $\sigma_y/\rho g z$ and σ_x/σ_y computed from equation (12) for several degrees of rock anisotropy with E/E' and G/G' ranging between 1 and 3, $\nu = \nu' = 0.25$ and for a dip angle ψ equal to 30°. Compared to the isotropic solution, e.g. $\sigma_x/\rho g z = \sigma_y/\rho g z = 0.333$ which is represented by point I in Figs 2a–c, both σ_x and σ_y increase with E/E' and G/G'. For a fixed value of G/G', the stresses increase as E/E' increases, that is as the rock mass becomes more deformable in directions normal to the planes of transverse isotropy. Note that for a fixed value of E/E', the stress σ_x parallel to the dip direction of the planes of transverse isotropy depends strongly on the value of G/G'. On the other hand, the stress σ_y parallel to the strike of the planes of transverse isotropy is not much affected by the value of G/G'. An increase of G/G' indicates that the rock mass becomes more deformable in shear in planes normal to the planes of transverse isotropy. For a fixed value of G/G', the stress ratio σ_x/σ_y decreases as E/E' increases.

Figures 3a–c show, respectively, the variations of $\sigma_x/\rho g z$, $\sigma_y/\rho g z$ and σ_x/σ_y with E/E' for $\nu' = 0.15$, 0.25 and 0.35, $G/G' = 1$, $\nu = 0.25$ and for a dip angle ψ equal to 30°. Compared to the isotropic solution represented by point I in Figs 3a–c, the horizontal stresses depend strongly on the value of the Poisson's ratio ν'. For a fixed value of E/E', the stresses and the stress ratio σ_x/σ_y increase with ν'. Recall that Poisson's ratio ν' controls the lateral straining in the planes of transverse isotropy resulting from a stress acting normal to those planes.

Figures 4a–c show, respectively, the variations of $\sigma_x/\rho g z$, $\sigma_y/\rho g z$ and σ_x/σ_y with the dip angle ψ for $E/E' = 1$, 2 and 3, $\nu = \nu' = 0.25$ and $G/G' = 1$. These figures indicate that values of σ_x and σ_y larger than those predicted with the isotropic solution can be induced in rock masses with shallow dipping strata. As the dip angle of the strata increases, the difference between the stresses calculated with the isotropic and anisotropic solutions decreases. A similar trend can be seen in Figs 5a–c for which G/G' is now equal to 3. Note that the values of the stress ratio σ_x/σ_y in Figs 4c and and 5c strongly depend on the dip angle of the rock strata.
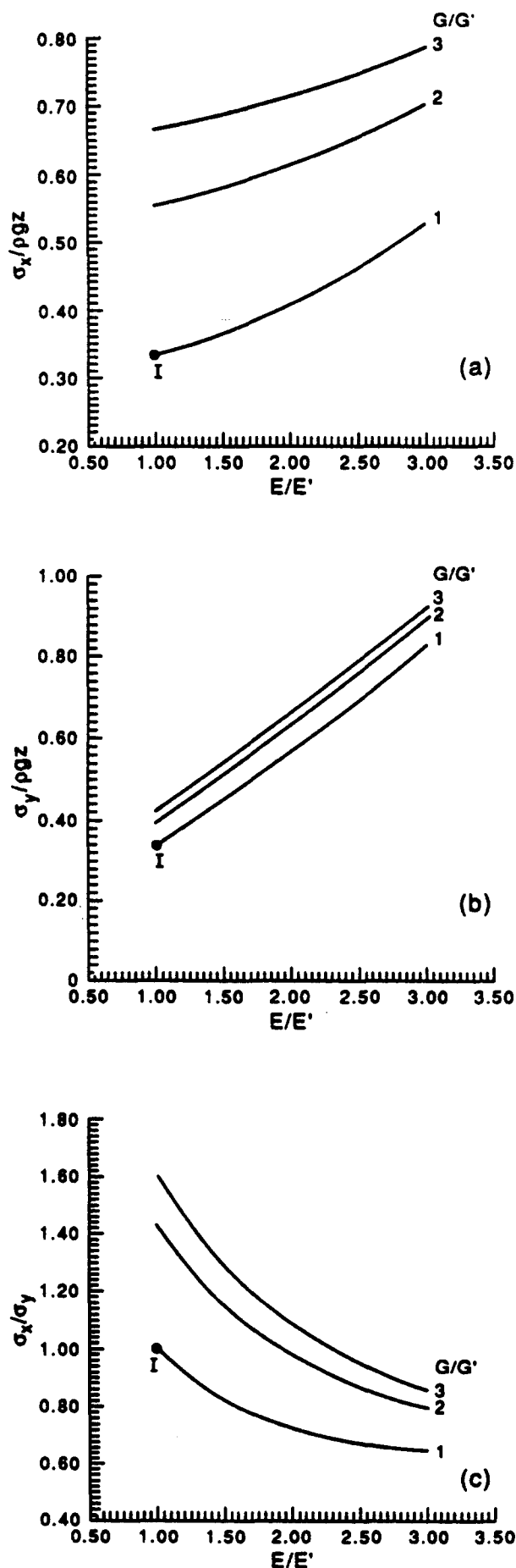

Fig. 2. Variations of: (a) $\sigma_x/\rho g z$; (b) $\sigma_y/\rho g z$; and (c) σ_x/σ_y, with E/E' for different values of G/G', $\nu = \nu' = 0.25$ and $\psi = 30°$.
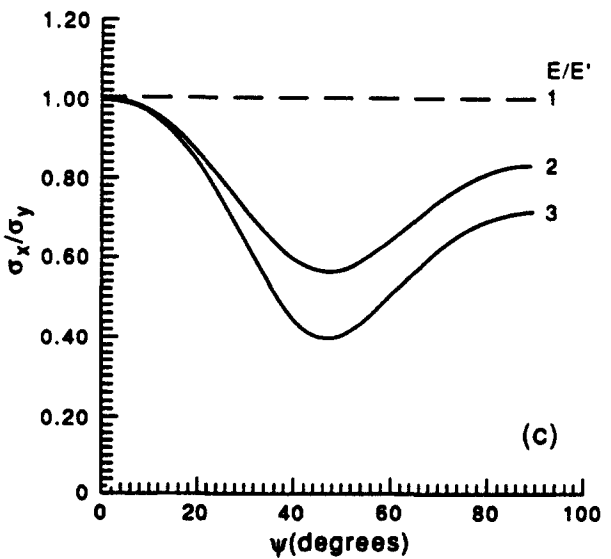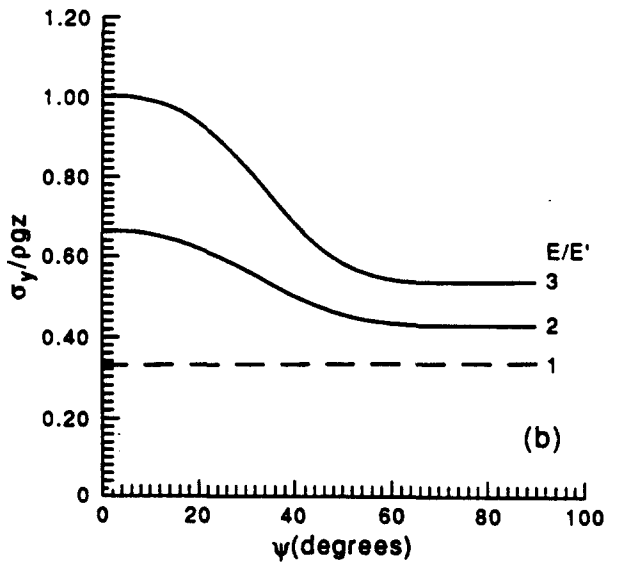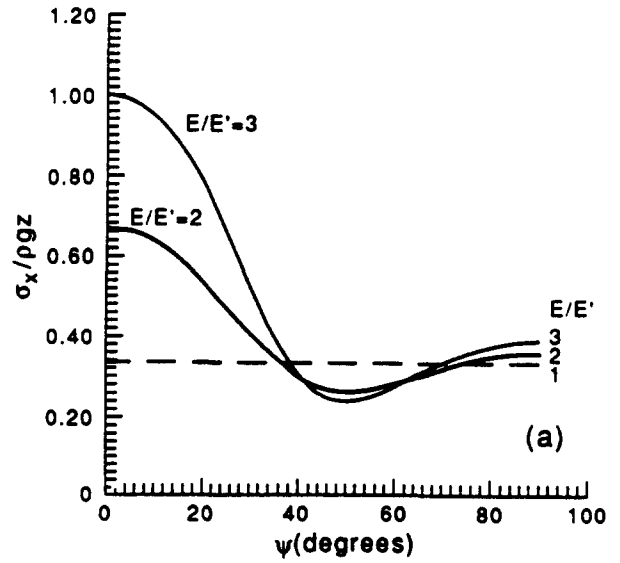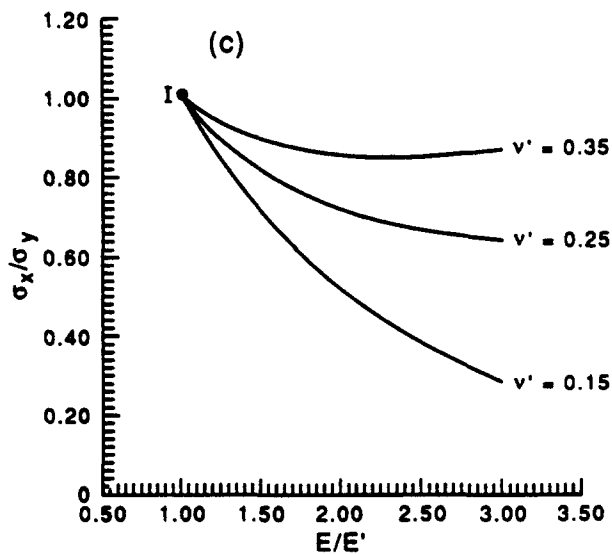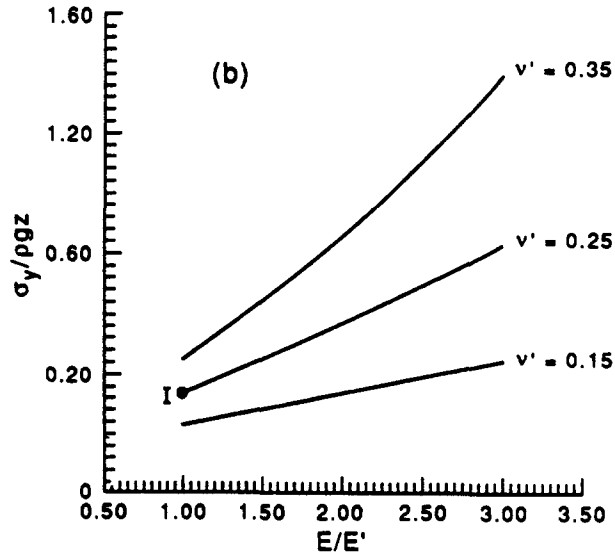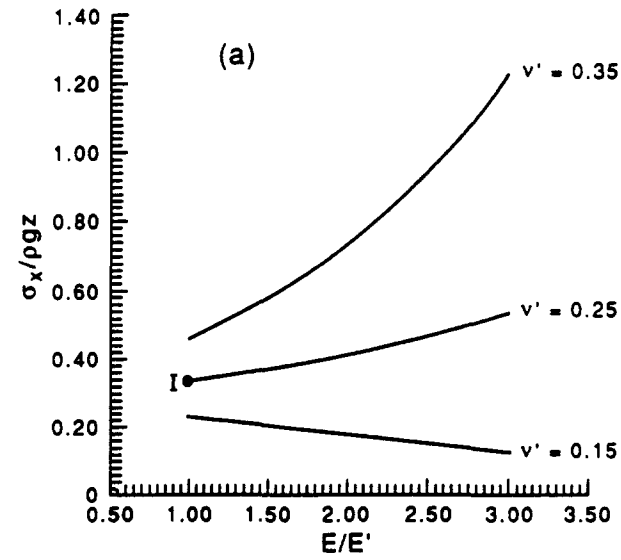
Fig. 3. Variations of: (a) $\sigma_x/\rho gz$; (b) $\sigma_y/\rho gz$; and (c) σ_x/σ_y with E/E' for $\nu' = 0.15$, 0.25 and 0.35, $G/G' = 1$, $\nu = 0.25$ and $\psi = 30°$.

Fig. 4. Variations of: (a) $\sigma_x/\rho gz$; (b) $\sigma_y/\rho gz$; and (c) σ_x/σ_y with ψ for $E/E' = 1$, 2 and 3, $\nu = \nu' = 0.25$ and $G/G' = 1$. The isotropic solution is shown as a dashed line.

Fig. 5. Variations of: (a) $\sigma_x/\rho gz$; (b) $\sigma_y/\rho gz$; and (c) σ_x/σ_y with ψ for $E/E' = 1, 2$ and 3, $v = v' = 0.25$ and $G/G' = 3$. The isotropic solution is shown as a dashed line.

TYPES OF GRAVITY-INDUCED STRESS FIELDS IN ANISOTROPIC ROCK MASSES

Domains of variation for the stresses

Figures 2–5 show that for transversely isotropic rock masses, the ratios $\sigma_x/\rho gz$, $\sigma_y/\rho gz$ and σ_x/σ_y can be less than, equal or greater than unity depending on the values of E/E', G/G', v and v' and the dip angle ψ of the rock strata. In other words, the type of gravity-induced stress field and the ordering of the principal stresses in situ depend greatly on the fabric of rock masses and their anisotropic character.

The five and nine elastic properties of transversely isotropic and orthotropic rock masses, respectively, cannot be randomly selected. Indeed, some inequalities associated with the thermodynamic constraints that the rock strain energy remains positive-definite, must be satisfied [5, 6]. These inequalities, in turn, induce constraints on the possible domains of variation for the gravitational stress components as shown by Amadei et al. [1] for orthotropic and transversely isotropic rock masses with horizontal or vertical planes of symmetry. In this section, the constraints are further discussed for transversely isotropic rock masses with inclined strata and the geometry of Fig. 1.

Recall that if a rock mass were to be modelled as a linearly elastic isotropic material, the combination of equation (21) with the thermodynamic constraint $-1 < v < 0.5$ implies that the possible domain of variation for the horizontal stress $\sigma_x = \sigma_y = \sigma_h$ induced by gravitational loading is limited. Indeed, the ratio $\sigma_h/\rho gz$ can only vary between 0 and 1 as the Poisson's ratio varies between 0 and 0.5 (negative Poisson's ratios although thermodynamically admissible have not been measured in rocks). In other words, the horizontal stress can never exceed the vertical stress at any depth z unless $v > 0.5$ which is thermodynamically inadmissible.
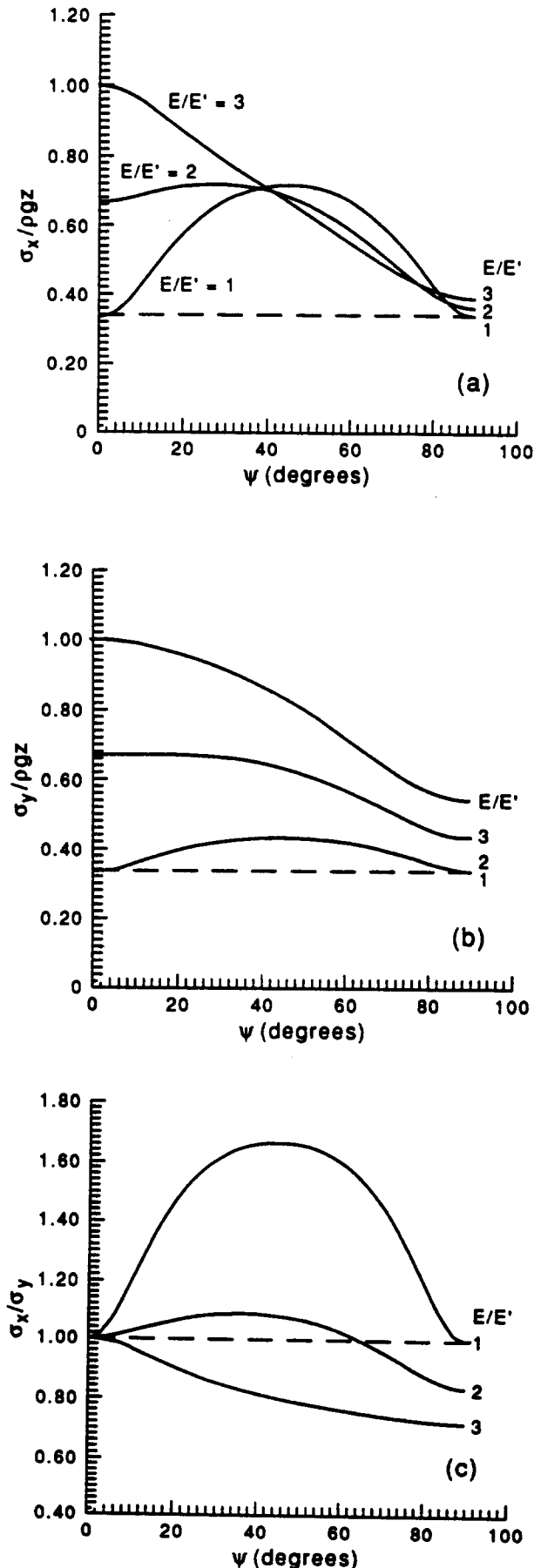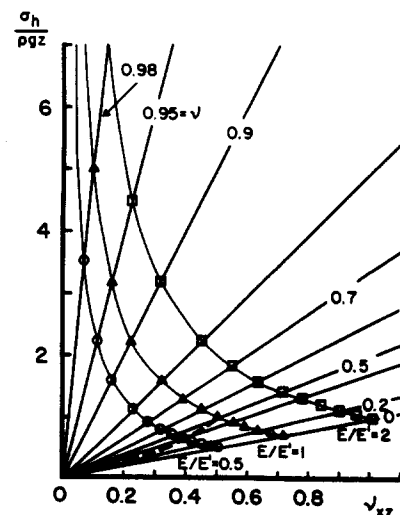

Fig. 6. Variations of the stress ratios $\sigma_h/\rho gz = \sigma_x/\rho gz$ with $v_{xz} = v'E/E'$ and v for a horizontally transverse isotropic rock mass. The limiting curves bounding the thermodynamically admissible stress domains are shown for $E/E' = 1, 2$ and 3. The isotropic solution is indicated by the dashed line (after Amadei et al. [1]).

If a rock mass is now modelled as transversely isotropic, its five elastic properties E, E', v, v' and G' must satisfy the following thermodynamic constraints [1]:

$$E, E', G' > 0, \qquad (22a)$$

$$-1 < v < 1, \qquad (22b)$$

$$-\sqrt{\frac{E'}{E} \cdot \frac{(1-v)}{2}} < v' < \sqrt{\frac{E'}{E} \cdot \frac{(1-v)}{2}}. \qquad (22c)$$

Considering only the positive part of the domains of variations for the Poisson's ratios v and v', the constraints in inequalities (22) can be substituted into equation (12) to determine the types of stress fields that are admissible in transversely isotropic rock masses.

Figure 6 shows the variation of the horizontal stress ratio $\sigma_h/\rho gz = \sigma_x/\rho gz = \sigma_y/\rho gz$ with $v'E/E'$ and v for a horizontally transverse isotropic rock mass ($\psi = 0°$). As shown by Amadei et al. [1], the horizontal stress can vary over a large region compared to the isotropic solution since the domains of variation for v and v' in inequalities (22) are not as restrictive as the domain of variation for v in the isotropic model. The region is bounded by a curve that depends on the value of E/E' and whose equation is obtained by combining the positive part of inequality (22c) with equation (19). Figure 6 shows that horizontal stresses larger than the vertical stress are admissible for horizontally layered rock masses.

Figures 7 and 8 show the domain of variation of $\sigma_x/\rho gz$ and $\sigma_y/\rho gz$ with E/E' and v' for transversely
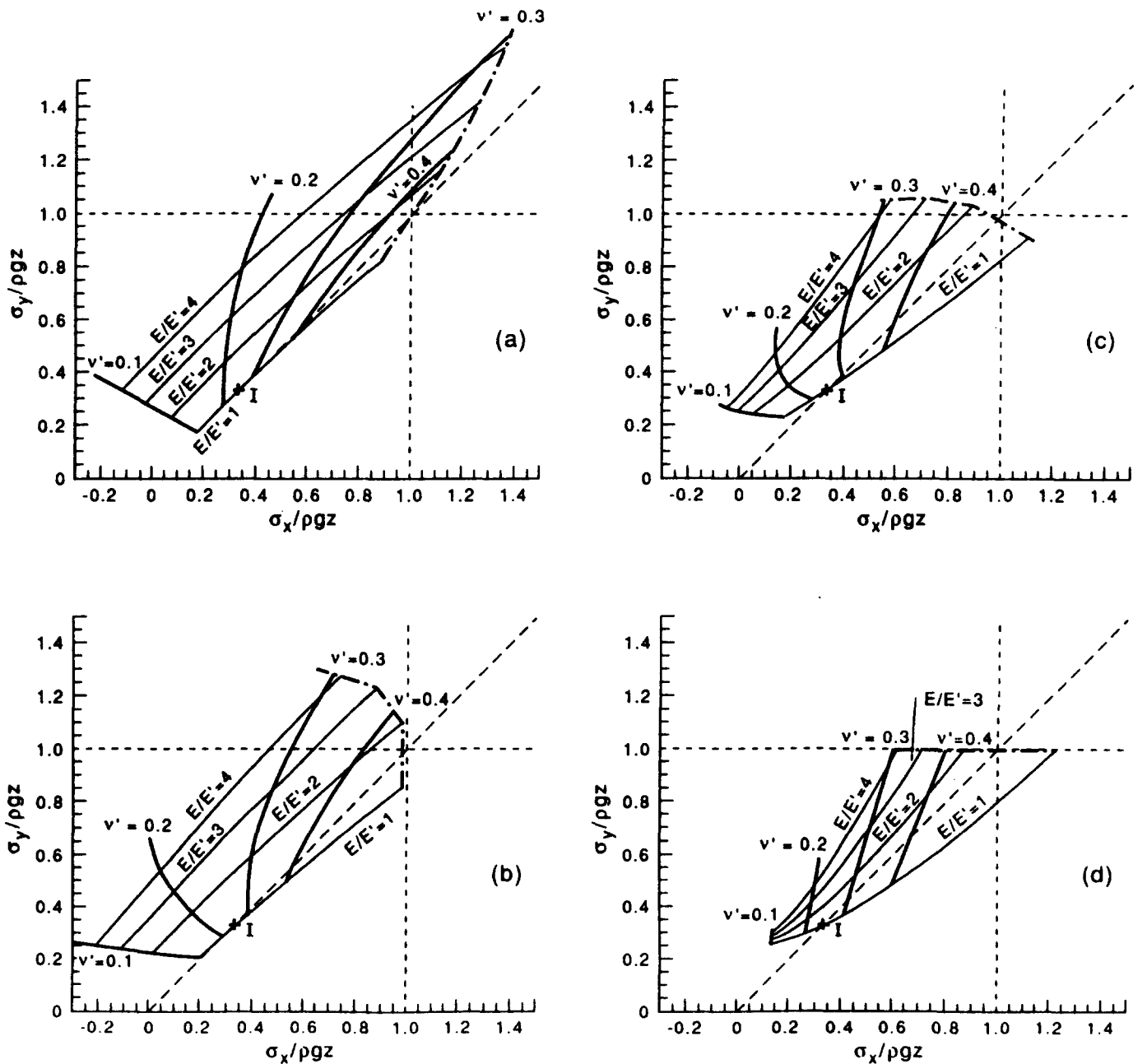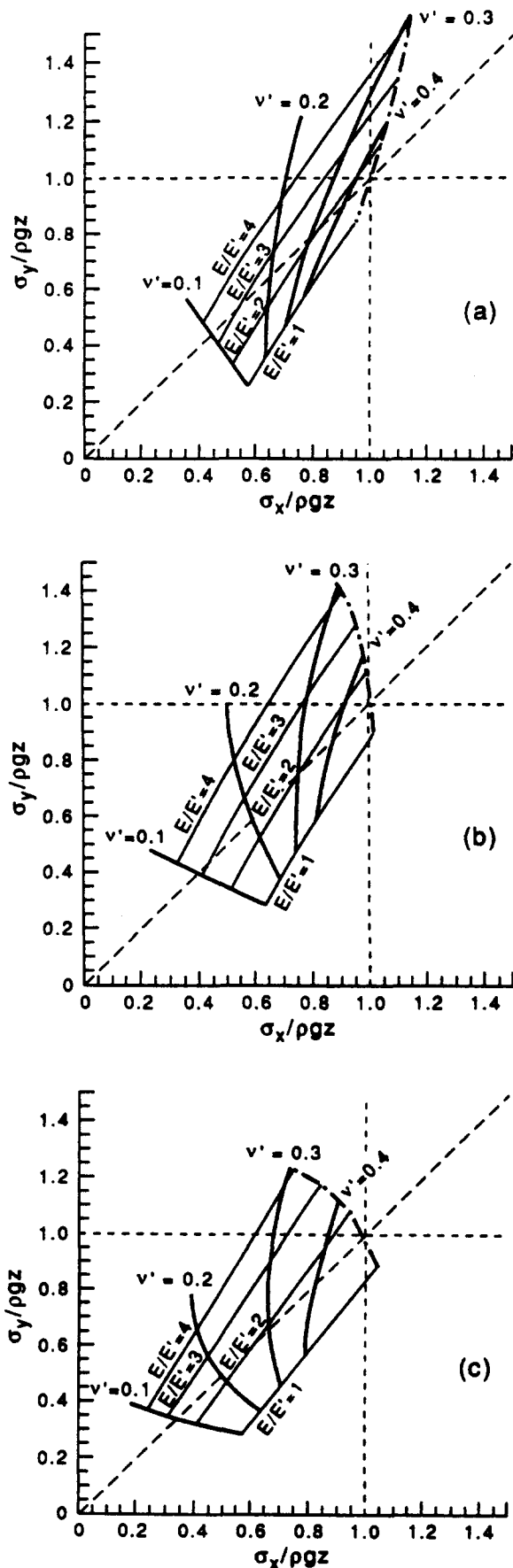

Fig. 7. Variations of the stress ratios $\sigma_x/\rho gz$ and $\sigma_y/\rho gz$ with E/E' and v' for $G/G' = 1$ and $v = 0.25$. Transversely isotropic rock masses with inclined strata dipping at angles ψ of 30, 45, 60 and 90° in (a–d), respectively. The limiting curve corresponding to the positive part of inequality (22c) is indicated by the dotted dashed line.

isotropic rock masses with inclined strata ($0 < \psi \leqslant 90°$) and for G/G' equal to 1 and 3, respectively. E/E' varies between 1 and 4, $v = 0.25$ and v' varies between 0.1 and 0.4. In Figs 7 and 8, the constraint associated with the positive part of inequality (22c) is indicated as dotted dashed lines.

Figures 7a–d show the admissible stress fields when $G/G' = 1$ and for strata dipping at angles ψ of 30, 45, 60 and 90°, respectively. It appears that the stress component σ_y acting parallel to the strata is in general larger than σ_x. However, as ψ increases, values of σ_x larger than σ_y becomes possible for values of E/E' between 1 and 2 and for Poisson's ratios v' larger than 0.3. Compared to the isotropic solution respresented by point I, horizontal stresses larger than the vertical stress $\rho g z$ are thermodynamically admissible. However, this becomes less admissible as ψ increases, that is, as the rock strata become steeper. Note also that for low values of v' and large values of E/E', tensile stresses can develop in the x-direction. This phenomenon will be discussed further in the next section.

The trends observed in Figs 7a–c can also be found in Figs 8a–c where G/G' is now equal to 3. Comparison of Figs 7 and 8 shows that the likelihood for having tensile stresses in the x-direction when E/E' ranges between 1 and 4 vanishes as G/G' increases from 1 to 3. Another parametric study not presented here has shown that for values ranging between 0.15 and 0.35, the Poisson's ratio v has little effect on the stress variations shown in Figs 7 and 8.

Tensile stresses under gravitational loading

Figure 7 shows that for a transversely isotropic rock mass, the x-component of the stress field induced by gravity could be tensile. This only takes place for rock masses with inclined strata and for certain values of the rock elastic properties, in particular v' and E/E'.

Fig. 8. Variations of the stress ratios $\sigma_x/\rho g z$ and $\sigma_y/\rho g z$ with E/E' and v' for $G/G' = 3$ and $v = 0.25$. Transversely isotropic rock masses with inclined strata dipping at angles ψ of 30, 45, and 60° in (a–c), respectively. The limiting curve corresponding to the positive part of inequality (22c) is indicated by the dotted dashed line.

Fig. 9. Range of E/E' and v' for which the stress σ_x is tensile (shaded region) when $G/G' = 1$, $v = 0.25$ and ψ ranges between 15 and 75°. The curve corresponding to the positive part of inequality (22c) is indicated as a dotted line.
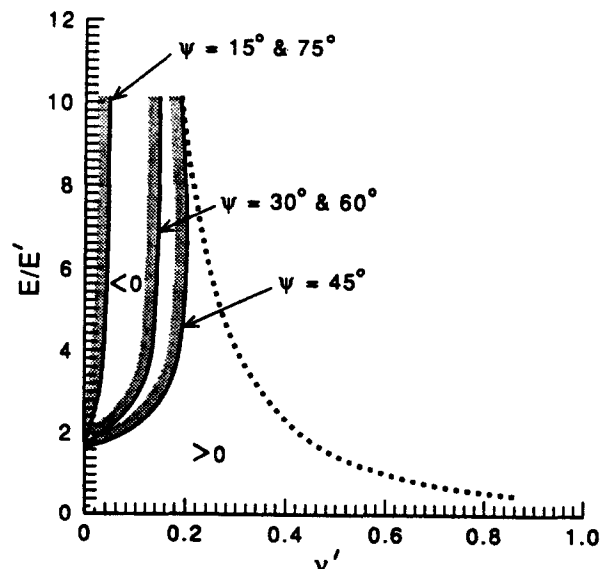
Fig. 10. Range of E/E' and v' for which the stress σ_x is tensile (shaded region) when $G/G' = 3$, $v = 0.25$ and ψ ranges between 15 and 75°. The curve corresponding to the positive part of inequality (22c) is indicated as a dotted line.
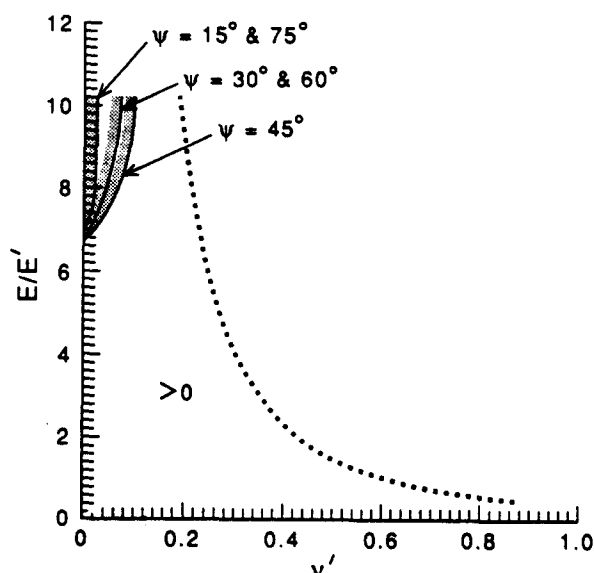
Because of its importance, the potential for developing tensile stresses under gravity alone was investigated further for transversely isotropic rock masses with E/E' ranging between 1 and 10 and v' ranging between 0 and 1.0. Figure 9 shows in a v', E/E' space, the range of elastic properties for which σ_x is tensile when $G/G' = 1$, $v = 0.25$ and for dip angles ranging between 15 and 75°. The curve corresponding to the positive part of inequality (22c) also appears in Fig. 9 as a dotted line. It appears that the range of elastic properties for which σ_x is tensile is maximum when the dip angle ψ is equal to 45°. Tensile stresses can occur for values of E/E' as low as 2 and values of v' as large as 0.2.

Figure 10 is similar to Fig. 9 but G/G' is now equal to 3. Comparison of Figs 9 and 10 indicates that as G/G' increases, tensile stresses can only occur for anisotropic rock masses with larger values of E/E' and smaller values of v'. Another parametric study not presented here has shown that for values ranging between 0.15 and 0.35, the Poisson's ratio v has little effect on the extent of the domains in Figs 9 and 10 where σ_x is tensile.

CONCLUSION

The closed-form solutions proposed in this paper can be used to predict the in situ stress field induced by gravitational loading of generally anisotropic, orthotropic and transversely isotropic homogeneous rock masses with strata that are inclined with respect to a horizontal ground surface. The stress field is multiaxial and is strongly correlated to the rock mass structure.

For all anisotropic rock masses, the vertical stress is always a principal stress and is equal to the weight of the overlying rock. Its magnitude is independent of anisotropy. The two horizontal principal stress components are not equal and their magnitude and orientation in the horizontal plane depend on the anisotropic character of the rock mass. In particular, for orthotropic and transversely isotropic rock masses with dipping strata, the horizontal stresses parallel to the strike and dip direction of the strata are always principal stresses and are not equal. They depend on the value of the dip angle and the nine or five elastic properties of orthotropic or transversely isotropic rock masses, respectively. These properties appear in the form of eight or four dimensionless quantities.

The expressions for the gravity-induced horizontal stresses differ if the horizontal displacements in an anisotropic rock mass are allowed or not. Using the assumption that the components of stress, strain and displacements, the ground surface boundary conditions and the rock mass deformability properties do not vary in the horizontal plane results in general expressions for gravity-induced stresses that can be used for all rock mass anisotropy types and orientations. This assumption makes the rock mass deform under a condition of no lateral strain but the horizontal displacements do not vanish and vary with depth only. Using the additional assumption that the two horizontal displacements must also vanish results in a uniaxial strain condition and creates constraints on the type of rock mass anisotropy for which gravity-induced stresses can be determined. The no lateral strain condition presented in this paper is not as restrictive as the uniaxial strain condition. For isotropic, transversely isotropic and orthotropic rock masses, the no lateral strain condition reduces to the uniaxial strain condition when the planes of anisotropy are either horizontal or vertical.

For anisotropic rock masses, the gravity-induced stress field is three-dimensional. Depending on the rock mass anisotropic properties and the orientation of the rock strata with respect to the ground surface, different stress states are thermodynamically admissible for which the strain energy of the rock always remains positive-definite. The horizontal stress components can be larger, equal or less than the vertical stress. For transversely isotropic rock masses, it was found in this paper that tensile horizontal stresses could develop in the dip direction of the rock strata for $v' < 0.2$, $E/E' > 2$, G/G' close to 1 and strata that are neither horizontal nor vertical. Such domains of variations for the elastic properties are not uncommon for intact anisotropic rocks. For larger values of G/G', tensile stresses were found to develop for larger values of E/E'. For instance, when $G/G' = 3$, E/E' must be at least equal to 7 for tensile stresses to appear. These conditions could probably be found near the surface or regularly jointed rock masses for which confinement is small and the anisotropy created by systems of joint surfaces is high. The possibility of generating tensile stresses under gravity alone opens new hypotheses for the formation of fractures in rock masses. This should be corroborated with field observations.

For orthotropic rock masses, the induced stress field is again multiaxial. However, the domains of variations for the stress components have not been investigated in detail in this paper due to the complex nature of

the thermodynamic constraints on the rock elastic parameters [1].

Gravity-induced stress fields in anisotropic rock masses depend on the type, degree and orientation of the rock anisotropy with respect to the ground surface. They also depend on how the rock mass properties and the degree of anisotropy vary with depth. The effect of such variations on gravity-induced stresses has already been discussed by Amadei and Savage [7] and Amadei *et al.* [2] for horizontally regularly jointed and layered rock masses. The solution presented in this paper can also be used to predict stresses in rock masses for which the deformability properties vary with depth only. This can take place if one or several of the five or nine elastic properties of transversely isotropic and orthotropic rock masses and/or the dip angle of the rock strata vary with depth. If the rock mass density ρ also varies with depth, then $\rho g z$ must be replaced by $g \int \rho \, dz$ in the expressions for the gravity stresses proposed in this paper. The effect of variations of rock mass density and deformability with depth on the *in situ* stress field is a subject of current research by the authors.

REFERENCES

1. Amadei B., Savage W. Z. and Swolfs H. S. Gravitational stresses in anisotropic rock masses. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.* **24**, 5–14 (1987).
2. Amadei B., Swolfs H. S. and Savage W. Z. Gravity induced stresses in stratified rock masses. *Rock Mech.* **21**, 1–20 (1988).
3. Terzaghi K. and Richart F. E. Stresses in rocks about cavities. *Geotechnique* **3**, 57–90 (1952).
4. Gerrard C. M. Background to mathematical modelling in geomechanics: the role of fabric and stress history. *Proc. Int. Symp. on Numerical Methods*, Karlsruhe, pp. 33–120 (1975).
5. Lempriere B. M. Poisson's ratios in orthotropic materials. *J. Am. Inst. Aeronaut. Astronaut.* **6**, 2226–2227 (1968).
6. Pickering D. J. Anisotropic elastic parameters for soils. *Geotechnique* **20**, 271–276 (1970).
7. Amadei B. and Savage W. Z. Gravitational stresses in regularly jointed rock masses. *Proc. Int. Symp. on Fundamentals of Rock Joints*, Bjorkliden, Sweden, pp. 463–473 (1985).
8. Amadei B. *Rock Anisotropy and The Theory of Stress Measurements, Lecture Notes in Engineering Series.* Springer, New York (1983).
9. Goodman R. E. *Introduction to Rock Mechanics.* Wiley, New York (1980).

APPENDIX

Matrices (A) and (H) in equations (1) and (11), respectively, are related as follows [8]:

$$(A) = (T_\sigma)^t (H)(T_\sigma), \tag{A1}$$

where (T_σ) is a (6×6) coordinate transformation matrix for stress (see Goodman [9], p. 404). Using equation (10) for matrix (H), the components a_{ij} of (A) are equal to:

$$a_{11} = \sin^2 \psi \left(\frac{\sin^2 \psi}{E_n} - \frac{v_m}{E_s} \cos^2 \psi \right) + \cos^2 \psi$$
$$\times \left(-\frac{v_m}{E_s} \sin^2 \psi + \frac{\cos^2 \psi}{E_s} \right) + \frac{\sin^2 2\psi}{4G_{ns}},$$

$$a_{12} = -\frac{v_m}{E_t} \sin^2 \psi - \frac{v_{tt}}{E_t} \cos^2 \psi,$$

$$a_{13} = \sin^2 \psi \left(\frac{\cos^2 \psi}{E_n} - \frac{v_m}{E_s} \sin^2 \psi \right) + \cos^2 \psi$$
$$\times \left(-\frac{v_m}{E_s} \cos^2 \psi + \frac{\sin^2 \psi}{E_s} \right) - \frac{\sin^2 2\psi}{4G_{ns}},$$

$$a_{23} = -\frac{v_m}{E_t} \cos^2 \psi - \frac{v_{tt}}{E_t} \sin^2 \psi,$$

$$a_{33} = \cos^2 \psi \left(\frac{\cos^2 \psi}{E_n} - \frac{v_m}{E_s} \sin^2 \psi \right) + \sin^2 \psi$$
$$\times \left(-\frac{v_m}{E_s} \cos^2 \psi + \frac{\sin^2 \psi}{E_s} \right) + \frac{\sin^2 2\psi}{4G_{ns}},$$

$$a_{15} = -\sin 2\psi \, \sin^2 \psi \left(\frac{1}{E_n} + \frac{v_m}{E_s} \right) + \sin 2\psi \, \cos^2 \psi$$
$$\times \left(\frac{1}{E_s} + \frac{v_m}{E_s} \right) - \frac{\sin 2\psi \, \cos 2\psi}{2G_{ns}},$$

$$a_{35} = -\sin 2\psi \, \cos^2 \psi \left(\frac{1}{E_n} + \frac{v_m}{E_s} \right) + \sin 2\psi \, \sin^2 \psi$$
$$\times \left(\frac{1}{E_s} + \frac{v_m}{E_s} \right) + \frac{\sin 2\psi \, \cos 2\psi}{2G_{ns}},$$

$$a_{44} = \frac{\sin^2 \psi}{G_{st}} + \frac{\cos^2 \psi}{G_{nt}},$$

$$a_{66} = \frac{\cos^2 \psi}{G_{st}} + \frac{\sin^2 \psi}{G_{nt}},$$

$$a_{46} = \sin \psi \, \cos \psi \left(\frac{1}{G_{st}} - \frac{1}{G_{nt}} \right),$$

$$a_{55} = \sin^2 2\psi \left(\frac{1}{E_n} + \frac{1}{E_s} + 2\frac{v_m}{E_s} \right) + \frac{\cos^2 2\psi}{G_{ns}},$$

$$a_{25} = \sin 2\psi \left(\frac{v_{nt}}{E_n} - \frac{v_{st}}{E_s} \right),$$

$$a_{22} = \frac{1}{E_t},$$

$$a_{14} = a_{16} = a_{24} = a_{26} = a_{34} = a_{36} = a_{45} = a_{56} = 0. \tag{A2}$$

When ψ is equal to 0 or 90°, the following coefficients also vanish: a_{15}, a_{25}, a_{35} and a_{46}.

For the geometry of Fig. 1, the components of matrices (A) and (C) are related as follows. Let D be equal to:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{35} \\ a_{51} & a_{52} & a_{53} & a_{55} \end{vmatrix}. \tag{A3}$$

Then,

$$c_{11} = \frac{1}{D} \begin{vmatrix} a_{22} & a_{23} & a_{25} \\ a_{23} & a_{33} & a_{35} \\ a_{25} & a_{35} & a_{55} \end{vmatrix},$$

$$c_{12} = -\frac{1}{D} \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ a_{23} & a_{33} & a_{35} \\ a_{25} & a_{35} & a_{55} \end{vmatrix},$$

$$c_{13} = \frac{1}{D} \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ a_{22} & a_{23} & a_{25} \\ a_{25} & a_{35} & a_{55} \end{vmatrix},$$

$$c_{15} = -\frac{1}{D} \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ a_{22} & a_{23} & a_{25} \\ a_{23} & a_{33} & a_{35} \end{vmatrix},$$

$$c_{22} = \frac{1}{D} \begin{vmatrix} a_{11} & a_{13} & a_{15} \\ a_{31} & a_{33} & a_{35} \\ a_{15} & a_{35} & a_{55} \end{vmatrix},$$

$$c_{23} = -\frac{1}{D} \begin{vmatrix} a_{11} & a_{13} & a_{15} \\ a_{21} & a_{23} & a_{25} \\ a_{51} & a_{53} & a_{55} \end{vmatrix},$$

$$c_{25} = \frac{1}{D} \begin{vmatrix} a_{11} & a_{13} & a_{15} \\ a_{21} & a_{23} & a_{25} \\ a_{13} & a_{33} & a_{35} \end{vmatrix},$$

$$c_{33} = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{51} & a_{52} & a_{55} \end{vmatrix},$$

$$c_{35} = -\frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{13} & a_{32} & a_{35} \end{vmatrix},$$

$$c_{55} = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \qquad \text{(A4)}$$

and

$$c_{44} = -\frac{a_{66}}{a_{46}^2 - a_{44} a_{66}},$$

$$c_{66} = -\frac{a_{44}}{a_{46}^2 - a_{44} a_{66}},$$

$$c_{46} = \frac{a_{46}}{a_{46}^2 - a_{44} a_{66}}. \qquad \text{(A5)}$$

Also,

$$c_{14} = c_{16} = c_{24} = c_{26} = c_{34} = c_{36} = c_{45} = c_{56} = 0. \qquad \text{(A6)}$$