STRESSES IN ANISOTROPIC ROCK MASS WITH IRREGULAR TOPOGRAPHY

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ABSTRACT: This paper presents a new analytical method for determining the state of stress in a homogeneous, general anisotropic, and elastic half-space limited by an irregular and smooth outer boundary. The half-space represents a rock mass with an irregular and continuous topography. The rock mass is subject to gravity and surface tractions. The stresses are determined assuming a condition of generalized plane strain, and are expressed in terms of three analytical functions following Lekhnitskii's complex function method. These analytical functions are determined using a numerical conformal mapping method and an integral equation method. As an illustrative example, it is shown how the proposed method can be used to determine the state of stress in long isolated and symmetric ridges and valleys in orthotropic or transversely isotropic rock masses. It is found that the magnitude of the stresses is of the order of the characteristic stress $\rho g|b|$, where ρ is the rock density, g is the gravitational acceleration, and $|b|$ is the height of the ridge or depth of the valley.

INTRODUCTION

Determination of the stress field due to gravity in a half-space is of importance in many geoscience related fields, such as geophysics, geomechanics, and glaciology. For rocks, Terzaghi and Richart (1952) first proposed closed-form solutions for gravitational stresses in laterally restrained horizontal rock masses with homogeneous, linearly elastic, and isotropic properties. Later on, Amadei et al. (1987) modified those closed-form solutions for orthotropic and transversely isotropic rock masses with horizontal or vertical anisotropy. Both homogeneous and stratified rock masses were considered. More recently, Amadei and Pan (1992) proposed closed-form solutions for gravitational stresses in generally anisotropic, orthotropic, and transversely isotropic rock masses with inclined strata.

All the closed-form solutions just mentioned are limited to rock masses with a horizontal ground surface. The effect of surface topography on gravitational stresses has been addressed in the past using two types of analytical methods. One is the exact conformal mapping method, as studied by Perloff et al. (1967), Ter-Martirosyan et al. (1974), by Savage et al. (1985), and Savage and Swolfs (1986). However, this approach is restricted to isotropic media, to a very few topographic features for which conformal mapping functions can be found exactly, and to two-dimensional problems. The other approach for two- and three-dimensional problems in isotropic media is the perturbation method discussed by McTigue and Mei (1981, 1987) and Liu and Zoback (1992). Liao et al. (1992) also used the perturbation method for two-dimensional problems in anisotropic media. The advantage of the perturbation method is that it can handle any smooth topographic feature.

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However, the solutions derived with that method are restricted to topographies with small slopes not exceeding 10%.

In this paper, we present a new analytical method for determining the state of stress in a homogeneous, general anisotropic, and elastic half-space limited by an irregular and smooth outer boundary. The half-space represents a rock mass with an irregular and continuous topography that is subject to gravity and surface loads. The stresses are determined assuming a condition of generalized plane strain. The solutions presented in this paper first make use of the closed-form solutions for the stresses in anisotropic rock masses with a horizontal ground surface proposed by Amadei and Pan (1992). Then, the analytical function method of Lekhnitskii (1963) is followed to obtain expressions for the stresses in an anisotropic half-space with an irregular boundary. The stresses depend on three analytical functions that are determined using a numerical conformal mapping method (Trummer 1986) and an integral equation method (Muskhelishvili 1972). The numerical conformal mapping method, to the writers' knowledge, has seldom been applied in the past to solve elasticity problems. Finally, numerical examples are presented for the stress distribution in a long isolated ridge consisting of transversely isotropic rock.

STATEMENT OF PROBLEM

Consider the equilibrium of an anisotropic half-space with the geometry of Fig. 1. The half-space represents a rock mass with an irregular topography. The medium in the half-space is assumed to be linearly elastic, homogeneous, anisotropic, and continuous, with a uniform density ρ. It is subjected to gravity, $\mathbf{g}$, and surface tractions, with components t_x, t_y, and t_z in a x, y, z coordinate system attached to the half-space such that the x and z axes are in the horizontal plane and the y axis is pointing upward. The half-space geometry, the surface tractions, and the medium's elastic properties are assumed to be independent of the z direction. The boundary

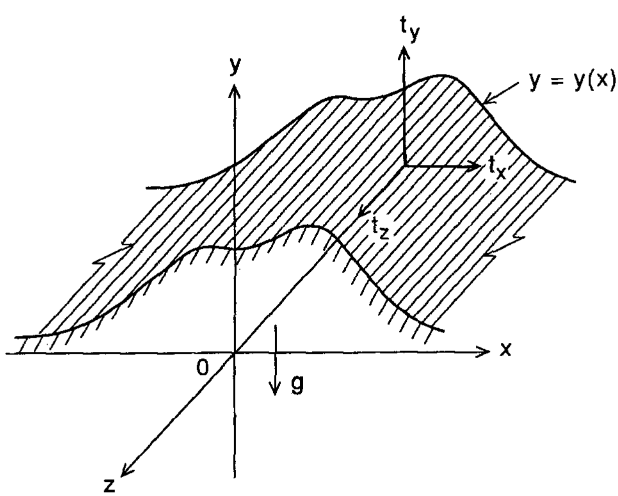

FIG. 1. Geometry of Problem; Half Space Limited by Boundary Curve $y = y(x)$ and Subject to Gravity, g, and Surface Tractions t_x, t_y and t_z

curve of the half-space is defined by an analytic function $y = y(x)$ or in parametric form $x = x(t)$, $y = y(t)$.

The problem is to find the magnitude and distribution of the stresses induced by gravity and surface loading of the half-space. Since the geometry of the problem is independent of the z coordinate and the medium is homogeneous, the stresses can be determined assuming a condition of generalized plane strain, e.g., all planes normal to the z axis are assumed to warp identically. The stresses and strains induced by gravity and the surface loads must satisfy the following equations:

Equations of Equilibrium

$$\sigma_{xx,x} + \sigma_{xy,y} = 0 \qquad (1a)$$

$$\sigma_{xy,x} + \sigma_{yy,y} - \rho g = 0 \qquad (1b)$$

$$\sigma_{xz,x} + \sigma_{yz,y} = 0 \qquad (1c)$$

where the derivative with respect to the coordinate variable is expressed by a subscript prime followed by the variable.

Constitutive Relations

$$\mathbf{e} = \mathbf{a}\boldsymbol{\sigma} \qquad (2)$$

or

$$\boldsymbol{\sigma} = \mathbf{c}\mathbf{e} \qquad (3)$$

where

$$\mathbf{e} = (e_{xx}, e_{yy}, e_{zz}, 2e_{yz}, 2e_{xz}, 2e_{xy})^T \qquad (4)$$

are the strain components, and

$$\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy})^T \qquad (5)$$

are the stress components; $\mathbf{a} = 6 \times 6$ symmetric compliance matrix with 21 independent components $a_{ij}(i, j = 1\text{–}6)$; and $\mathbf{c}$ = corresponding stiffness matrix with components $c_{ij}(i, j = 1\text{–}6)$, such that $\mathbf{a} = \mathbf{c}^{-1}$, the inverse of $\mathbf{c}$. In (4) and (5), the superscript T indicates the transpose of the matrix. In this paper, tensile normal stresses are taken to be positive.

Compatibility Conditions

$$e_{xz,y} - e_{yz,x} = 0 \qquad (6a)$$

$$e_{xx,yy} + e_{yy,xx} = 2e_{xy,xy} \qquad (6b)$$

Boundary Conditions on $y = y(x)$

$$\sigma_{xx}\cos(n, x) + \sigma_{xy}\cos(n, y) = t_x \qquad (7a)$$

$$\sigma_{xy}\cos(n, x) + \sigma_{yy}\cos(n, y) = t_y \qquad (7b)$$

$$\sigma_{xz}\cos(n, x) + \sigma_{yz}\cos(n, y) = t_z \qquad (7c)$$

where $\cos(n, x)$ and $\cos(n, y)$ = direction cosines of the outward normal, $\mathbf{n}$, of the boundary curve $y = y(x)$; and t_x, t_y and t_z = given boundary tractions.

FORMULATION

Following the approach used by Savage et al. (1985), the stress field is expressed as the sum of two parts:

$$\sigma_{ij} = \sigma_{ij}^h + \sigma_{ij}^p \quad (8)$$

where σ_{ij}^h satisfies the equations of equilibrium (1) without gravity, and σ_{ij}^p = particular solution of those equations. Obviously, the total stress field σ_{ij} is also required to satisfy the constitutive relations [(2) or (3)], the compatibility conditions (6), and the boundary conditions (7).

As shown in a recent paper by Amadei and Pan (1992), the stress components σ_{ij}^p can be expressed as

$$\sigma_{xx}^p = c_1 \rho g y \quad (9a)$$

$$\sigma_{yy}^p = \rho g y \quad (9b)$$

$$\sigma_{xz}^p = c_2 \rho g y \quad (9c)$$

$$\sigma_{zz}^p = c_3 \rho g y \quad (9d)$$

$$\sigma_{xy}^p = \sigma_{yz}^p = 0 \quad (9e)$$

where

$$c_1 = \frac{1}{D}\left[c_{12}(c_{44}c_{66} - c_{46}^2) - c_{14}(c_{24}c_{66} - c_{26}c_{46}) + c_{16}(c_{24}c_{46} - c_{26}c_{44})\right] \quad (10a)$$

$$c_2 = \frac{1}{D}\left[c_{25}(c_{44}c_{66} - c_{46}^2) - c_{45}(c_{24}c_{66} - c_{26}c_{46}) + c_{56}(c_{24}c_{46} - c_{26}c_{44})\right] \quad (10b)$$

$$c_3 = \frac{1}{D}\left[c_{23}(c_{44}c_{66} - c_{46}^2) - c_{34}(c_{24}c_{66} - c_{26}c_{46}) + c_{36}(c_{24}c_{46} - c_{26}c_{44})\right] \quad (10c)$$

and

$$D = c_{22}(c_{44}c_{66} - c_{46}^2) - c_{24}(c_{42}c_{66} - c_{62}c_{46}) + c_{26}(c_{24}c_{46} - c_{26}c_{44}) \quad (11)$$

The stress field defined in (9) was derived for a flat-lying horizontal half-space under gravity alone, assuming that the displacement components are uniform in the horizontal xz plane (no lateral strain condition). The stress field was obtained from the displacements and therefore satisfies the constitutive relations and the compatibility conditions. Also, it is easy to show that this stress field is a solution of the inhomogeneous (1).

The homogeneous solution σ_{ij}^h must satisfy the homogeneous part of the equations of equilibrium (1), the constitutive relations (2) or (3), the compatibility conditions (6), and be such that when combined with the particular solution (9), the total stresses σ_{ij} satisfy the imposed traction boundary conditions (7) on the curve $y = y(x)$. To find such a homogeneous solution, the complex function method developed by Lekhnitskii (1963) was applied first. The stress components σ_{ij}^h are assumed to take the following form:

$$\sigma_{xx}^h = F_{,yy} \quad (12a)$$

$$\sigma^h_{yy} = F_{,xx} \qquad (12b)$$

$$\sigma^h_{xy} = -F_{,xy} \qquad (12c)$$

$$\sigma^h_{xz} = \Psi_{,y} \qquad (12d)$$

$$\sigma^h_{yz} = -\Psi_{,x} \qquad (12e)$$

$$\sigma^h_{zz} = -\frac{1}{a_{33}}(a_{13}\sigma^h_{xx} + a_{23}\sigma^h_{yy} + a_{34}\sigma^h_{yz} + a_{35}\sigma^h_{xz} + a_{36}\sigma^h_{xy}) \qquad (12f)$$

where the two stress functions $F(x, y)$ and $\Psi(x, y)$ must be determined. In (12f), the stress component σ^h_{zz} is calculated assuming no strain in the z direction (generalized plane-strain condition). Substituting (12) into the constitutive relation (2) and then into the compatibility conditions (6) results in the following two coupled equations:

$$L_4F + L_3\Psi = 0 \qquad (13a)$$

$$L_3F + L_2\Psi = 0 \qquad (13b)$$

where L_2, L_3, and L_4 = differential operators given by

$$L_2 = \beta_{44}\partial_{xx} - 2\beta_{45}\partial_{xy} + \beta_{55}\partial_{yy} \qquad (14a)$$

$$L_3 = -\beta_{24}\partial_{xxx} + (\beta_{25} + \beta_{46})\partial_{xxy} - (\beta_{14} + \beta_{56})\partial_{xyy} + \beta_{15}\partial_{yyy} \qquad (14b)$$

$$L_4 = \beta_{22}\partial_{xxxx} - 2\beta_{26}\partial_{xxxy} + (2\beta_{12} + \beta_{66})\partial_{xxyy} - 2\beta_{16}\partial_{xyyy} + \beta_{11}\partial_{yyyy} \qquad (14c)$$

and β_{ij} is related to a_{ij} as follows

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \qquad i, j = 1, 2, 4, 5, 6 \qquad (15)$$

The general solution to (13) was proposed by Lekhnitskii (1963) and can be expressed using three analytical functions $F_k(z_k)$ ($k = 1, 2, 3$), such that

$$F = 2\,\mathrm{Re}[F_1(z_1) + F_2(z_2) + F_3(z_3)] \qquad (16a)$$

$$\Psi = 2\,\mathrm{Re}\left[\lambda_1F_1'(z_1) + \lambda_2F_2'(z_2) + \frac{1}{\lambda_3}F_3'(z_3)\right] \qquad (16b)$$

where

$$z_k = x + \mu_k y \qquad k = 1, 2, 3 \qquad (17)$$

and

$$\lambda_1 = -\frac{l_3(\mu_1)}{l_2(\mu_1)} \qquad (18a)$$

$$\lambda_2 = -\frac{l_3(\mu_2)}{l_2(\mu_2)} \qquad (18b)$$

$$\lambda_3 = -\frac{l_3(\mu_3)}{l_4(\mu_3)} \qquad (18c)$$

with

$$l_2(\mu) = \beta_{55}\mu^2 - 2\beta_{45}\mu + \beta_{44} \tag{19a}$$

$$l_3(\mu) = \beta_{15}\mu^3 - (\beta_{14} + \beta_{56})\mu^2 + (\beta_{25} + \beta_{46})\mu - \beta_{24} \tag{19b}$$

$$l_4(\mu) = \beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22} \tag{19c}$$

μ_k ($k = 1, 2, 3$) are the three complex roots with positive imaginary parts of the following equation:

$$l_4(\mu)l_2(\mu) - l_3^2(\mu) = 0 \tag{20}$$

Finally, $F_k'(z_k)$ denote the derivatives of $F_k(z_k)$ with respect to the complex variables z_k for $k = 1, 2$, and 3.

Substituting (16) into (12) gives the following expressions for the stresses σ_{ij}^h:

$$\sigma_{xx}^h = 2\,\mathrm{Re}[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2) + \mu_3^2\lambda_3\Phi_3'(z_3)] \tag{21a}$$

$$\sigma_{yy}^h = 2\,\mathrm{Re}[\Phi_1'(z_1) + \Phi_2'(z_2) + \lambda_3\Phi_3'(z_3)] \tag{21b}$$

$$\sigma_{xy}^h = -2\,\mathrm{Re}[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2) + \mu_3\lambda_3\Phi_3'(z_3)] \tag{21c}$$

$$\sigma_{xz}^h = 2\,\mathrm{Re}[\mu_1\lambda_1\Phi_1'(z_1) + \mu_2\lambda_2\Phi_2'(z_2) + \mu_3\Phi_3'(z_3)] \tag{21d}$$

$$\sigma_{yz}^h = -2\,\mathrm{Re}[\lambda_1\Phi_1'(z_1) + \lambda_2\Phi_2'(z_2) + \Phi_3'(z_3)] \tag{21e}$$

$$\sigma_{zz}^h = -\frac{1}{a_{33}}(a_{13}\sigma_{xx}^h + a_{23}\sigma_{yy}^h + a_{34}\sigma_{yz}^h + a_{35}\sigma_{xz}^h + a_{36}\sigma_{xy}^h) \tag{21f}$$

where

$$\Phi_\alpha(z_\alpha) = F_\alpha'(z_\alpha) \qquad \alpha = 1, 2 \tag{22a}$$

$$\Phi_3(z_3) = \frac{F_3'(z_3)}{\lambda_3} \tag{22b}$$

are three new analytical functions that will be determined from the boundary conditions (7) as discussed in the following section.

First, substituting (9) and (12) into (8), the total stress components are equal to

$$\sigma_{xx} = F_{,yy} + c_1\rho g y \tag{23a}$$

$$\sigma_{yy} = F_{,xx} + \rho g y \tag{23b}$$

$$\sigma_{xy} = -F_{,xy} \tag{23c}$$

$$\sigma_{xz} = \Psi_{,y} + c_2\rho g y \tag{23d}$$

$$\sigma_{yz} = -\Psi_{,x} \tag{23e}$$

$$\sigma_{zz} = -\frac{1}{a_{33}}(a_{13}\sigma_{xx}^h + a_{23}\sigma_{yy}^h + a_{34}\sigma_{yz}^h + a_{35}\sigma_{xz}^h + a_{36}\sigma_{xy}^h) + c_3\rho g y \tag{23f}$$

Second, substituting these stress components into (7) and using the relations

$$\cos(n, y) = x'(s) \tag{24a}$$

$$\cos(n, x) = -y'(s) \tag{24b}$$

the boundary conditions take the following form when evaluated at the surface:

$$(F_{,yy} + c_1\rho g y)y'(s) + F_{,xy}x'(s) = -t_x \quad \text{(25a)}$$

$$F_{,xy}y'(s) + (F_{,xx} + \rho g y)x'(s) = t_y \quad \text{(25b)}$$

$$(\Psi_{,y} + c_2\rho g y)y'(s) + \Psi_{,x}x'(s) = -t_z \quad \text{(25c)}$$

where s = arc length along the boundary curve $y = y(x)$; and $x'(s)$ and $y'(s)$ = total derivatives of x and y with respect to s, respectively.

Finally, integrating (25) with respect to s, and using (16) and (22), the three unknown analytical functions $\Phi_k(z_k)$ $(k = 1, 2, 3)$ appearing in (21) must satisfy the following system of three equations:

$$2\,\mathrm{Re}[\Phi_1(z_1) + \Phi_2(z_2) + \lambda_3\Phi_3(z_3)] = -\rho g \int_0^s yx'(s)\,ds + \int_0^s t_y\,ds \quad \text{(26a)}$$

$$2\,\mathrm{Re}[\mu_1\Phi_1(z_1) + \mu_2\Phi_2(z_2) + \mu_3\lambda_3\Phi_3(z_3)] = -c_1\rho g \int_0^s yy'(s)\,ds - \int_0^s t_x\,ds \quad \text{(26b)}$$

$$2\,\mathrm{Re}[\lambda_1\Phi_1(z_1) + \lambda_2\Phi_2(z_2) + \Phi_3(z_3)] = -c_2\rho g \int_0^s yy'(s)\,ds - \int_0^s t_z\,ds \quad \text{(26c)}$$

Note that in deriving (26*a*)–(26*c*), the three constants of integration were set equal to zero because only simply connected contours are considered here (Lekhnitskii 1963).

In summary, the stresses in an anisotropic half-space bounded by a curve with equation $y = y(x)$ are obtained by adding (9) and (21). The three analytical functions $\Phi_k'(z_k)$ $(k = 1, 2, 3)$ appearing in (21) can be determined from the boundary conditions (26*a*)–(26*c*). It is obvious that the determination of these analytical functions depends mainly upon the geometry of the boundary curve $y = y(x)$ and the applied loads acting on the boundary of the half-space. Our next task is therefore to discuss the problem related to the determination of these three analytical functions.

CONFORMAL MAPPING AND INTEGRAL EQUATION METHODS

Eqs. (26*a*)–(26*c*) actually represent a representative problem, which is to find three functions that are analytical in a region and have given values on the boundary of that region. One of the common representative theorems for two-dimensional regions is the Cauchy integral theorem (Muskhelishvili 1953). The latter states that if $f(z)$ is an analytical function of the complex variable $z = x + iy$ in the lower half-plane $y \leq 0$ with $f(\infty) = 0$, then

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{t - z}\,dt = -f(z) \quad \text{(27a)}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{f(t)}}{t - z}\,dt = 0 \quad \text{(27b)}$$

where an overbar is used to denote the corresponding conjugate complex value.

Without loss of generality, it is assumed that the boundary curve of the half-space can always be expressed as follows:

$$y = y(x) \quad \text{(28)}$$

or in parametric form as

$$x = x(t) \quad \text{(29}a\text{)}$$

$$y = y(t) \equiv y[x(t)] \quad \text{(29}b\text{)}$$

Again, without loss of generality, the parameter t is assumed to vary from $-\infty$ to $+\infty$. In parametric form, the complex expression for the boundary curve will be

$$z(t) = x(t) + iy(t) \quad \text{(30)}$$

Similarly, the complex variables z_k ($k = 1, 2, 3$) appearing in (26) will have the following values on the boundary curve:

$$z_k(t) = x(t) + \mu_k y(t) \qquad k = 1, 2, 3 \quad \text{(31)}$$

If, for the boundary curve defined in (29), we can find a set of conformal mappings

$$z_k = Z_k(\zeta_k) \qquad k = 1, 2, 3 \quad \text{(32)}$$

that map the lower half-planes bounded by (31) onto the lower flat half-planes $\operatorname{Im} \zeta_k \leq 0$ ($k = 1, 2, 3$), then (26) can be reduced to a system of integral equations for three new analytical functions $\Psi_k(\zeta_k)$ ($k = 1, 2, 3$). Actually, by mapping onto the ζ_k planes, the three stress functions Φ_k are replaced by the new analytical functions Ψ_k such that

$$\Phi_k(z_k) = \Phi_k[Z_k(\zeta_k)] \equiv \Psi_k(\zeta_k) \qquad k = 1, 2, 3 \quad \text{(33)}$$

Also, the derivative of Ψ_k with respect to ζ_k is

$$\Psi_k'(\zeta_k) = \Phi_k'(z_k)Z_k'(\zeta_k) \qquad k = 1, 2, 3 \quad \text{(34)}$$

Let t_k be the value of ζ_k on the boundary curve ($\operatorname{Im} \zeta_k = 0$), then (34) reduces to

$$\Psi_k'(t_k) = \Phi_k'(z_k)Z_k'(t_k) \qquad k = 1, 2, 3 \quad \text{(35)}$$

where z_k is defined in (31).

Differentiating both sides of (26) with respect to the parameter t, the boundary conditions are now equal to

$$\Psi_1'(t_1)t_1'(t) + \Psi_2'(t_2)t_2'(t) + \lambda_3\Psi_3'(t_3)t_3'(t) + \overline{\Psi_1'(t_1)}t_1'(t) + \overline{\Psi_2'(t_2)}t_2'(t)$$
$$+ \overline{\lambda_3\Psi_3'(t_3)}t_3'(t) = -\rho g y(t)x'(t) + t_y(t)s'(t) \quad \text{(36}a\text{)}$$

$$\mu_1\Psi_1'(t_1)t_1'(t) + \mu_2\Psi_2'(t_2)t_2'(t) + \mu_3\lambda_3\Psi_3'(t_3)t_3'(t) + \overline{\mu_1\Psi_1'(t_1)}t_1'(t)$$
$$+ \overline{\mu_2\Psi_2'(t_2)}t_2'(t) + \overline{\mu_3\lambda_3\Psi_3'(t_3)}t_3'(t) = -c_1\rho g y(t)y'(t) - t_x(t)s'(t) \quad \text{(36}b\text{)}$$

$$\lambda_1\Psi_1'(t_1)t_1'(t) + \lambda_2\Psi_2'(t_2)t_2'(t) + \Psi_3'(t_3)t_3'(t) + \overline{\lambda_1\Psi_1'(t_1)}t_1'(t) + \overline{\lambda_2\Psi_2'(t_2)}t_2'(t)$$

$$+\overline{\Psi_3'(t_3)}t_3'(t) = -c_2\rho g y(t)y'(t) - t_z(t)s'(t) \quad \text{(36c)}$$

where $x'(t)$, $y'(t)$, $s'(t)$, $t_1'(t)$, $t_2'(t)$, and $t_3'(t)$ = total derivatives of the real functions x, y, s, t_1, t_2, and t_3 (with respect to t), respectively.

Because analytical functions remain analytical after conformal mapping, the three functions $\Psi_k'(k = 1, 2, 3)$ in (36) are analytical in the lower half-planes Im $\zeta_k \le 0$. By applying the Cauchy integrals similar to the formulas (27), after some manipulations we obtain the following system of three integral equations:

$$b_{11}\Psi_1'(\zeta_1) + \frac{b_{12}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_2'(t_2)t_2'(t_1)\,dt_1}{t_1-\zeta_1} + \frac{b_{13}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_3'(t_3)t_3'(t_1)\,dt_1}{t_1-\zeta_1}$$

$$= \frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{f_1(t)t'(t_1)\,dt_1}{t_1-\zeta_1} \quad \text{(37a)}$$

$$b_{21}\Psi_2'(\zeta_2) + \frac{b_{22}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_1'(t_1)t_1'(t_2)\,dt_2}{t_2-\zeta_2} + \frac{b_{23}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_3'(t_3)t_3'(t_2)\,dt_2}{t_2-\zeta_2}$$

$$= \frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{f_2(t)t'(t_2)\,dt_2}{t_2-\zeta_2} \quad \text{(37b)}$$

$$b_{31}\Psi_3'(\zeta_3) + \frac{b_{32}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_1'(t_1)t_1'(t_3)\,dt_3}{t_3-\zeta_3} + \frac{b_{33}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_2'(t_2)t_2'(t_3)\,dt_3}{t_3-\zeta_3}$$

$$= \frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{f_3(t)t'(t_3)\,dt_3}{t_3-\zeta_3} \quad \text{(37c)}$$

where the coefficients b_{ij} $(i, j = 1, 2, 3)$ and the functions $f_i(t)$ $(i = 1, 2, 3)$ are given by (69) and (70) in Appendix I.

Let ζ_k approach the boundary from the lower half-planes, and let's make use of the Plemelj formulae (Muskhelishvili 1972). Eqs. (37*a*)–(37*c*) then reduce to a system of three singular integral equations with Cauchy kernels

$$b_{11}\Psi_1'(\tau_1) + \frac{b_{12}}{2}\Psi_2'(\tau_2)t_2'(\tau_1) + \frac{b_{13}}{2}\Psi_3'(\tau_3)t_3'(\tau_1) + \frac{b_{12}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_2'(t_2)t_2'(t_1)\,dt_1}{t_1-\tau_1}$$

$$+ \frac{b_{13}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_3'(t_3)t_3'(t_1)\,dt_1}{t_1-\tau_1} = \frac{f_1(\tau)t'(\tau_1)}{2} + \frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{f_1(t)t'(t_1)\,dt_1}{t_1-\tau_1} \quad \text{(38a)}$$

$$b_{21}\Psi_2'(\tau_2) + \frac{b_{22}}{2}\Psi_1'(\tau_1)t_1'(\tau_2) + \frac{b_{23}}{2}\Psi_3'(\tau_3)t_3'(\tau_2) + \frac{b_{22}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_1'(t_1)t_1'(t_2)\,dt_2}{t_2-\tau_2}$$

$$+ \frac{b_{23}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_3'(t_3)t_3'(t_2)\,dt_2}{t_2-\tau_2} = \frac{f_2(\tau)t'(\tau_2)}{2} + \frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{f_2(t)t'(t_2)\,dt_2}{t_2-\tau_2} \quad \text{(38b)}$$

$$b_{31}\Psi_3'(\tau_3) + \frac{b_{32}}{2}\Psi_1'(\tau_1)t_1'(\tau_3) + \frac{b_{33}}{2}\Psi_2'(\tau_2)t_2'(\tau_3) + \frac{b_{32}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_1'(t_1)t_1'(t_3)\,dt_3}{t_3-\tau_3}$$

$$+ \frac{b_{33}}{2\pi i}\int_{+\infty}^{-\infty}\frac{\Psi_2'(t_2)t_2'(t_3)\,dt_3}{t_3-\tau_3} = \frac{f_3(\tau)t'(\tau_3)}{2} + \frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{f_3(t)t'(t_3)\,dt_3}{t_3-\tau_3} \quad \text{(38c)}$$

In (38*a*)–(38*c*), τ is a fixed point on the $t(-\infty, +\infty)$ axis and τ_k $(k = 1,$

2, 3) are fixed points on the t_k (Im $\zeta_k = 0$) axes. The variables $t'(t_j)$ and $t'_k(t_j)$ $(k, j = 1, 2, 3)$ are, respectively, the total derivatives of t and t_k with respect to the variable t_j $(-\infty, +\infty)$ and are equal to

$$t'(t_j) = \frac{Z'_j(t_j)}{x'(t) + \mu_j y'(t)} \quad \text{(39a)}$$

$$t'_k(t_j) = \frac{Z'_j(t_j)}{Z'_k(t_k)} \cdot \frac{x'(t) + \mu_k y'(t)}{x'(t) + \mu_j y'(t)} \quad \text{(39b)}$$

The three integral (38*a*)–(38*c*) can be discretized and solved for the boundary values of the three analytical functions $\Psi'_k(t_k)$ by the method proposed by Sarkar et al. (1988). The infinite integrals appearing in (38*a*)–(38*c*) are transformed to the circumference of unit discs by the inversion of mapping (51). Then, the interior values of the analytical functions, Ψ'_k, are calculated using the Cauchy integral theorem (27). Finally, the stress functions $\Phi'_k(z_k)$ are obtained using (34).

Substituting the stress functions $\Phi'_k(z_k)$ into (21) and adding (21) to (9), the six stress components are equal to

$$\sigma_{xx} = 2\,\mathrm{Re}[\mu_1^2\Phi'_1(z_1) + \mu_2^2\Phi'_2(z_2) + \mu_3^2\lambda_3\Phi'_3(z_3)] + c_1\rho g y \quad \text{(40a)}$$

$$\sigma_{yy} = 2\,\mathrm{Re}[\Phi'_1(z_1) + \Phi'_2(z_2) + \lambda_3\Phi'_3(z_3)] + \rho g y \quad \text{(40b)}$$

$$\sigma_{xy} = -2\,\mathrm{Re}[\mu_1\Phi'_1(z_1) + \mu_2\Phi'_2(z_2) + \mu_3\lambda_3\Phi'_3(z_3)] \quad \text{(40c)}$$

$$\sigma_{xz} = 2\,\mathrm{Re}[\mu_1\lambda_1\Phi'_1(z_1) + \mu_2\lambda_2\Phi'_2(z_2) + \mu_3\Phi'_3(z_3)] + c_2\rho g y \quad \text{(40d)}$$

$$\sigma_{yz} = -2\,\mathrm{Re}[\lambda_1\Phi'_1(z_1) + \lambda_2\Phi'_2(z_2) + \Phi'_3(z_3)] \quad \text{(40e)}$$

$$\sigma_{zz} = -\frac{2}{a_{33}}\,\mathrm{Re}\{[a_{13}\mu_1^2 + a_{23} - a_{34}\lambda_1 + a_{35}\mu_1\lambda_1 - a_{36}\mu_1]\Phi'_1(z_1)$$
$$+ [a_{13}\mu_2^2 + a_{23} - a_{34}\lambda_2 + a_{35}\mu_2\lambda_2 - a_{36}\mu_2]\Phi'_2(z_2) + [a_{13}\lambda_3\mu_3^2$$
$$+ a_{23}\lambda_3 - a_{34} + a_{35}\mu_3 - a_{36}\mu_3\lambda_3]\Phi'_3(z_3)\} + c_3\rho g y \quad \text{(40f)}$$

Eqs. (40*a*)–(40*f*) indicate that, in general, at each point in the half-space the stress state is three-dimensional and the principal stress components are inclined with respect to the x, y, and z axes.

When determining the stress components, the three integrals on the right-hand side of (38*a*)–(38*c*) must be determined. It is noteworthy that for these integrals to converge, the boundary curve (29) must be such that

$$\lim_{t\to\pm\infty} |y(t)x'(t)| = a_1 < \infty; \qquad \lim_{t\to\pm\infty} |y(t)y'(t)| = a_2 < \infty \quad \text{(41)}$$

where a_1 and a_2 = two constants. Also, similar restrictions must be exerted on t_x, t_y, and t_z. Under these conditions, the system of singular integral (38) can be solved for the analytical functions involved. The remaining task is to determine the conformal mapping functions $Z'_k(\zeta_k)$ appearing in (34) and (35).

NUMERICAL CONFORMAL MAPPING

Several methods have been proposed in the literature for the numerical conformal mapping of a bounded, simply connected domain onto a unit disc. Generally, these methods can be divided into two types: expansion and integral equation methods (Papamichael et al. 1986). Common expansion methods include the Bergman kernel method and the closely related Ritz variational method, which have been discussed in detail by Papamichael and Kokkinos (1981). However, because expansion methods are numerically unstable, integral equation methods are more effective for numerical conformal mapping. Among these methods, the one based on the Szegö kernel has been found to be more effective, stable, and accurate (Trummer 1986; Kerzman and Trummer 1986; O'Donnell and Rokhlin 1989). The Szegö kernel method is summarized in the following section, and a detailed description of the method can be found in Trummer (1986).

We assume that Ω is a bounded simply connected domain in the complex z-plane, and that $F = R(z)$ is the function that maps Ω conformally onto the unit disc $\|F\| < 1$, subject to the normalization $R(c) = 0$, $R'(c) > 0$, where $c \in \Omega$ is an arbitrary point. We also assume that $\partial\Omega$, the boundary of Ω, $\in C^2$, has the parametric expression $z(t)$, $0 \leq t \leq \beta$ with $z'(t) \equiv dz/dt \neq 0$. Then the mapping function and its derivative can be expressed by the Szegö kernel as

$$R(z) = -i \frac{z'(t)}{|z'(t)|} \frac{R'(z)}{|R'(z)|} \qquad z \in \partial\Omega \quad \text{(42a)}$$

$$R'(z) = \frac{2\pi}{s(c, c)} s^2(z, c) \qquad z \in \bar{\Omega} \quad \text{(42b)}$$

where $\bar{\Omega} = \Omega + \partial\Omega =$ closure of Ω. The Szegö kernel $s(z, c)$ is the unique solution to the integral equation

$$s(z, c) + \int_{w \in \partial\Omega} A(z, w) s(w, c)\, d\sigma_w = \overline{H(c, z)} \qquad z \in \partial\Omega \quad \text{(43)}$$

with

$$A(w, z) = \overline{H(z, w)} - H(w, z) \qquad w, z \in \partial\Omega,\ w \neq z \quad \text{(44a)}$$

$$A(w, z) = 0 \qquad w = z \in \partial\Omega \quad \text{(44b)}$$

$$H(w, z) = \frac{1}{2\pi i} \frac{z'(t)}{|z'(t)|} \frac{1}{z - w} \qquad w \in \bar{\Omega},\ z \in \partial\Omega,\ w \neq z \quad \text{(44c)}$$

Using a parametric form $z(t)$ for the boundary, the integral (43) becomes

$$\phi(t) + \int_0^\beta k(t, \tau) \phi(\tau)\, d\tau = \psi(t) \qquad 0 \leq t \leq \beta \quad \text{(45)}$$

where

$$\phi(t) = \sqrt{|z'(t)|}\, s[z(t), c] \quad \text{(46a)}$$

$$\psi(t) = \sqrt{|z'(t)|}\, \overline{H[c, z(t)]} \quad \text{(46b)}$$

$$k(t, \tau) = \sqrt{|z'(t)|}\, A[z(t), z(\tau)] \sqrt{|z'(\tau)|} \quad \text{(46c)}$$

The integral (45) can be solved numerically by the Nyström's method (Atkinson 1976). Because of the periodicity of all the functions in (45), we can choose the trapezoidal rule (Davis and Rabinowitz 1984) for the Nyström's method to obtain

$$\phi(t_i) + \frac{\beta}{n}\sum_{j=1}^{n} k(t_i, t_j)\phi(t_j) = \psi(t_i) \quad t_i = \frac{(i-1)\beta}{n} \qquad i = 1, 2, \ldots n \;\ldots\; (47)$$

Eq. (47) is a complex system of linear equations, and can be solved efficiently by the generalized biconjugate gradient method (Sarkar et al. 1988). Once $\phi(t)$ is obtained at discretized points, the Szegö kernel can be obtained from (46*a*). To get the mapping function and its derivative from (42*a*) and (42*b*), we also need to calculate $s(c, c)$, which is equal to

$$s(c, c) = \int_{\partial\Omega} \overline{s(z, c)} s(z, c)\, d\sigma_z = \int_0^{\beta} |\phi(t)|^2\, dt \;\ldots\; (48)$$

To calculate the mapping functions and their derivatives that appear in our problem of interest, three successive conformal mappings are carried out

Mapping 1

$$z_k \Rightarrow w_k \qquad k = 1, 2, 3 \qquad w_k = \frac{z_k + iA_k}{z_k - iA_k} \;\ldots\; (49)$$

which maps the irregular half planes of interest onto irregular bounded domains w_k. The quantity A_k represents complex constants that can be chosen such that mapping (49) is conformal, and z_k ($k = 1, 2, 3$) are the complex variables in the irregular half-planes with boundary values given by (31).

Mapping 2

$$w_k \Rightarrow F_k \qquad k = 1, 2, 3 \qquad F_k = F_k(w_k) \;\ldots\; (50)$$

which maps the irregular bounded domains onto unit discs F_k. This is done using the numerical conformal mapping method previously described. The accuracy of this mapping can be evaluated by calculating $|\|F_k\| - 1|$, which for the numerical examples presented in the following section is at least equal to 10^{-15} for $n = 200$ in (47). This mapping method is therefore very accurate.

Mapping 3

$$F_k \Rightarrow \zeta_k \qquad k = 1, 2, 3 \qquad \zeta_k = i\,\frac{F_k + 1}{F_k - 1} \;\ldots\; (51)$$

which maps the unit discs onto flat half planes ζ_k.

PLANE-STRAIN SOLUTION

If there is a plane of symmetry normal to the z axis of Fig. 1, then the generalized plane-strain solution previously described reduces to a plane-strain solution. For this special case

$$c_{46} = c_{56} = c_{4i} = c_{5i} = 0 \qquad \text{for } i = 1, 2, 3 \;\ldots\; (52a)$$

$$\beta_{46} = \beta_{56} = \beta_{4i} = \beta_{5i} = 0 \qquad \text{for } i = 1, 2, 3 \;\ldots\; (52b)$$

$$l_3(\mu) = \lambda_1 = \lambda_2 = \lambda_3 = 0 \;\ldots\; (52c)$$

Substituting these relations into the general solution gives that c_2 in (9*c*) always vanishes. Furthermore, the problem of finding the stresses can be decomposed as the sum of two uncoupled problems.

Plane Problem

For which the stress components are equal to

$$\sigma_{xx} = 2\,\mathrm{Re}[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)] + c_1\rho g y \quad (53a)$$

$$\sigma_{yy} = 2\,\mathrm{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)] + \rho g y \quad (53b)$$

$$\sigma_{xy} = -2\,\mathrm{Re}[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)] \quad (53c)$$

$$\sigma_{zz} = -\frac{2}{a_{33}}\,\mathrm{Re}[(a_{13}\mu_1^2 + a_{23} - a_{36}\mu_1)\Phi_1'(z_1)$$

$$+ (a_{13}\mu_2^2 + a_{23} - a_{36}\mu_2)\Phi_2'(z_2)] + c_3\rho g y \quad (53d)$$

where μ_1 and μ_2 and their conjugates are the roots of $l_4(\mu) = 0$ in (19*c*) and

$$\Phi_1'(z_1) = \frac{\Psi_1'(\zeta_1)}{Z_1'(\zeta_1)} \quad (54a)$$

$$\Phi_2'(z_2) = \frac{\Psi_2'(\zeta_2)}{Z_2'(\zeta_2)} \quad (54b)$$

which account for the effect of gravity and the surface-load components acting in the x, y plane.

In (54), the boundary values of Ψ_1' and Ψ_2' are obtained by solving the integral (38*a*) and (38*b*) with b_{ij} $(i, j = 1, 2, 3)$ and $f_i(t)$ $(i = 1, 2, 3)$ being given by (71) and (72) and in Appendix I.

Antiplane Problem

For which the stress components are equal to

$$\sigma_{xz} = 2\,\mathrm{Re}[\mu_3\Phi_3'(z_3)] \quad (55a)$$

$$\sigma_{yz} = -2\,\mathrm{Re}[\Phi_3'(z_3)] \quad (55b)$$

where μ_3 and its conjugate are the roots of $l_2(\mu) = 0$ in (19*a*). The expression $\Phi_3'(z_3)$ is such that its boundary value is equal to

$$\Phi_3'(z_3) = \frac{\Psi_3'(t_3)}{Z_3'(t_3)} = \frac{-1}{Z_3'(t_3)}\left[\frac{t_z(t)s'(t)t'(t_3)}{2} + \frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{t_z(\tau)s'(\tau)t'(\tau_3)\,d\tau_3}{\tau_3 - t_3}\right] \quad (56)$$

This function accounts for the effect of the surface-load component parallel to the z axis. In the absence of such load, both σ_{xz} and σ_{yz} vanish. In that case, two of the principal stresses induced by gravity and the surface tractions t_x and t_y are located in the x, y plane, and σ_{zz} is the third principal stress.

STRESSES IN ISOLATED RIDGE OR VALLEY

As an illustrative example, consider a long isolated symmetric ridge with the geometry of Fig. 2(*a*). The topography of the ridge is expressed in parametric form as follows:

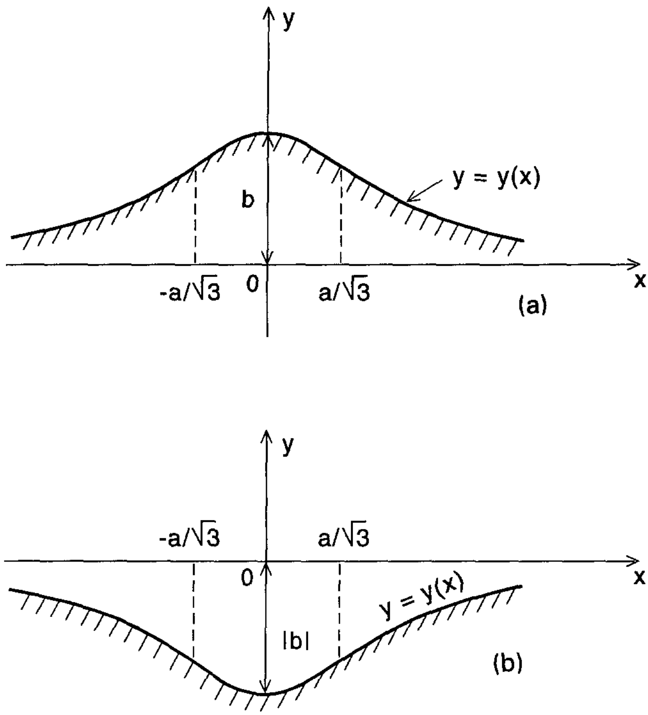

FIG. 2. (a) Symmetric Ridge of Height b; (b) Symmetric Valley of Depth $|b|$

$$x = t \qquad (-\infty < t < +\infty) \quad \ldots\ldots \quad (57a)$$

$$y = \frac{a^2 b}{(t^2 + a^2)} \quad \ldots\ldots \quad (57b)$$

where b = ridge height and is assumed to be positive. If b is negative, (57) corresponds to a long isolated symmetric valley where $|b|$ is the depth of the valley [Fig. 2(*b*)]. The inflection points of the boundary curve are located at $x = \pm a/\sqrt{3}$ and $y = 0.75b$, at which the slopes are equal to $\pm(3b\sqrt{3})/(8a)$.

For the geometry of Figs. 2(*a*) and 2(*b*), (31) becomes

$$z_k(t) = t + \frac{a^2 b \mu_k}{t^2 + a^2} \qquad k = 1, 2, 3; \qquad -\infty < t < \infty \quad \ldots\ldots \quad (58)$$

After carrying out the analytical conformal mapping defined in (49), the unbounded curves (58) become bounded curves in the w_k planes as

$$w_k(t) = \frac{t + \dfrac{a^2 b \mu_k}{t^2 + a^2} + iA_k}{t + \dfrac{a^2 b \mu_k}{t^2 + a^2} - iA_k} \qquad k = 1, 2, 3; \qquad -\infty < t < \infty \quad \ldots\ldots \quad (59)$$

Introducing a new parameter, θ, which varies over a finite interval $(-\pi/2, \pi/2)$ such that $t = a \tan \theta$, (59) can be reduced to

$$w_k(\theta) = \frac{a\cdot\sin\theta + b\mu_k\cos^3\theta + iA_k\cos\theta}{a\cdot\sin\theta + b\mu_k\cos^3\theta - iA_k\cos\theta} \qquad k = 1, 2, 3;$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad \ldots\ldots\ldots\ldots \quad (60)$$

The derivative of (60) with respect to θ is equal to

$$\frac{dw_k(\theta)}{d\theta} = \frac{2iA_k(2b\mu_k\sin\theta\cos^3\theta - a)}{(a\cdot\sin\theta + b\mu_k\cos^2\theta - iA_k\cos\theta)^2} \qquad k = 1, 2, 3;$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad \ldots\ldots\ldots\ldots \quad (61)$$

Finally, the relationship between z_k and ζ_k, and the functions $Z'_k(\zeta_k)$ in (34) can be expressed as

$$\zeta_k = i\frac{F_k(w_k) + 1}{F_k(w_k) - 1} \qquad k = 1, 2, 3 \quad \ldots\ldots\ldots\ldots \quad (62)$$

and

$$\frac{1}{Z'_k(\zeta_k)} = \frac{-4A_k}{[F_k(w_k) - 1]^2}\frac{dF_k}{dw_k}\frac{1}{(z_k - iA_k)^2} \qquad k = 1, 2, 3 \quad \ldots\ldots \quad (63)$$

with

$$w_k = \frac{z_k + iA_k}{z_k - iA_k} \qquad k = 1, 2, 3 \quad \ldots\ldots\ldots\ldots \quad (64)$$

In (62) and (63), the functions $F_k(w_k)$ ($k = 1, 2, 3$) and their derivatives are determined using the numerical conformal mapping method previously discussed.

For the geometry of Figs. 2(*a*) and 2(*b*), $t'(t_j)$ and $t'_k(t_j)$ defined in (39) take the following form:

$$t'(t_j) = \frac{Z'_j(t_j)}{1 - \dfrac{2a^2bt\mu_j}{(a^2 + t^2)^2}} \qquad \ldots\ldots\ldots\ldots \quad (65a)$$

$$t'_k(t_j) = \frac{Z'_j(t_j)}{Z'_k(t_k)}\cdot\frac{(a^2 + t^2)^2 - 2a^2bt\mu_k}{(a^2 + t^2)^2 - 2a^2bt\mu_j} \qquad \ldots\ldots\ldots\ldots \quad (65b)$$

The rock mass in the ridge of Fig. 2(*a*) is assumed to be orthotropic in a n, s, t cartesian coordinate system. That coordinate system is attached to planes of anisotropy in the medium, and its orientation with respect to the x, y, z coordinate system, and therefore the ridge, is defined by the dip azimuth β and the dip angle ψ, as shown in Fig. 3. The t-axis is located in the xz plane. The constitutive equation for the rock in the n, s, t coordinate system is given by the following equation:

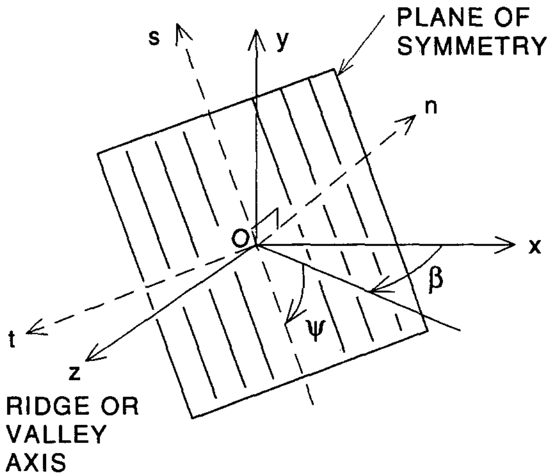

FIG. 3. Orientation of Planes of Symmetry with Respect to x, y, z Coordinate System Attached to Ridge or Valley

$$\begin{bmatrix} e_{nn} \\ e_{ss} \\ e_{tt} \\ 2e_{st} \\ 2e_{nt} \\ 2e_{ns} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_n} & -\frac{\nu_{sn}}{E_s} & -\frac{\nu_{tn}}{E_t} & 0 & 0 & 0 \\ -\frac{\nu_{ns}}{E_n} & \frac{1}{E_s} & -\frac{\nu_{ts}}{E_t} & 0 & 0 & 0 \\ -\frac{\nu_{nt}}{E_n} & -\frac{\nu_{st}}{E_s} & \frac{1}{E_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{st}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{nt}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{ns}} \end{bmatrix} \begin{bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{tt} \\ \sigma_{st} \\ \sigma_{nt} \\ \sigma_{ns} \end{bmatrix} \quad \text{........ (66)}$$

or in a more compact matrix form

$$\mathbf{e}_{nst} = \mathbf{h}\boldsymbol{\sigma}_{nst} \quad \text{........ (67)}$$

Nine independent elastic constants are needed to describe the deformability of the rock in the n, s, t coordinate system. The quantity E_n, E_s, and E_t are the Young's moduli in the n, s, and t (or 1, 2, and 3) directions, respectively. The quantity G_{ns}, G_{nt}, and G_{st} are the shear moduli in planes parallel to the ns, nt, and st planes, respectively. Finally, ν_{ij} $(i, j = n, s, t)$ are the Poisson's ratios that characterize the normal strains in the symmetry directions j when a stress is applied in the symmetry directions i. Because of symmetry of the compliance matrix $\mathbf{h}$, Poisson's ratios ν_{ij} and ν_{ji} are such that $\nu_{ij}/E_i = \nu_{ji}/E_j$.

Eqs. (66) and (67) still apply if the medium is transversely isotropic in one of the three ns, nt, or st planes. In that case, only five independent elastic constants are needed to describe the deformability of the medium in the n, s, t coordinate system. Let's call these constants E, E', ν, ν', and G'

with the following definitions: (1) E and E' are Young's moduli in the plane of transverse isotropy and in direction normal to it, respectively; (2) ν and ν' are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel and normal to it, respectively; and (3) G' is the shear modulus in planes normal to the plane of transverse isotropy. Relationships exist among E, E', ν, ν', and G' and the coefficients of matrix **h** in (67). For instance, for transverse isotropy in the st plane

$$\frac{1}{E_n} = \frac{1}{E'}; \frac{1}{E_s} = \frac{1}{E_t} = \frac{1}{E}; \frac{1}{G_{ns}} = \frac{1}{G_{nt}} = \frac{1}{G'}; \frac{\nu_{ns}}{E_n} = \frac{\nu_{nt}}{E_n} = \frac{\nu'}{E'}; \frac{\nu_{st}}{E_s} = \frac{\nu_{ts}}{E_t}$$

$$= \frac{\nu}{E}; \frac{1}{G_{st}} = \frac{1}{G} = \frac{2(1+\nu)}{E} \quad \text{(68)}$$

For known orientations of the planes of anisotropy with respect to the x, y, and z axes, the constitutive relation of the medium in the x, y, z coordinate system and defined in (2) or (3) can be obtained by using second-order tensor coordinate transformation rules. Because of the linear relationships existing between coefficients a_{ij} and h_{ij} of matrices **a** and **h** in (2) and (67), respectively, it can be shown that the ratios between the stresses σ_{ij} and a characteristic stress $\rho g|b|$ depend on the following eight dimensionless quantities:

$$\frac{E_s}{E_n}; \frac{E_s}{E_t}; \nu_{sn}; \nu_{tn}; \nu_{ts}; \frac{E_s}{G_{st}}; \frac{E_s}{G_{nt}}; \frac{E_s}{G_{ns}}$$

If the medium is transversely isotropic with, for instance, transverse isotropy in the st plane, and using (68), the stress ratios are found to depend only on four dimensionless terms

$$\frac{E}{E'}; \nu; \nu'; \frac{G}{G'}$$

The stress ratios $\sigma_{ij}/\rho g|b|$ also depend on: (1) The orientation angles β and ψ of the planes of transverse isotropy with respect to the x, y, and z

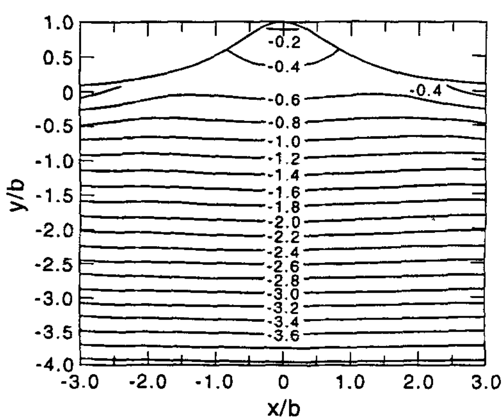

FIG. 4(a). Contour Plots of $\sigma_{xx}/\rho gb$ for Transversely Isotropic Case with β = 0°, E/E' = 3, G/G' = 3, $\nu = \nu'$ = 0.25, and ψ = 0°

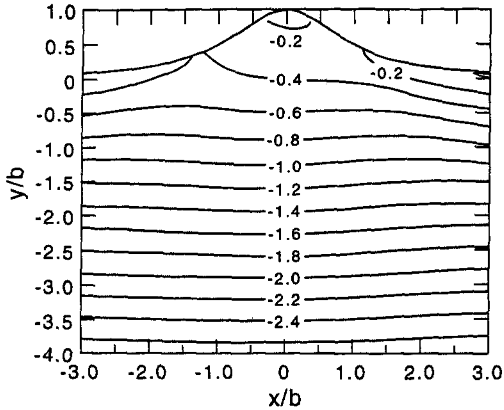

FIG. 4(b). Contour Plots of $\sigma_{xx}/\rho g b$ for Transversely Isotropic Case with $\beta = 0°$, $E/E' = 3$, $G/G' = 3$, $\nu = \nu' = 0.25$, and $\psi = 45°$

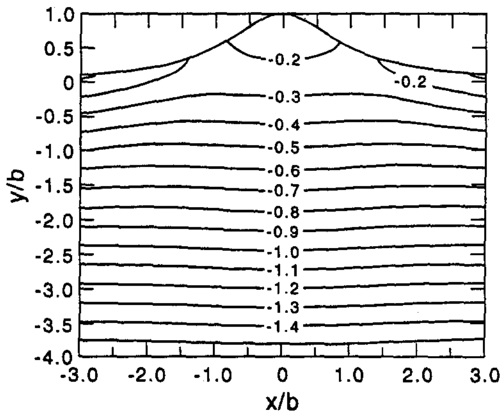

FIG. 4(c). Contour Plots of $\sigma_{xx}/\rho g b$ for Transversely Isotropic Case with $\beta = 0°$, $E/E' = 3$, $G/G' = 3$, $\nu = \nu' = 0.25$, and $\psi = 90°$

axes attached to the ridge or valley; (2) the coordinates $(x/|b|, y/|b|)$ of the points at which the stresses are calculated; and (3) the ratios $a/|b|$ and $b/|b|$ describing the geometry of the ridge or valley. If the ridge has any surface loads, the stress ratios also depend on three dimensionless quantities $t_x/\rho g|b|$, $t_y/\rho g|b|$, and $t_z/\rho g|b|$.

As a numerical example, consider the symmetric ridge of Fig. 2(a) with $a/b = 1$, and gravity being the only active force. The slopes at the inflection points of the ridge are equal to 0.65. The rock in the ridge is assumed to be transversely isotropic in the st plane of Fig. 3. The planes of transverse isotropy are assumed to strike parallel to the ridge axis ($\beta = 0°$), and dip at an angle ψ of 0° (horizontal anisotropy), 45° (inclined anisotropy), or 90° (vertical anisotropy). The stress distribution in the ridge was determined for a transversely isotropic rock mass with $E/E' = 3$, $G/G' = 3$, and $\nu = \nu' = 0.25$. Figs. 4(a), 4(b), and 4(c) show the contour diagrams of the horizontal stress ratio $\sigma_{xx}/\rho g b$ only for $\psi = 0°$, 45°, and 90°, respectively.

These figures show that the orientation of the anisotropy has a strong effect on the distribution of horizontal stresses. At a given depth below the ridge, the horizontal stress is much higher when the rock mass has horizontal anisotropy [Fig. 4(*a*)], than when the anisotropy is vertical [Fig. 4(*c*)]. This is because the rock mass is three times more deformable in the vertical direction than in the horizontal direction. This increase in stress can also be observed when the planes of anisotropy are inclined at 45° [Fig. 4(*b*)]. In that case, the stress distribution is no longer symmetric with respect to the vertical axial plane of the ridge.

CONCLUSION

The analytical conformal mapping method has been used in the past to solve some problems of elasticity for which the governing equation is equivalent to a harmonic or a biharmonic equation (Nehari 1952; Muskhelishvili 1953). However, that method is limited to a small family of problems with irregular geometries for which exact conformal mapping functions can be found. For more complex elasticity problems, it is common practice to use numerical methods, such as the finite element, finite difference, or boundary element methods. When determining gravitational stresses in rock masses with non-planar topographies, the analytical conformal mapping method can only be used for simple isolated symmetric ridges or valleys in isotropic media.

In this paper, we propose a new analytical method for determining the stresses in a homogeneous, general anisotropic, and elastic half-space, with any irregular but smooth boundary. The half-space is subject to gravity and surface loads. The stresses are determined assuming a condition of generalized plane strain, and are expressed in terms of three analytical functions using Lekhnitskii's method. These functions are determined using a numerical conformal mapping method and an integral equation method. When applied to rock masses, the proposed method is limited to rock masses in two-dimensional conditions.

To the writer's knowledge, the numerical conformal mapping method has rarely been used in the past for solving elasticity problems. The method is indeed very powerful when combined with the integral equation method, since the solutions are expressed in terms of analytical functions that are continuous and differentiable. Because of this analytical property, more accurate results can be obtained. Further, the method presented in this paper could be a substitute for numerical methods when solving two-dimensional elasticity problems with complex geometries. This application is being investigated by the writers.

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APPENDIX I. VARIABLES IN (37)–(38)

Coefficients b_{ij} and functions $f_i(t)$ in (37)–(38)

$$b_{11} = (\overline{\mu_2} - \mu_1)(\overline{\lambda_2\lambda_3} - 1) - (\overline{\mu_2 - \mu_3})\overline{\lambda_3}(\overline{\lambda_2} - \lambda_1) \quad \text{(69a)}$$

$$b_{12} = (\overline{\mu_2} - \mu_2)(\overline{\lambda_2\lambda_3} - 1) - (\overline{\mu_2 - \mu_3})\overline{\lambda_3}(\overline{\lambda_2} - \lambda_2) \quad \text{(69}b\text{)}$$

$$b_{13} = (\overline{\mu_2} - \mu_3)\lambda_3(\overline{\lambda_2\lambda_3} - 1) - (\overline{\mu_2 - \mu_3})\overline{\lambda_3}(\overline{\lambda_2}\lambda_3 - 1) \quad \text{(69}c\text{)}$$

$$b_{21} = (\overline{\lambda_1\lambda_3} - 1)(\overline{\mu_1} - \mu_2) - (\overline{\lambda_1} - \lambda_2)\overline{\lambda_3}(\overline{\mu_1 - \mu_3}) \quad \text{(69}d\text{)}$$

$$b_{22} = (\overline{\lambda_1\lambda_3} - 1)(\overline{\mu_1} - \mu_1) - (\overline{\lambda_1} - \lambda_1)\overline{\lambda_3}(\overline{\mu_1 - \mu_3}) \quad \text{(69}e\text{)}$$

$$b_{23} = (\overline{\lambda_1\lambda_3} - 1)\lambda_3(\overline{\mu_1} - \mu_3) - (\overline{\lambda_1}\lambda_3 - 1)\overline{\lambda_3}(\overline{\mu_1 - \mu_3}) \quad \text{(69}f\text{)}$$

$$b_{31} = (\overline{\lambda_1 - \lambda_2})\lambda_3(\overline{\mu_1} - \mu_3) - (\overline{\lambda_1}\lambda_3 - 1)(\overline{\mu_1 - \mu_2}) \quad \text{(69}g\text{)}$$

$$b_{32} = (\overline{\lambda_1 - \lambda_2})(\overline{\mu_1} - \mu_1) - (\overline{\lambda_1} - \lambda_1)(\overline{\mu_1 - \mu_2}) \quad \text{(69}h\text{)}$$

$$b_{33} = (\overline{\lambda_1 - \lambda_2})(\overline{\mu_1} - \mu_2) - (\overline{\lambda_1} - \lambda_2)(\overline{\mu_1 - \mu_2}) \quad \text{(69}i\text{)}$$

$$f_1(t) = [\overline{\mu_2}(\overline{\lambda_2\lambda_3} - 1) - (\overline{\mu_2 - \mu_3})\overline{\lambda_2\lambda_3}]u(t) + (\overline{\lambda_2\lambda_3} - 1)v(t)$$
$$- (\overline{\mu_2 - \mu_3})\overline{\lambda_3}w(t) \quad \text{(70}a\text{)}$$

$$f_2(t) = [\overline{\mu_1}(\overline{\lambda_1\lambda_3} - 1) - (\overline{\mu_1 - \mu_3})\overline{\lambda_1\lambda_3}]u(t) + (\overline{\lambda_1\lambda_3} - 1)v(t)$$
$$- (\overline{\mu_1 - \mu_3})\overline{\lambda_3}w(t) \quad \text{(70}b\text{)}$$

$$f_3(t) = [\overline{\mu_1}(\overline{\lambda_1 - \lambda_2}) - \overline{\lambda_1}(\overline{\mu_1 - \mu_2})]u(t) + (\overline{\lambda_1 - \lambda_2})v(t)$$
$$- (\overline{\mu_1 - \mu_2})w(t) \quad \text{(70}c\text{)}$$

where $u(t) = -\rho g y(t)x'(t) + t_y(t)s'(t)$; $v(t) = c_1\rho g y(t)y'(t) + t_x(t)s'(t)$; and $w(t) = c_2\rho g y(t)y'(t) + t_z(t)s'(t)$.

If there is a plane of symmetry normal to the z axis, (69) and (70) reduce to

$$b_{11} = \mu_1 - \overline{\mu_2} \quad \text{(71}a\text{)}$$

$$b_{12} = \mu_2 - \overline{\mu_2} \quad \text{(71}b\text{)}$$

$$b_{13} = 0 \quad \text{(71}c\text{)}$$

$$b_{21} = \mu_2 - \overline{\mu_1} \quad \text{(71}d\text{)}$$

$$b_{22} = \mu_1 - \overline{\mu_1} \quad \text{(71}e\text{)}$$

$$b_{23} = 0 \quad \text{(71}f\text{)}$$

$$b_{31} = \overline{\mu_1 - \mu_2} \quad \text{(71}g\text{)}$$

$$b_{32} = 0 \quad \text{(71}h\text{)}$$

$$b_{33} = 0 \quad \text{(71}i\text{)}$$

and

$$f_1(t) = -\overline{\mu_2}u(t) - v(t) \quad \text{(72}a\text{)}$$

$$f_2(t) = -\overline{\mu_1}u(t) - v(t) \quad \text{(72}b\text{)}$$

$$f_3(t) = -(\overline{\mu_1 - \mu_2})w(t) = -(\overline{\mu_1 - \mu_2})t_z(t)s'(t) \quad \text{(72}c\text{)}$$

APPENDIX II. REFERENCES

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APPENDIX III. NOTATION

The following symbols are used in this paper:

A_k = complex constants;
$A(w, z)$ = complex function in (44);
$\mathbf{a}$ = global compliance matrix;
a_{ij} = components of compliance matrix $\mathbf{a}$;
a, b = constants defining ridge or valley geometry in (57);
a_1, a_2 = real constants;
b_{ij} = complex constants defined in Appendix I;
$\mathbf{c}$ = stiffness matrix;
c^2 = curve class of second continuously derivative;
c_{ij} = components of stiffness matrix $\mathbf{c}$;
E, E' = Young's moduli;
E_i = Young's moduli in local *nst* coordinate system;
$\mathbf{e}$ = strain matrix;
e_{ij} = components of strain matrix $\mathbf{e}$;
$F(x, y)$ = stress function;
$F_k \Rightarrow \zeta_k$ = mappings from F_k planes onto ζ_k planes;
$f'(x)$ = total derivative df/dx;
$f_j(t)$ = complex functions defined in Appendix I;
$f_{,x}$ = partial derivative $\partial f/\partial x$;
$\bar{f}$ = conjugate of f;
G, G' = shear moduli;
G_{ij} = shear moduli in local *nst* coordinate system;
g = gravitational acceleration;
$H(w, z)$ = complex function in (44);
$\mathbf{h}$ = local compliance matrix;
$\operatorname{Im} f$ = imaginary part of f;
$k(t, \tau)$ = complex function in (46);
L_2, L_3, L_4 = differential operators in (14);
l_2, l_3, l_4 = complex parameters in (19);
nst = local coordinate system;
$\operatorname{Re} f$ = real part of f;
$s(t)$ = arc length along boundary curve;
$s(z, c)$ = Szegö kernel in (43);
t = also used as parametric variable;
t_k, τ_k = real variables; boundary values of complex variables ζ_k;
$t_k(t)$ = real functions in (36);
$t(t_k)$ = real functions in (37);
$t_i(t_k)$ = real functions in (37) and (38);
t_x, t_y, t_z = surface traction components;
$w_k \Rightarrow F_k$ = mappings from w_k planes onto F_k planes;
xyz = global coordinate system;
$y = y(x)$ = boundary curve of topography;
z = also used to represent the complex plane;
$z(t) = x(t) + iy(t)$ = complex expression of $y = y(x)$;
$z_k(t) = x(t) + \mu_k y(t)$ = boundary curve of z_k planes;

$z_k = Z_k(\zeta_k)$ = mapping functions;
$z_k \Rightarrow w_k$ = mappings from z_k planes onto w_k planes;
β = dip azimuth;
β = also used as an upper bound for variable t;
β_{ij} = elastic constants defined in (15);
$\zeta_k = Z_k^{-1}(z_k)$ = mappings from z_k planes onto ζ_k planes;
θ = real variable in (60) and (61);
$\lambda_1, \lambda_2, \lambda_3$ = complex parameters in (18);
μ_k = complex roots of (20);
ν_{ij} = Poisson's ratios in local *nst* coordinate system;
ν, ν' = Poisson's ratios;
ρ = rock mass density;
$\boldsymbol{\sigma}$ = stress matrix;
σ_{ij} = components of stress matrix $\boldsymbol{\sigma}$;
σ_{ij}^h = stresses for homogeneous solution;
σ_{ij}^p = stresses for particular solution;
$\Phi_k(z_k)$ = complex functions;
$\Phi(t)$ = complex function in (46);
$\Psi_k(\zeta_k)$ = complex functions;
$\Psi(t)$ = complex function in (46);
$\Psi(x, y)$ = stress function;
ψ = dip angle;
Ω = a bounded simply connected domain;
$\bar{\Omega} = \Omega + \partial\Omega$ = closure of Ω; and
$\partial\Omega$ = boundary of Ω.