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A 3-D boundary element formulation of anisotropic elasticity with gravity

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This paper presents a new boundary element formulation for three-dimensional (3-D) linear elastic bodies with anisotropic properties (transverse isotropy with any orientation) subject to gravity. Green's functions for such anisotropic media are derived in exact closed-form. *Particular stresses and displacements related to the gravity body force are also presented in* exact closed-form *for generalized anisotropic media. The boundary integral equation with only weak singularities is formulated based on the principle of superposition and is such that* no numerical integration *associated either with the Green's functions for anisotropic media or with body forces is necessary. This new 3-D boundary element formulation has been programmed, and several numerical examples are presented and checked with existing closed-form solutions. It is found that even with a very coarse discretization, excellent results can be obtained with the proposed method.*

Keywords: boundary element method, anisotropic elasticity, gravity body force

1. Introduction

Since the pioneer work of Rizzo[1] for two-dimensional (2-D) linear elastostatics and Cruse[2] for 3-D linear elastostatics, the boundary element method (BEM) has rapidly developed and has been applied to various engineering problems. Although this method is most suitable for linear problems with homogeneous material properties, extension to various nonlinear problems in heterogeneous media is possible. It is noted, however, that by so doing, the beauty and main advantage of the BEM (i.e., discretization of the problem on boundaries only) are sometimes lost. For instance, when a body force or its equivalent is introduced, there is usually a volumetric integration term associated with it.[3,4] For some very simple cases, the volumetric integrals can usually be transformed into boundary integrals by the Galerkin vector approach[5]; otherwise one has to employ some approximate methods, such as the dual reciprocity and Fourier series expansion methods.[6,7] If, on the other hand, particular solutions corresponding to body forces are available, the exact method based on the principle of superposition can be used.[8-11] Apart from the body force problem, anisotropy in 3-D linear elastostatics is another. Until now, the Green's functions for 3-D anisotropic problems have been either calculated by numerical integration[11,12] or avoided by the dual reciprocity approximation technique.[13]

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This paper presents a BEM solution for 3-D finite and linear elastic bodies with anisotropic properties (transverse isotropy with any orientation). For this purpose, a new boundary element formulation with weak singularities is derived based on the principle of superposition and on the rigid body displacement method.[14-16] This new formulation involves the particular solutions of displacements and stresses related to the body force, which for the case of gravity can be obtained in exact closed-form.[17] The Green's functions involved are derived based on the solutions given by Pan and Chou[18] and are included in the BEM formulation in exact closed-form for the first time. Therefore, while the material anisotropic properties and the gravity body force are included in the new BEM formulation, the beauty and main advantage of the BEM are strictly preserved.

2. Formulation

Consider a finite anisotropic body as shown in *Figure 1* and let X, Y, Z (or X_1, X_2, X_3) be a global coordinate system. The finite body is subjected to body forces F_i ($i = 1, 2, 3$), boundary tractions on Γ_t and boundary displacements on Γ_u (*Figure 1*). Within the body, the displacement and stress fields must satisfy the following equations:

Equations of equilibrium

$$\sigma_{ij,j} + F_i = 0 \tag{1}$$

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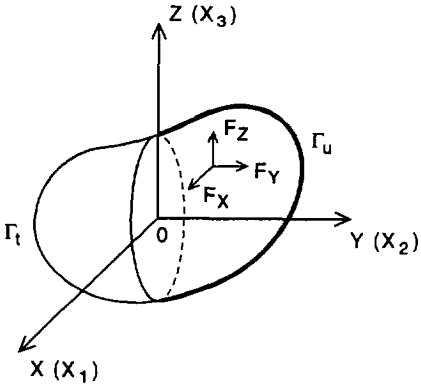

Figure 1. Geometry of the problem. A finite 3-D region with body forces F_i ($i = 1, 2, 3$). On boundaries Γ_u and Γ_t ($\Gamma = \Gamma_u + \Gamma_t$), displacements and tractions are described, respectively.

Constitutive relations

$$[\boldsymbol{\sigma}] = [\mathbf{c}][\mathbf{e}] \tag{2}$$

with

$$[\boldsymbol{\sigma}] = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{32}, \sigma_{31}, \sigma_{12}]^t \tag{3a}$$

$$[\mathbf{e}] = [e_{11}, e_{22}, e_{33}, 2e_{32}, 2e_{31}, 2e_{12}]^t \tag{3b}$$

and [**c**] being the elastic stiffness matrix of the anisotropic body.

Displacement and strain relations

$$e_{ij} = 0.5(u_{i,j} + u_{j,i}) \tag{4}$$

The displacements and stresses must also satisfy the following boundary conditions

$$u_j = \bar{u}_i \qquad X \in \Gamma_u \tag{5a}$$

$$\sigma_{ij} n_j = \bar{T}_i \qquad X \in \Gamma_t \tag{5b}$$

Because the problem is linear, the total displacements and stresses can be expressed as follows

$$u_i^{\mathrm{t}} = u_i^{\mathrm{h}} + u_i^{\mathrm{p}} \tag{6a}$$

$$\sigma_{ij}^{\mathrm{t}} = \sigma_{ij}^{\mathrm{h}} + \sigma_{ij}^{\mathrm{p}} \tag{6b}$$

$$T_i^{\mathrm{t}} = T_i^{\mathrm{h}} + T_i^{\mathrm{p}} \tag{6c}$$

where the superscript t denotes the total solution, h the homogenerous solution, and p a particular solution corresponding to the body forces F_i.

For the homogenerous part, the following integral equation holds at each internal point X_p^4:

$$u_i^{\mathrm{h}}(X_p) + \int_\Gamma T_{ij}^*(X_p, X) u_j^{\mathrm{h}}(X)\, d\Gamma(X) = \int_\Gamma U_{ij}^*(X_p, X) T_j^{\mathrm{h}}(X)\, d\Gamma(X) \tag{7}$$

with U_{ij}^* and T_{ij}^* being the Green's displacements and tractions.

Substituting equations (6a)–(6c) into equation (7) gives

$$\begin{aligned} u_i^{\mathrm{t}}(X_p) &+ \int_\Gamma T_{ij}^*(X_p, X) u_j^{\mathrm{t}}(X)\, d\Gamma(X) \\ &= \int_\Gamma U_{ij}^*(X_p, X) T_j^{\mathrm{t}}(X)\, d\Gamma(X) \\ &\quad + u_i^{\mathrm{p}}(X_p) + \int_\Gamma T_{ij}^*(X_p, X) u_j^{\mathrm{p}}(X)\, d\Gamma(X) \\ &\quad - \int_\Gamma U_{ij}^*(X_p, X) T_j^{\mathrm{p}}(X)\, d\Gamma(X) \end{aligned} \tag{8}$$

Before X_p is approached to the boundary, it is important to reduce the order of the singularity associated with T_{ij}^*. The singularity of T_{ij}^* on the left-hand side of equation (8) can be avoided by the rigid body displacement method,[14,15] whereas the one on the right-hand side can be weakened by the following identity, which is essentially equivalent to the rigid body displacement method[6]:

$$\int_\Gamma T_{ij}^*(X_p, X)\, d\Gamma(X) = -\delta_{ij} \tag{9}$$

Making use of this identity, equation (8) then becomes

$$\begin{aligned} u_i^{\mathrm{t}}(X_p) &+ \int_\Gamma T_{ij}^*(X_p, X) u_j^{\mathrm{t}}(X)\, d\Gamma(X) \\ &= \int_\Gamma U_{ij}^*(X_p, X) T_j^{\mathrm{t}}(X)\, d\Gamma(X) \\ &\quad + \int_\Gamma T_{ij}^*(X_p, X)\left[u_j^{\mathrm{p}}(X) - u_j^{\mathrm{p}}(X_p)\right] d\Gamma(X) \\ &\quad - \int_\Gamma U_{ij}^*(X_p, X) T_j^{\mathrm{p}}(X)\, d\Gamma(X) \end{aligned} \tag{10}$$

If X_p approaches a point Y on the boundary, we finally obtain the following boundary integral equation

$$\begin{aligned} b_{ij} u_j^{\mathrm{t}}(Y) &+ \int_\Gamma T_{ij}^*(Y, X) u_j^{\mathrm{t}}(X)\, d\Gamma(X) \\ &= \int_\Gamma U_{ij}^*(Y, X) T_j^{\mathrm{t}}(X)\, d\Gamma(X) \\ &\quad + \int_\Gamma T_{ij}^*(Y, X)\left[u_j^{\mathrm{p}}(X) - u_j^{\mathrm{p}}(Y)\right] d\Gamma(X) \\ &\quad - \int_\Gamma U_{ij}^*(Y, X) T_j^{\mathrm{p}}(X)\, d\Gamma(X) \end{aligned} \tag{11}$$

where b_{ij} are coefficients that depend only upon the local geometry of the boundary at Y.

It is noted that all the terms on the right-hand side of equation (11) have only weak singularities. Thus, this equation is integrable. Although the second term on the left-hand side of equation (11) has a strong singularity, it can be treated by the rigid body displacement method. At the same time, the calculation of b_{ij}, which is geometry dependent, can also be avoided.

Using the boundary conditions (5a)–(5b), equation (11) can be discretized and solved numerically for the unknown boundary displacements and tractions. Then equation (10) can be used to calculate the internal displacements. In order to calculate the internal stresses, we need to first take the derivative of equation (10) with respect to the internal coordinates X_p. This procedure results in the following equation:

$$u^{\mathrm{t}}_{i,k}(X_p) + \int_\Gamma T^*_{ij,k}(X_p, X)u^{\mathrm{t}}_j(X)\, d\Gamma(X)$$

$$= \int_\Gamma U^*_{ij,k}(X_p, X)T^{\mathrm{t}}_j(X)\, d\Gamma(X)$$

$$+ \int_\Gamma T^*_{ij,k}(X_p, X)[u^{\mathrm{p}}_j(X) - u^{\mathrm{p}}_j(X_p)]\, d\Gamma(X)$$

$$- \int_\Gamma T^*_{ij}(X_p, X)u^{\mathrm{p}}_{j,k}(X_p)\, d\Gamma(X)$$

$$- \int_\Gamma U^*_{ij,k}(X_p, X)T^{\mathrm{p}}_j(X)\, d\Gamma(X) \qquad (12)$$

Once $u^{\mathrm{t}}_{i,k}$ are obtained, equations (2), (3a), and (3b) can then be used to calculate the internal stresses. It is noted that in equations (11) and (12), the particular solutions of displacements and stresses (tractions), as well as the Green's displacements and stresses, need to be provided. In the following sections, we will present the particular solution associated with the gravity body force and the Green's functions for anisotropic media.

3. Particular solution related to gravity body force

A very simple and exact closed-form solution was provided by the authors[17] for gravity-induced stresses in a generalized anisotropic elastic half-space. That solution can also be adopted as a particular solution of stresses for our analysis here. Assuming that gravity acts in the $-Z$ (or $-X_3$) direction, the particular solution of stresses can be expressed as[17]:

$$\sigma^{\mathrm{p}}_{11} = c_1\gamma X_3 \qquad (13a)$$

$$\sigma^{\mathrm{p}}_{22} = c_2\gamma X_3 \qquad (13b)$$

$$\sigma^{\mathrm{p}}_{33} = \gamma X_3 \qquad (13c)$$

$$\sigma^{\mathrm{p}}_{12} = c_3\gamma X_3 \qquad (13d)$$

$$\sigma^{\mathrm{p}}_{32} = \sigma^{\mathrm{p}}_{31} = 0 \qquad (13e)$$

Making use of equations (2)–(4), the particular solution of displacements is obtained by integrating equations (13a)–(13e):

$$u^{\mathrm{p}}_1 = d_1\gamma X_3^2 \qquad (14a)$$

$$u^{\mathrm{p}}_2 = d_2\gamma X_3^2 \qquad (14b)$$

$$u^{\mathrm{p}}_3 = d_3\gamma X_3^2 \qquad (14c)$$

In equations (13)–(14), γ is the unit weight of the medium, and

$$c_1 = [c_{13}c_4 - c_{14}c_5 + c_{15}c_6]/d \qquad (15a)$$

$$c_2 = [c_{23}c_4 - c_{24}c_5 + c_{25}c_6]/d \qquad (15b)$$

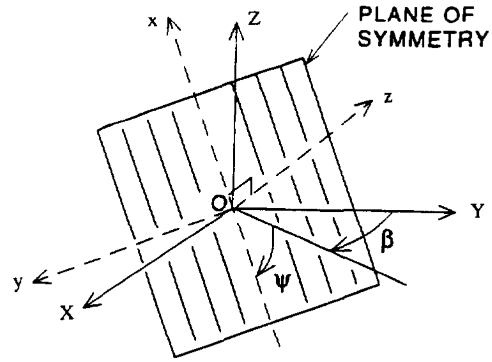

Figure 2. Relationship between global (X, Y, Z) and local (x, y, z) coordinate systems. β and ψ are the dip orientation and dip angle of the plane of transverse isotropy, respectively.

$$c_3 = [c_{36}c_4 - c_{46}c_5 + c_{56}c_6]/d \qquad (15c)$$

$$d_1 = 0.5c_6/d \qquad c_4 = c_{44}c_{55} - c_{45}^2 \qquad (15d)$$

$$d_2 = -0.5c_5/d \qquad c_5 = c_{34}c_{55} - c_{35}c_{45} \qquad (15e)$$

$$d_3 = 0.5c_4/d \qquad c_6 = c_{34}c_{45} - c_{35}c_{44} \qquad (15f)$$

$$d = c_{33}c_4 - c_{34}c_5 + c_{35}c_6 \qquad (15g)$$

It is easy to show that the particular solutions of stresses and displacements above satisfy equations (1)–(4) as well as the compatibility conditions.

4. Green's displacements and stresses

It is well known that there is no closed-form solution for the Green's displacements and stresses in an infinite and generalized anisotropic elastic medium. So far, the Green's solutions in 3-D for any kind of anisotropy have been obtained by numerical integration. However, if the medium is transversely isotropic, a closed-form solution is available in a coordinate system attached to the plane of transverse isotropy. Let (x, y, z) be that coordinate system as shown in *Figure 2*, the Green's displacements and stresses derived by Pan and Chou[18] can be expressed as

$$u^k_i = f_{ik}(x, y, z) \qquad (16)$$

and

$$\sigma^k_{ij} = g_{ijk}(x, y, z) \qquad (17)$$

The subscript i $(=x, y, z)$ in equation (16) denotes the displacement components, and ij $(=xx, yy, zz, yz, xz, xy)$ in equation (17) denotes the stress components. In both equations (16) and (17), the superscript k $(=x, y, z)$ denotes the components of point forces, and $f_{ik}(x, y, z)$ and $g_{ijk}(x, y, z)$ are the closed-form expressions that can be directly derived from the results given by Pan and Chou.[18] It is worthwhile to point out that in rederiving $f_{ik}(x, y, z)$ and $g_{ijk}(x, y, z)$, one error was found in Pan and Chou's formulation for the Green's stresses (Pan and Chou,[18] σ_{31} on p. 611). Once the Green's solutions are given in the local (x, y, z) coordinates, the corresponding

Green's solutions in the global (X, Y, Z) coordinates can be obtained by coordinate transformations as follows

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\cos\psi\sin\beta & -\cos\psi\cos\beta & \sin\psi \\ \cos\beta & -\sin\beta & 0 \\ \sin\psi\sin\beta & \sin\psi\cos\beta & \cos\psi \end{bmatrix} \times \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (18)$$

where β and ψ are the orientation angles of the plane of transverse isotropy (*Figure 2*).

The Green's displacements and stresses caused by the point forces in the local (x, y, z) coordinates are first transformed into those caused by the point forces in the global (X, Y, Z) coordinates:

$$[u_i^X, u_i^Y, u_i^Z]^t = [\mathbf{H}][u_i^x, u_i^y, u_i^z]^t \quad (19)$$

$$[\sigma_{ij}^X, \sigma_{ij}^Y, \sigma_{ij}^Z]^t = [\mathbf{H}][\sigma_{ij}^x, \sigma_{ij}^y, \sigma_{ij}^z]^t \quad (20)$$

where $i = x, y, z$ in equation (19) and $ij = xx, yy, zz, yz, xz, xy$ in equation (20). The elements of matrix $[\mathbf{H}]$ are given in the Appendix.

With equations (19) and (20), the Green's displacements and stresses in the global (X, Y, Z) coordinates can be determined by the following transformations:

$$[u_X^k, u_Y^k, u_Z^k]^t = [\mathbf{H}][u_x^k, u_y^k, u_z^k]^t \quad (21)$$

$$[\sigma_{XX}^k, \sigma_{YY}^k, \sigma_{ZZ}^k, \sigma_{YZ}^k, \sigma_{XZ}^k, \sigma_{XY}^k]^t = [\mathbf{Q}][\sigma_{xx}^k, \sigma_{yy}^k, \sigma_{zz}^k, \sigma_{yz}^k, \sigma_{xz}^k, \sigma_{xy}^k]^t \quad (22)$$

In equations (21) and (22), $k = X, Y, Z$ denotes the component of point forces in the global (X, Y, Z) coordinates, and $[\mathbf{Q}]$ is a 6×6 matrix whose components are given in the Appendix.

The Green's displacements and stresses in the local (x, y, z) coordinates are also differentiated with respect to (x, y, z), and transformed into the global (X, Y, Z) coordinates using transformations similar to equations (19)–(22).

5. Numerical examples

The boundary integral equation (11) was solved numerically using a 3-D BEM program written by the authors. In this program, the boundary of the continuum is assumed to be discretized into nine-node quadrilateral curved elements. The displacements and tractions on the boundary of each element are expressed in terms of their corresponding node values by the following equations

$$u_i = \sum_{k=1}^{9} \phi_k u_i^k \qquad i = 1, 2, 3 \quad (23a)$$

$$T_i = \sum_{k=1}^{9} \phi_k T_i^k \qquad i = 1, 2, 3 \quad (23b)$$

where $\phi_k (k = 1 \text{ to } 9)$ are nine shape functions.[4] Assuming isoparametric elements, the coordinates at any point in an element are related to the element nodal coordinates as follows

$$X_i = \sum_{k=1}^{9} \phi_k X_i^k \qquad i = 1, 2, 3 \quad (24)$$

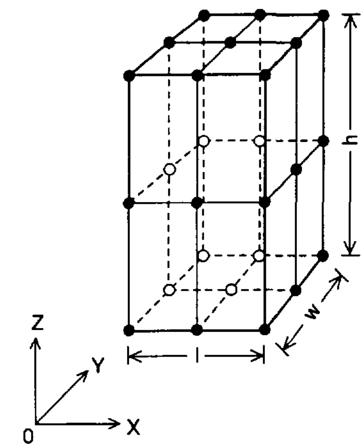

Figure 3. Block of $l \times w \times h$ discretized with six nine-node quadrilateral elements with a total of 26 nodes.

In the numerical examples presented below, we consider the simple example of an anisotropic block (parallelepiped) which is subject to a uniform compression or gravity (*Figure 4a, 4b,* and *4c*). The block is $l \times w \times h$ m^3 and its geometry is shown in *Figure 3*. Its surface is discretized using only 6 nine-node quadratic elements with a total of 26 nodes.

In the first example, the block ($l = w = h = 1$ m) is subject to a uniaxial compression of 1 kN/m^2 acting on two of its sides in the Z (vertical) direction (*Figure 4a*). The medium is transversely isotropic with $E = 12 \times 10^4$ kN/m^2, $E' = 4 \times 10^4$ kN/m^2, $\nu = \nu' = 0.25$, $G = 4.8 \times 10^4$ kN/m^2, $G' = 1.6 \times 10^4$ kN/m^2, where E and E' are Young's moduli in the plane of transverse isotropy and in the direction normal to it, respectively; ν and ν' are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel and normal to it, respectively; and $G = 0.5E/(1 + \nu)$ and G' are the shear moduli in the plane of transverse isotropy and in planes normal to this plane, respectively (*Figure 2*).

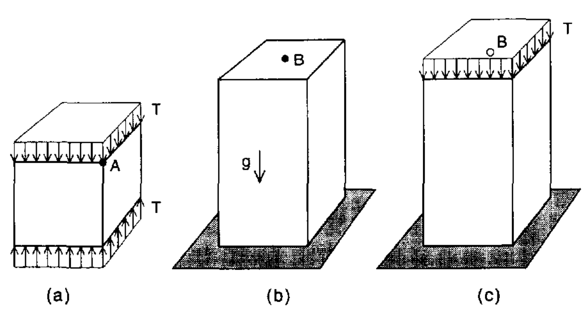

Figure 4. Block under uniform vertical compression on its upper and bottom faces in (a), under gravity and resting on a rigid base in (b), and on a rigid base under uniform vertical compression on its upper face in (c).

Table 1. Block under uniform compression (surface displacements $\times 10^{-5}$ m).

	Case I ($\beta=\psi=0$)		Case II ($\beta=0$, $\psi=90$)		Case III ($\beta=0$, $\psi=45$)		Case IV ($\beta=60$, $\psi=45$)	
	Numerical	Exact	Numerical	Exact	Numerical	Exact	Numerical	Exact
u_X	0.3125	0.3125	0.1042	0.1042	0.2083	0.2083	−0.5496	−0.5496
u_Y	−0.3125	−0.3125	−0.3125	−0.3125	−1.3542	−1.3542	−0.7032	−0.7031
u_Z	−1.2500	−1.2500	−0.4167	−0.4167	−1.0417	−1.0417	−1.0417	−1.0417

Table 2. Block under uniform compression (internal displacements and stresses) (transversely isotropic Case IV: $\beta=60$, $\psi=45$).

	u_Z ($\times 10^{-5}$ m)		σ_{XX} (kN/m^2)		σ_{YY} (kN/m^2)		σ_{ZZ} (kN/m^2)	
Z (m)	Numerical	Exact	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.75	−0.5208	−0.5209	0.0011	0.0000	−0.0040	0.0000	−1.001	−1.000
0.625	−0.2604	−0.2604	−0.0000	0.0000	−0.0000	0.0000	−1.000	−1.000
0.5	0.0000	0.0000	−0.0000	0.0000	0.0000	0.0000	−1.000	−1.000
0.375	0.2604	0.2604	−0.0000	0.0000	−0.0000	0.0000	−1.000	−1.000
0.25	0.5208	0.5208	0.0014	0.0000	−0.0044	0.0000	−0.998	−1.000

For this case, exact solutions for stresses and displacements are available[19] and can be expressed as follows:

$$\sigma_{ZZ} = -T \qquad (25a)$$

$$\sigma_{ij} = 0 \text{ for } ij \neq ZZ \qquad (25b)$$

$$u_X = -T(a_{13}X + a_{36}Y + a_{35}Z) + \omega_Y Z - \omega_Z Y + u_X^0 \qquad (26a)$$

$$u_Y = -T(a_{23}Y + a_{34}Z) + \omega_Z X - \omega_X Z + u_Y^0 \qquad (26b)$$

$$u_Z = -Ta_{33}Z + \omega_X Y - \omega_Y X + u_Z^0 \qquad (26c)$$

where u_X^0, u_Y^0, u_Z^0, ω_X, ω_Y, ω_Z are six constants which can be determined by six independent displacement boundary conditions; and a_{ij} are the elements of the elastic compliance matrix $[\mathbf{a}]$ ($[\mathbf{a}] = [\mathbf{c}]^{-1}$).

Displacements in the X, Y, and Z directions at the corner A of *Figure 4a* are given in *Table 1* for several anisotropic cases and are compared with the exact values. This table indicates that the 3-D BEM program reproduces essentially the exact solution. This is also true for the internal displacements and stresses as shown in *Table 2* for the transversely isotropic Case IV ($\beta=60$; $\psi=45$). The internal values are calculated along the vertical central line of the block. *Table 2* also indicates that in calculating the internal displacements and stresses, the position of the internal point cannot be too close to the boundary.

The second example corresponds to a block ($l = w = 1$ m, $h = 2$ m) resting on a rigid base and subject to gravity (*Figure 4b*) or to a uniform vertical compression (*Figure 4c*). For these two cases, no exact closed-form solution is available. However, simple solutions exist on the basis of St. Venant's principle (Lekhnitskii,[19] pp. 75–79). The results of the analysis are reported in *Table 3*. The upper row in *Table 3* gives the vertical displacements at the center of the upper surface of the block (point B in *Figures 4b* and *4c*) induced by gravity (g) with $\gamma = 27$ kN/m^3, and the lower row gives the vertical displacements at point B induced by a uniform vertical compression (T) equal to -1 kN/m^2. For both loading conditions, the elastic constants for the block are the same, i.e., $\nu = 0.25$, and $G = 1.6 \times 10^4$ kN/m^2. In *Table 3*, the first column gives the approximate results based on St. Venant's principle,[19] the second gives the results obtained using the Galerkin vector approach,[4,5] whereby the volumetric integrals entering into the BEM formulation are transformed

Table 3. Isotropic block on a rigid base (surface displacements).

	St. Venant's	Galerkin's	Present
u_Z ($\times 10^{-2}$ m) (g)	−0.135	−0.135	−0.133
u_Z ($\times 10^{-4}$ m) (T)	−0.500	−0.498	−0.493

Table 4. Block on rigid base (internal displacements and stresses) (transversely isotropic Case IV: $\beta=60$, $\psi=45$).

	Gravity				Uniform vertical compression			
	u_Z ($\times 10^{-2}$ m)		σ_{ZZ} ($\times 10^2$ kN/m^2)		u_Z ($\times 10^{-4}$ m)		σ_{ZZ} (kN/m^2)	
Z	Numerical	S.V.	Numerical	S.V.	Numerical	S.V.	Numerical	S.V.
1.5	−0.094	−0.105	−0.133	−0.135	−0.287	−0.312	−0.996	−1.000
1.25	−0.084	−0.096	−0.200	−0.203	−0.233	−0.260	−0.997	−1.000
1.0	−0.070	−0.084	−0.272	−0.270	−0.179	−0.208	−1.008	−1.000
0.75	−0.054	−0.068	−0.360	−0.338	−0.125	−0.156	−1.044	−1.000
0.5	−0.034	−0.049	−0.460	−0.405	−0.074	−0.104	−1.091	−1.000

into boundary integrals, and the last column gives the results obtained with the present method. It is obvious that all three approaches lead to comparable results.

Finally, *Table 4* shows a comparison between the internal displacement and stress in the vertical direction calculated by the present program (Numerical) and those obtained with the St. Venant's solution (S.V.). Again, for the gravity case, $\gamma = 27$ kN/m^3, and for the uniform vertical compression case, $T = -1$ kN/m^2. The block, however, is transversely isotropic with properties being the same as in Case IV ($\beta = 60$, $\psi = 45$) above. The internal displacements and stresses are calculated along the central line of the block. Along this line, the solution based on St. Venant's principle is expected to give reasonably accurate results, as shown in *Table 4*, where even for the point near the rigid base ($Z = 0.5$), the results from St. Venant's principle are still close to those calculated with the present program. However, the purpose of *Table 4* is not to justify the valibility of St. Venant's principle but to demonstrate the validity of the present program. It is noteworthy that the Galerkin vector approach,[4,5] which transforms the volumetric integrals into the boundary integrals is only applicable to isotropic media. For anisotropic media, the Galerkin vector approach does not work and volumetric integration is unavoidable.

6. Conclusions

A new boundary element formulation is derived in this paper for any oriented transversely isotropic elastic media. The particular solutions for the gravity body force are given in a simple and exact closed-form and are incorporated rigorously into the boundary element formulation. The Green's functions are also derived in exact closed-form for a transversely isotropic elastic medium with planes of elastic symmetry having any orientation.

Several numerical examples were studied and carefully checked with existing closed-form solutions. It is found that even with a very coarse discretization, excellent results could be obtained.

While the present program can be used to calculate the displacements and stresses in an anisotropic and gravity-loaded finite body, it can be extended to calculate the displacements and stresses induced by excavations in an infinite anisotropic medium subjected to triaxial far-field stresses. This is currently under investigation, as well as the fracture propagation in anisotropic media.

Appendix A

Elements of matrix **[H]** in equations (19)–(21):

$$\begin{bmatrix} -\cos\psi\sin\beta & \cos\beta & \sin\psi\sin\beta \\ -\cos\psi\cos\beta & -\sin\beta & \sin\psi\cos\beta \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \quad \text{(A1)}$$

Elements of matrix **[Q]** in equation (22):

$$\begin{bmatrix} H_{11}^2 & H_{12}^2 & H_{13}^2 & 2H_{12}H_{13} & 2H_{11}H_{13} & 2H_{11}H_{12} \\ H_{21}^2 & H_{22}^2 & H_{23}^2 & 2H_{22}H_{23} & 2H_{21}H_{23} & 2H_{21}H_{22} \\ H_{31}^2 & H_{32}^2 & H_{33}^2 & 2H_{32}H_{33} & 2H_{31}H_{33} & 2H_{31}H_{32} \\ H_{21}H_{31} & H_{22}H_{32} & H_{23}H_{33} & H_{22}H_{33}+H_{32}H_{23} & H_{23}H_{31}+H_{33}H_{21} & H_{21}H_{32}+H_{31}H_{22} \\ H_{31}H_{11} & H_{32}H_{12} & H_{33}H_{13} & H_{12}H_{33}+H_{32}H_{13} & H_{13}H_{31}+H_{33}H_{11} & H_{11}H_{32}+H_{31}H_{12} \\ H_{11}H_{21} & H_{12}H_{22} & H_{13}H_{23} & H_{12}H_{23}+H_{22}H_{13} & H_{13}H_{21}+H_{23}H_{11} & H_{11}H_{22}+H_{21}H_{12} \end{bmatrix} \quad \text{(A2)}$$

where H_{ij} are elements of matrix **[H]** given in (A1).

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