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A symmetric boundary integral approach to transient poroelastic analysis

E. Pan, G. Maier

Abstract The problem of the transient quasi-static analysis of a poroelastic body subjected to a history of external actions is formulated in terms of four boundary integral equations, using time-dependent Green's functions of the "free" poroelastic space. Some of these Green's functions, not available in the literature are derived "ad hoc". The boundary integral operator constructed is shown to be symmetric with respect to a time-convolutive bilinear form so that the boundary solution is characterized by a variational property and its approximation preserving symmetry can be achieved by a Galerkin boundary element procedure.

1 Introduction

Poroelasticity is concerned with heterogeneous media consisting of an elastic solid skeleton saturated by a diffusing pore fluid. Its range of application covers a variety of important real-life problems, such as design of nuclear reactor cores, exploitation of oil or gas deposits, simulations of living bone behaviour to orthopaedical surgery purposes, control of filtration leakage from reservoirs, and manufacturing process design for composite materials. Poroelasticity is now the subject of a fairly abundant literature, stemming from Terzaghi's concept of effective stress and Biot's linear consolidation theory (1941, 1957, 1962). From a computational mechanics point of view, poroelastic analysis has been conducted using either the finite element method or the traditional (collocation) boundary element method.

The comprehensive state-of-the-art review by Cheng and Detournay (1993) provides abundant, clearly presented information, updated to 1991, on constitutive models, problem formulations, solution methodologies and geotechnical applications. Systematic presentations of coupled problems from the computational standpoint and of the phenomenological theory of porous media from the continuum mechanics standpoint are contained in Lewis and Schrefler (1987) and in the recent treatise by Coussy (1995), respectively. Among the noteworthy publications on poroelasticity, somehow related to the present study, we quote here those due to Manolis and Beskos (1989), Pan (1991), Zhang and Cowin (1995), Chiou and Chi (1994), Chen and Dargush (1995). The last two of the above contributions specifically concern boundary integral equation (BIE)-boundary element (BE) methods based on traditional approaches which make use of a single kind of fundamental solution (mostly "single layer" concentrated sources), the first two provide fundamental solutions for both single and double layer sources.

In the last few years, novel "symmetric" boundary integral equation formulations have been proposed for a number of boundary-value (BV) and boundary-initial-value (BIV) problems. Most of these formulations are based on a suitably combined use of both "static" (or "single layer") and "kinematic" (or "double layer") sources. On such basis, symmetry-preserving boundary element methods have been developed by a weighted-residual Galerkin approach and proven to be computationally advantageous in various circumstances.

Representative contributions of this research line for time-independent problems are contained in papers by Maier and Polizzotto (1987), Polizzotto (1988), Sirtori et al. (1992), Bonnet and Bui (1993), Maier et al. (1993), Balakrishna et al. (1994). Basic concepts of the symmetric Galerkin boundary element method (SGBEM), with emphasis of computational aspects, are lucidly presented, and inserted within the broader BIE-BE context, in recent books by Kane (1994) and Bonnet (1995).

This paper (anticipated in a communication of the ICES-IABEM Conference in the Hawaii, 1995) is intended to provide a symmetric BIE formulation for linear poroelasticity. The approach adopted here, presumably for the first time in "coupled problems", is inspired by the symmetric BIE-BE approach used in elastodynamics (Maier et al., 1991) and viscoelasticity (Carini et al., 1991). It is based on fundamental solutions in transient isotropic poroelasticity which either were available in the literature (Cheng and Predeleanu, 1987; Detournay and Cheng, 1987; Pan, 1991; Smith and Booker, 1993) or have been generated "ad hoc" in the present investigation for both two- and three-dimensional problems.

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E. Pan
Dept. of Civil Engineering, Univ. of Colorado, Boulder, CO 80309, USA

G. Maier
Dept. of Structural Engineering, Politecnico of Milan, 20133, Milan, Italy

Correspondence to: G. Maier

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The governing equations for transient poroelasticity adopted herein are those for the classical fully-coupled Biot porous medium. Time-dependent discontinuities of total traction and pore pressure and, simultaneously, of cumulative fluid flux ("single layer" sources) and solid displacements ("double layer" sources) are considered on the boundary of a homogeneous poroelastic body embedded in an unbounded poroelastic space. Using time-dependent Green's functions for this "free" space, the analysis of the transient response of the poroelastic body to external actions is formulated in terms of four BIEs governing the time histories of the boundary unknowns. The integral operator thus generated can be proven to be symmetric in both space and time with respect to a time-convolutive bilinear form. A variational principle is then derived, which characterizes the boundary solutions in time. Boundary element discretization in space and time by a Galerkin weighted-residual approach leads to linear algebraic equations with a symmetric coefficient matrix.

Though not discussed herein, computational benefits of this symmetry are reasonably expected as they have been pointed out in other contexts (Sirtori et al. 1992, Maier et al. 1993, Balakrishna et al. 1994, Kane 1994).

2
Problem formulation

The boundary-initial value (BIV) problem considered herein concerns the quasi-static (no inertia effects) response of an isotropic poroelastic body to a given history of external actions over a time interval $0 \le t \le T$. The body occupies a two- or three-dimensional open domain Ω with a smooth boundary Γ. The governing equations are formulated as follows. Tensorial notation is adopted with the index summation rule and with commas denoting derivatives with respect to Cartesian space coordinates x.

Equilibrium relates total stresses σ_{ij} to bulk (solid and fluid) body forces F_i:

$$\sigma_{ij,j} + F_i = 0 \quad \text{in } \Omega \tag{1}$$

The transport law relates the fluid flux q_i (or specific discharge: volume of fluid per unit time and unit surface normal to the i-th axis) to the pressure p and to the fluid body force f_i, according to Darcy's law for seepage through porous media:

$$q_i = -k(p_{,i} - f_i) \quad \text{in } \Omega \tag{2}$$

where k is the (constant) permeability coefficient, which captures the combined effects of the pore geometry and fluid viscosity. This law presumes constant fluid density ρ, so that $f_i = \rho g_i$ is an assigned constant, with g_i being the i-th gravity component.

For the solid skeleton the geometry compatibility of strains e_{ij} with respect to the displacement field u_i is assumed linear:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega \tag{3}$$

For the fluid phase the mass conservation (or continuity) law involves the given source density γ (injected fluid volume per unit time and unit volume of porous medium), the specific discharge q_i and the fluid content ζ (fluid volume per unit volume of porous medium):

$$\frac{d\zeta}{dt} + q_{i,i} = \gamma \quad \text{in } \Omega \tag{4}$$

The coupled constitutive laws for the solid and fluid phase are expressed according to Rice and Cleary (1976), in the form:

$$\sigma_{ij} + \alpha p \delta_{ij} = 2Ge_{ij} + \frac{2G\nu}{1-2\nu} e\delta_{ij} \quad \text{in } \Omega \tag{5}$$

$$p = -\frac{2GB(1+\nu_u)}{3(1-2\nu_u)} e + \frac{2GB^2(1-2\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-2\nu_u)} \zeta \quad \text{in } \Omega \tag{6}$$

where $e = e_{ii}$ denotes the solid volumetric strain and δ_{ij} the Kronecker delta.

Five material parameters intervene in the above constitution, namely: the bulk shear modulus G; the drained and undrained Poisson's ratios ν and ν_u, respectively; the Biot coefficient of effective stress α; and the Skempton pore pressure coefficient B. Among them, only four parameters are independent because of the following relation (see e. g. Cheng and Predeleanu, 1987):

$$\alpha = \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)} \tag{7}$$

The essential (Dirichlet) and natural (Neumann) boundary conditions are formulated on two disjointed and complementary parts of Γ. Namely, for the solid phase ($\Gamma_u \cup \Gamma_t = \Gamma$):

$$u_i = \overline{u}_i(\boldsymbol{x}_h, t), \ \boldsymbol{x}_h \in \Gamma_u; \qquad \sigma_{ij} n_j = \overline{t}_i(\boldsymbol{x}_h, t), \quad \boldsymbol{x}_h \in \Gamma_t \tag{8a, b}$$

and for the fluid ($\Gamma_p \cup \Gamma_q = \Gamma$):

$$p = \overline{p}(\boldsymbol{x}_h, t), \ \boldsymbol{x}_h \in \Gamma_p; \qquad q_i n_i = \overline{q}(\boldsymbol{x}_h, t), \quad \boldsymbol{x}_h \in \Gamma_q \tag{9a, b}$$

The initial conditions referred to herein read:

$$u_i(\boldsymbol{x}_h, t) = \overline{u_i}; \ p(\boldsymbol{x}_h, t) = \overline{p}, \quad \text{in } \Omega \text{ at } t = 0 \tag{10}$$

A stress field σ_{ij} complying with the balance Eqs. (1) and (8b) at $t = 0$ (instead of displacements), and/or a flux field q_i complying with the continuities equations (4) and (9b) at $t = 0$ (instead of pressure), represent alternative initial conditions which would preserve the well-posedness of the problem.

It will be convenient in what follows to express the fluid mass conservation in the time-integral form alternative to Eq. (4):

$$\zeta + v_{i,i} = \zeta_0 + Q \tag{11}$$

Here, Q denotes the cumulative fluid volume injected per unit mixture volume up to time t:

$$Q = \int_0^t \gamma \, \mathrm{d}t' \tag{12}$$

In Eq. (11), ζ_0 is the initial fluid content, which has no role in the problem if full saturation of the solid is a priori guaranteed throughout Ω. With this hypothesis, we can set $\zeta_0 = 0$, namely ζ will be conceived henceforth as variation of the fluid content per unit bulk volume. The time-cumulative flux components

$$v_i = \int_0^t q_i \, \mathrm{d}t' \tag{13}$$

can be interpreted, and will be referred to, as "relative fluid displacements".

Consistent with Eq. (11), the natural (Neumann) boundary condition (9b) will be replaced by

$$v \equiv v_i n_i = \overline{v}(\boldsymbol{x}_h, t), \quad \boldsymbol{x}_h \in \Gamma_v \equiv \Gamma_q \tag{14}$$

where v represents the time-cumulative outward normal flux, i.e. the relative fluid displacement vector v_i projected on normal n_i:

$$v = \int_0^t q_i n_i \mathrm{d}t' \equiv \int_0^t q \mathrm{d}t' \tag{15}$$

The linear BIV problem defined (in a "strong", differential form) by Eqs. (1)–(10) [with the alternatives (11) and (14)] will be re-formulated in terms of a peculiar system of boundary integral equations (BIE) in the next section and this system will be shown to be self-adjoint (or "symmetric") in the subsequent sections.

3
A symmetric BIE formulation

In the BIV problem of Section 2, for the sake of simplicity, the following further restrictions will be adopted henceforth, but might easily be relaxed without conceptually affecting the conclusions.

(i) The solution concerns the transient response to a perturbation of an initially stationary, time-independent state. In other terms, the initial (at $t = 0$) fields satisfy the governing equations of the steady state, so that they can be ignored, i.e. $\overline{u_i} = 0$ and $\overline{p} = 0$ can and will be set in equations (10). This hypothesis merely avoids cumbersome domain integrals of data in the sequel.

(ii) The external actions act on the boundary Γ only, namely: $F_i = 0, f_i = 0, \gamma = 0$ (and, hence $Q = 0$) in Ω. Thus the relevant domain integral data will not show up in subsequent developments, but could be trivially recovered.

(iii) The domain Ω contains a homogeneous poroelastic medium. The extension of the present approach to zone-wise inhomogeneous system could be carried out according to the path of reasoning proposed for the GSBEM in elastodynamics by Maier et al., (1991).

Henceforth, matrix notation will be frequently adopted, bold-face symbols denoting matrices (or vectors) and a superscript "t" transposition.

Let the domain Ω be thought of as embedded in a homogeneous poroelastic "free" space Ω_∞. We denote by Γ^- and Γ^+ two (smooth) surfaces infinitely close to Γ: the former inside Ω with unit outward normal $\boldsymbol{v}^-$ (coincident with the outward normal vector $\boldsymbol{v}$ to Γ, alternatively denoted by $\boldsymbol{n}$); the latter outside Ω and conceived as the boundary of the exterior, complementary domain $\Omega_{\infty^-}\ \Omega$, endowed with outward normal $\boldsymbol{v}^+ = -\boldsymbol{v}^- = -\boldsymbol{v}$. Marking by superscripts + and − quantities defined over Γ^+ and Γ^-, respectively, let the following discontinuities be defined across Γ at all points $\boldsymbol{\xi} \in \Gamma$:

$$\Delta\, \boldsymbol{t} = \boldsymbol{t}^+ + \boldsymbol{t}^-; \qquad \Delta\, \boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^- \tag{16a}$$

$$\Delta\, p = p^+ - p^-; \qquad \Delta\, v = -(v^+ + v^-) \tag{16b}$$

If these discontinuities are now conceived as external actions (or "sources") acting on the unbounded poroelastic space Ω_∞ as functions of $\boldsymbol{\xi} \in \Gamma$ ("source point" or "load point") and of time $\tau (0 \le \tau \le T)$, their effects in the field point $\boldsymbol{x} (\neq \boldsymbol{\xi})$ at time $t > \tau$, can be evaluated by superposition through Green's functions of Ω_∞, which are provided in analytical form in Appendix A and discussed in Section 4. These two-points functions or kernels will be gathered in matrices (scalars in some cases) denoted by $\boldsymbol{G}_{hk}$. The former subscript $h (= u, t, v, p)$ is intended to specify the nature of the effect at $(\boldsymbol{x}, t)$, corresponding to solid phase displacements, total tractions, cumulative normal flux and pore pressure, respectively. The latter subscript $k (= u, t, v, p)$ specify the nature of the discontinuity source concentrated at $(\boldsymbol{\xi}, \tau)$, precisely: subscript u corresponds to the static traction discontinuity $\Delta \boldsymbol{t}$ as the cause; t to the kinematic source, i.e. to displacement discontinuity $\Delta \boldsymbol{u}$; v recalls the pressure jump Δp as the cause; p refers to the jump Δv of the time-integrated normal flux (note that each second subscript uses the symbol of the conjugate of the relevant source quantity).

The arguments of the above influence functions will be omitted later for brevity, unless needed to avoid ambiguity. They read: $(\boldsymbol{x}, \boldsymbol{\xi}; t - \tau; \oplus, \otimes)$, where $\oplus$ becomes $\boldsymbol{n}$ and/or $\otimes$ becomes $\boldsymbol{v}$ whenever $\boldsymbol{G}_{hk}$ depends also on the outward normal $\boldsymbol{n}$ in $\boldsymbol{x}$ (it does for $h = t$ and v) and/or the outward normal $\boldsymbol{v}$ in $\boldsymbol{\xi}$ (it does for $k = t$ and v, i.e. for concentrated jumps of displacements and pressure across Γ in $\boldsymbol{\xi}$).

Adopting the above symbology, we can express formally the announced superposition for each one of the four kinds of effects in the free space Ω_∞ due to discontinuity sources of the four kinds specified by Eqs. (16a) and (16b) and distributed on Γ. This effect superposition can be cast into the following concise form:

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{t} \\ v \\ p \end{bmatrix}_{(x,t)} = \int_0^t \int_\Gamma \begin{bmatrix} \boldsymbol{G}_{uu}(\boldsymbol{x},\xi;t-\tau,\oplus,\otimes) & \boldsymbol{G}_{ut}(\boldsymbol{x},\xi;t-\tau,\oplus,\boldsymbol{v}) & \boldsymbol{G}_{uv}(\boldsymbol{x},\xi;t-\tau,\oplus,\boldsymbol{v}) & \boldsymbol{G}_{up}(\boldsymbol{x},\xi;t-\tau,\oplus,\otimes) \\ \boldsymbol{G}_{tu}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\otimes) & \boldsymbol{G}_{tt}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\boldsymbol{v}) & \boldsymbol{G}_{tv}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\boldsymbol{v}) & \boldsymbol{G}_{tp}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\otimes) \\ \boldsymbol{G}_{vu}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\otimes) & \boldsymbol{G}_{vt}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\boldsymbol{v}) & G_{vv}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\boldsymbol{v}) & G_{vp}(\boldsymbol{x},\xi;t-\tau,\boldsymbol{n},\otimes) \\ \boldsymbol{G}_{pu}(\boldsymbol{x},\xi;t-\tau,\oplus,\otimes) & \boldsymbol{G}_{pt}(\boldsymbol{x},\xi;t-\tau,\oplus,\boldsymbol{v}) & G_{pv}(\boldsymbol{x},\xi;t-\tau,\oplus,\boldsymbol{v}) & G_{pp}(\boldsymbol{x},\xi;t-\tau,\oplus,\otimes) \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{t} \\ \Delta \boldsymbol{u} \\ \Delta p \\ \Delta v \end{bmatrix}_{(\xi,\tau)} \mathrm{d}\Gamma \mathrm{d}\tau \tag{17}$$

Equation (17) concerning the poroelastic space Ω_∞ is linked to the BIV problem formulated in Section 2 for the actual system in $\overline{\Omega} = \Omega \cup \Gamma$, if the following provisions are taken.

(a). The effects on the l.h.s. of Eq. (17) are evaluated on $\boldsymbol{x} \in \Gamma^-$ and identified with the four actual boundary fields, each one of which is known or unknown on disjointed complementary parts of Γ, according to the boundary conditions (8), (9a) and (14). Only the given fields (barred symbols) and the relevant parts of Γ are preserved on the l.h.s. of Eq. (17), which thus is reduced to the vector of data shown below on the r.h.s of Eq. (18).

(b). The external domain $\Omega_\infty - \Omega$ is imposed to be unperturbed: namely, in the expressions (16a) and (16b) of the discontinuities all the former addends (marked by + as concerning Γ^+) are set to zero. The latter addends (minus, on Γ^-) in Eqs. (16a) and (16b) are identified with the given fields (barred symbols) on the relevant portions of Γ according to conditions (8), (9a) and (14), and with the unknown fields on their corresponding complements of Γ.

By means of the above interventions in Eq. (17), one arrives at the following system of integral equations which governs the solution, on the boundary alone, to the BIV problem of Section 2, subject to the weak restrictions adopted at the beginning of the present Section:

$$\begin{aligned} \begin{bmatrix} \overline{\boldsymbol{u}} \text{ on } \Gamma_u \\ -\overline{\boldsymbol{t}} \text{ on } \Gamma_t \\ -\overline{v} \text{ on } \Gamma_v \\ \overline{p} \text{ on } \Gamma_p \end{bmatrix}_{(x,t)} &= \int_0^t \int_{\Gamma_k} \begin{bmatrix} \boldsymbol{G}_{uu} & -\boldsymbol{G}_{ut} & -\boldsymbol{G}_{uv} & -\boldsymbol{G}_{up} \\ -\boldsymbol{G}_{tu} & \boldsymbol{G}_{tt} & \boldsymbol{G}_{tv} & \boldsymbol{G}_{tp} \\ -\boldsymbol{G}_{vu} & \boldsymbol{G}_{vt} & G_{vv} & G_{vp} \\ \boldsymbol{G}_{pu} & -\boldsymbol{G}_{pt} & -G_{pv} & -G_{pp} \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \text{ on } \Gamma_u \\ \boldsymbol{u} \text{ on } \Gamma_t \\ p \text{ on } \Gamma_v \\ v \text{ on } \Gamma_p \end{bmatrix}_{(\xi,\tau)} \mathrm{d}\Gamma \mathrm{d}\tau \\ &+ \int_0^t \int_{\Gamma_k} \begin{bmatrix} \boldsymbol{G}_{uu} & -\boldsymbol{G}_{ut} & -\boldsymbol{G}_{uv} & -\boldsymbol{G}_{up} \\ -\boldsymbol{G}_{tu} & \boldsymbol{G}_{tt} & \boldsymbol{G}_{tv} & \boldsymbol{G}_{tp} \\ -\boldsymbol{G}_{vu} & \boldsymbol{G}_{vt} & G_{vv} & G_{vp} \\ \boldsymbol{G}_{pu} & -\boldsymbol{G}_{pt} & -G_{pv} & -G_{pp} \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{t}} \text{ on } \Gamma_t \\ \overline{\boldsymbol{u}} \text{ on } \Gamma_u \\ \overline{p} \text{ on } \Gamma_p \\ \overline{v} \text{ on } \Gamma_v \end{bmatrix}_{(\xi,\tau)} \mathrm{d}\Gamma \mathrm{d}\tau \end{aligned} \tag{18}$$

The subscript k which identifies the space-integration domain Γ_k, in Eq. (18) and elsewhere later, is meant to equal the second (column) subscript (u, t, v, p in turn) of the integrand kernels $\boldsymbol{G}_{hk}$.

Denote by $\boldsymbol{Y}$ the vector which collects unknowns on the r.h.s. of Eq. (18); by $\boldsymbol{L}$ the relevant integral operator, by $\boldsymbol{D}$ a vector which gathers all data, i.e. the difference between the original boundary data on the l.h.s. of Eq. (18) and their integral transforms on the r.h.s.. Thus the BIE (18) can be compactly rewritten in the form:

$$\boldsymbol{L}\boldsymbol{Y}(\boldsymbol{\xi},\tau) = \boldsymbol{D}(\boldsymbol{x},t) \tag{19}$$

where $0 \le \tau \le t \le T$, and $\boldsymbol{x} \in \Gamma_h, \boldsymbol{\xi} \in \Gamma_k$, with subscripts h and k specified in Eq. (18).

The operator $\boldsymbol{L}$ is said to be symmetric if it satisfies the condition

$$< \boldsymbol{Y}', \boldsymbol{L}\boldsymbol{Y} > = < \boldsymbol{Y}, \boldsymbol{L}\boldsymbol{Y}' >, \quad \forall \boldsymbol{Y}, \boldsymbol{Y}' \tag{20}$$

where $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ are two arbitrary vectors of variable fields defined on their respective parts of Γ like in Eq. (18), and $< \boldsymbol{Y}', \boldsymbol{L}\boldsymbol{Y} >$ denotes a convolutive bilinear form (convolutive with respect to time t) associated to the operator $\boldsymbol{L}$, namely:

$$< \boldsymbol{Y}', \boldsymbol{L}\boldsymbol{Y} > = \int_{\Gamma_h} \int_0^T \boldsymbol{Y}'^t(x, T-t)\boldsymbol{L}\boldsymbol{Y}(\xi,t)\mathrm{d}t\mathrm{d}\Gamma_x \tag{21}$$

Here in Γ_h subscript $h = u, t, v, p$ equals the first (row) index of the integrand kernel $\boldsymbol{G}_{kh}$ according to Eq. (18).

Symmetry of $\boldsymbol{L}$ in the above sense will be proven in Section 5, after focusing in the next Section on some basic properties of the Green's functions which are contained in $\boldsymbol{L}$ according to Eq. (18).

4
Reciprocal properties of the Green's functions for the poroelastic space

The linear, coupled constitutive law for poroelastic materials, Eqs. (5) and (6), exhibits the following reciprocal property of Betti type (see, e.g. Cheng and Predeleanu, 1987):

$$\sigma^1_{ij} e^2_{ij} + p^1 \zeta^2 = \sigma^2_{ij} e^1_{ij} + p^2 \zeta^1 \tag{22}$$

where superscripts 1 and 2 denote two independent systems of field quantities. More meaningfully, it reflects the path-independent (i.e. conservative, holonomic, thermodynamically reversible) nature of the poroelastic material. In other terms, the above property implies the exact-differential character of the work increment:

$$\delta W = \sigma_{ij}\delta e_{ij} + p\delta\zeta \tag{23}$$

The constitutive reciprocity expressed by Eq. (22) leads to a Betti-like theorem and to other far-reaching consequences in linear poroelasticity theory. In particular, it provides the basis for the proof, given below, of the reciprocal property of the Green's functions concerned herein.

Starting from Betti's reciprocal property (22) and considering the unbounded domain Ω_∞, Pan (1991) showed that, for given discontinuities on Γ of solid and fluid displacements ($\Delta u_i(\xi,\tau)$ with normal $v_k(\xi)$, and $\Delta v(\xi,\tau)$, respectively), the consequent solid displacements at $(\boldsymbol{x}, t)$ can be expressed by the following representation formula:

$$\begin{aligned} u_j(\boldsymbol{x},t) = \int_\Gamma \int_0^t &[-\Delta u_i(\xi,\tau)\sigma^j_{ik}(\boldsymbol{x},\xi;t-\tau)v_k(\xi) \\ &+\Delta v(\xi,\tau)p^j(\boldsymbol{x},\xi;t-\tau)]\mathrm{d}\tau\mathrm{d}\Gamma_\xi \end{aligned} \tag{24}$$

In this equation σ^j_{ik} and p^j are the total stress and pressure, respectively, at $(\boldsymbol{x}, t)$ due to an instantaneous point force (or, equivalently, to a concentrated, traction discontinuity) of unit impulse in the j-direction at (ξ,τ). From Eq. (24) one immediately obtains the Green's functions for solid displacements corresponding to the unit discontinuity source $\Delta u_i(\xi,\tau)$ with normal $v_k(\xi)$, and $\Delta v(\xi,\tau)$, respectively:

$$u^{ik}_j(\boldsymbol{x},\xi;t-\tau) = -\sigma^j_{ik}(\boldsymbol{x},\xi;t-\tau) \tag{25a}$$

$$u^v_j(\boldsymbol{x},\xi;t-\tau) = p^j(\boldsymbol{x},\xi;t-\tau) \tag{25b}$$

The Green's functions for the point force, i.e., those on the r.h.s of Eqs. (25a) and (25b) can be found in Cheng and Predeleanu (1987) or Pan (1991). By observing the analytical expressions of those Green's functions (see Appendix A) and exchanging in them the positions of the source and field points, one notices that:

$$u^{ik}_j(\boldsymbol{x},\xi;t-\tau) = \sigma^j_{ik}(\xi,\boldsymbol{x};t-\tau) \tag{26a}$$

$$u^v_j(\boldsymbol{x},\xi;t-\tau) = -p^j(\xi,\boldsymbol{x};t-\tau) \tag{26b}$$

Equation (26b) can be re-interpreted in terms of the influence functions used for effect superpositions in Sec. 3 and re-written in the notation adopted there. Thus Eq. (26b) yields the following reciprocity relation between matrix-valued kernels in the former terms on the r.h.s.of Eq. (18), i.e. in the integral operator $\boldsymbol{L}$:

$$\boldsymbol{G}^t_{up}(\boldsymbol{x},\xi;t-\tau;\oplus,\otimes) = -\boldsymbol{G}_{pu}(\xi,\boldsymbol{x};t-\tau;\oplus,\otimes) \tag{27}$$

Equation (26a) can be multiplied on both sides by $v_k(\xi)$ to give

$$u^{ik}_i(\boldsymbol{x},\xi;t-\tau)v_k(\xi) = \sigma^j_{ik}(\xi,\boldsymbol{x};t-\tau)\boldsymbol{v}_k(\xi) \tag{28}$$

which is equivalent to the further relation between two other matrix-valued kernels in Eq. (18):

$$\boldsymbol{G}^t_{ut}(\boldsymbol{x},\xi;t-\tau;\oplus,\boldsymbol{v}) = \boldsymbol{G}_{tu}(\xi,\boldsymbol{x};t-\tau,\boldsymbol{v},\otimes) \tag{29}$$

The formulae gathered in Appendix A and paths of reasoning similar to the above ones which led to Eqs. (27) and (29) can be followed to derive reciprocity relations for each pair of off-diagonal kernels (hk and kh) of the former matrix in Eq. (18) and for the kernels on its main diagonal, namely.

$$\begin{aligned} \boldsymbol{G}^t_{hk}(\boldsymbol{x},\xi;t-\tau;\oplus,\otimes) &= \boldsymbol{G}_{kh}(\xi,\boldsymbol{x};t-\tau,\oplus,\otimes), \\ h,k &= u,t,v,p \end{aligned} \tag{30}$$

5 Symmetry of the boundary integral operator L and variational property on the boundary

In the bilinear form Eq. (21), following Maier et al. (1991), let us rearrange the integration sequences and make use of the Heaviside function $H(t-\tau)$ ($=1$ for $\tau < t$; $=0$ for $\tau > t, 0 \le H \le 1$ for $\tau = t$). Thus the time convolutive form of Eq. (21) can be expressed alternatively as follows:

$$< \boldsymbol{Y}', \boldsymbol{L}\boldsymbol{Y} >= \int_0^T \int_0^T [^*]H(t-\tau)\mathrm{d}\tau\mathrm{d}t \tag{31}$$

where the symbol $[^*]$ in the integrand means:

$$\begin{aligned} [^*] = \Big[&\int_{\Gamma_u}\int_{\Gamma_u} \boldsymbol{t}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{uu}\boldsymbol{t}(\xi,\tau) \\ &- \int_{\Gamma_u}\int_{\Gamma_t} \boldsymbol{t}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{ut}\boldsymbol{u}(\xi,\tau) \\ &- \int_{\Gamma_u}\int_{\Gamma_v} \boldsymbol{t}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{uv}p(\xi,\tau) \\ &- \int_{\Gamma_u}\int_{\Gamma_p} \boldsymbol{t}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{up}\boldsymbol{v}(\xi,\tau) \\ &- \int_{\Gamma_t}\int_{\Gamma_u} \boldsymbol{u}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tu}\boldsymbol{t}(\xi,\tau) \\ &+ \int_{\Gamma_t}\int_{\Gamma_t} \boldsymbol{u}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tt}\boldsymbol{u}(\xi,\tau) \\ &+ \int_{\Gamma_t}\int_{\Gamma_v} \boldsymbol{u}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tv}p(\xi,\tau) \\ &+ \int_{\Gamma_t}\int_{\Gamma_p} \boldsymbol{u}'^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tp}\boldsymbol{v}(\xi,\tau) \\ &- \int_{\Gamma_v}\int_{\Gamma_u} p'(\boldsymbol{x},T-t)\boldsymbol{G}_{vu}\boldsymbol{t}(\xi,\tau) \\ &+ \int_{\Gamma_v}\int_{\Gamma_t} p'(\boldsymbol{x},T-t)\boldsymbol{G}_{vt}\boldsymbol{u}(\xi,\tau) \\ &+ \int_{\Gamma_v}\int_{\Gamma_v} p'(\boldsymbol{x},T-t)G_{vv}p(\xi,\tau) \\ &+ \int_{\Gamma_v}\int_{\Gamma_p} p'(\boldsymbol{x},T-t)G_{vp}v(\xi,\tau) \\ &+ \int_{\Gamma_p}\int_{\Gamma_u} v'(\boldsymbol{x},T-t)\boldsymbol{G}_{pu}\boldsymbol{t}(\xi,\tau) \end{aligned}$$

$$- \int_{\Gamma_p}\int_{\Gamma_t} v'(\boldsymbol{x}, T-t)\boldsymbol{G}_{pt}\boldsymbol{u}(\xi,\tau)$$
$$- \int_{\Gamma_p}\int_{\Gamma_v} v'(\boldsymbol{x}, T-t)G_{pv}p(\xi,\tau) \tag{32}$$
$$- \int_{\Gamma_p}\int_{\Gamma_p} v'(\boldsymbol{x}, T-t)G_{pp}v(\xi,\tau)\Big]\mathrm{d}\xi\ \mathrm{d}\boldsymbol{x}$$

In Eq. (32), like Eq. (34) below, the dependencies of the Green's function $\boldsymbol{G}_{hk}$ or $\boldsymbol{G}_{hk}$ [e.g.: $\boldsymbol{G}_{hk}(\boldsymbol{x},\xi;t-\tau;\oplus,\otimes)$] are omitted for brevity.

Similarly, for $<\boldsymbol{Y},\boldsymbol{L}\boldsymbol{Y}'>$ we have:

$$<\boldsymbol{Y},\boldsymbol{L}\boldsymbol{Y}'>=\int_0^T\int_0^T[^{*\prime}]H(t-\tau)\mathrm{d}\tau\ \mathrm{d}t \tag{33}$$

where:

$$[^{*\prime}]=\Big[\int_{\Gamma_u}\int_{\Gamma_u}\boldsymbol{t}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{uu}\boldsymbol{t}'(\xi,\tau)$$
$$-\int_{\Gamma_u}\int_{\Gamma_t}\boldsymbol{t}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{ut}\boldsymbol{u}'(\xi,\tau)$$
$$-\int_{\Gamma_u}\int_{\Gamma_v}\boldsymbol{t}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{uv}p'(\xi,\tau)$$
$$-\int_{\Gamma_u}\int_{\Gamma_p}\boldsymbol{t}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{up}v'(\xi,\tau)$$
$$-\int_{\Gamma_t}\int_{\Gamma_u}\boldsymbol{u}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tu}\boldsymbol{t}'(\xi,\tau)$$
$$+\int_{\Gamma_t}\int_{\Gamma_t}\boldsymbol{u}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tt}\boldsymbol{u}'(\xi,\tau)$$
$$+\int_{\Gamma_t}\int_{\Gamma_v}\boldsymbol{u}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tv}p'(\xi,\tau)$$
$$+\int_{\Gamma_t}\int_{\Gamma_p}\boldsymbol{u}^t(\boldsymbol{x},T-t)\boldsymbol{G}_{tp}v'(\xi,\tau)$$
$$-\int_{\Gamma_v}\int_{\Gamma_u}p(\boldsymbol{x},T-t)\boldsymbol{G}_{vu}\boldsymbol{t}'(\xi,\tau)$$
$$+\int_{\Gamma_v}\int_{\Gamma_t}p(\boldsymbol{x},T-t)\boldsymbol{G}_{vt}\boldsymbol{u}'(\xi,\tau)$$
$$+\int_{\Gamma_v}\int_{\Gamma_v}p(\boldsymbol{x},T-t)G_{vv}p'(\xi,\tau)$$
$$+\int_{\Gamma_v}\int_{\Gamma_p}p(\boldsymbol{x},T-t)G_{vp}v'(\xi,\tau)$$
$$+\int_{\Gamma_p}\int_{\Gamma_u}v(\boldsymbol{x},T-t)\boldsymbol{G}_{pu}\boldsymbol{t}'(\xi,\tau)$$
$$-\int_{\Gamma_p}\int_{\Gamma_t}v(\boldsymbol{x},T-t)\boldsymbol{G}_{pt}\boldsymbol{u}'(\xi,\tau)$$
$$-\int_{\Gamma_p}\int_{\Gamma_v}v(\boldsymbol{x},T-t)G_{pv}p'(\xi,\tau)$$
$$-\int_{\Gamma_p}\int_{\Gamma_p}v(\boldsymbol{x},T-t)G_{pp}v'(\xi,\tau)\Big]\mathrm{d}\xi\ \mathrm{d}x \tag{34}$$

Taking into account for the Greezn's functions the symmetric properties in space (for any $t-\tau$) pointed out in Section 4, i.e. $\boldsymbol{G}^t_{up}(\boldsymbol{x},\xi;t-\tau;\oplus,\otimes) = -\boldsymbol{G}^t_{pu}(\xi,\boldsymbol{x};t-\tau;\oplus,\otimes)$, etc., it can be easily shown that this expression is exactly equal to that given in Eq. (31). Therefore, the symmetry property of the operator $\boldsymbol{L}$ in the sense of Eq. (20) (namely $<\boldsymbol{Y},\boldsymbol{L}\boldsymbol{Y}'>=<\boldsymbol{Y}',\boldsymbol{L}\boldsymbol{Y}>$, for any $\boldsymbol{Y}$ and $\boldsymbol{Y}'$) is arrived at.

As a consequence, the boundary source distributions Δt (on $\Gamma_u\times T$), $\Delta\boldsymbol{u}$ (on $\Gamma_t\times T$), and Δp (on $\Gamma_v\times T$), Δv (on $\Gamma_p\times T$), which solve the transient poroelastic problem in its integral formulation are characterized by the stationarity of the functional

$$F(\boldsymbol{Y}(\boldsymbol{x},t))\equiv\frac{1}{2}<\boldsymbol{L}\boldsymbol{Y},\boldsymbol{Y}>-<\boldsymbol{D},\boldsymbol{Y}> \tag{35}$$

This statement is proved by the simple traditional path of reasoning which follows. The variation of this functional reads:

$$\delta F=\frac{1}{2}<\boldsymbol{L}\delta\boldsymbol{Y},\boldsymbol{Y}>+\frac{1}{2}<\boldsymbol{L}\boldsymbol{Y},\delta\boldsymbol{Y}> \\ +\frac{1}{2}<\boldsymbol{L}\delta\boldsymbol{Y},\delta\boldsymbol{Y}>-<\boldsymbol{D},\delta\boldsymbol{Y}> \tag{36}$$

By virtue of the symmetry property, the first variation becomes

$$\delta^{(1)}F=<\boldsymbol{L}\boldsymbol{Y}-\boldsymbol{D},\delta\boldsymbol{Y}> \tag{37}$$

Thus, the stationarity of F, namely,

$$\delta^{(1)}F=0,\quad \text{for any } \delta\boldsymbol{Y} \tag{38}$$

is both necessary and sufficient for Eq. (19) to be verified, i.e. for $\boldsymbol{Y}$ to represent the boundary solution of the poroelastic problem.

6 Discretization in space and time

The unknown boundary fields of tractions $\boldsymbol{t}$, solid displacements $\boldsymbol{u}$, pressure p and fluid displacement v, can now be modeled over the time interval T and on the relevant boundary portions $\Gamma_u,\Gamma_t,\Gamma_v$ and Γ_p, respectively, subdivided into BE:

$$\boldsymbol{t}(\boldsymbol{x},t)=\boldsymbol{N}_t(\boldsymbol{x},t)\boldsymbol{X}_t\quad,\quad \boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{N}_u(\boldsymbol{x},t)\boldsymbol{X}_u \tag{39a}$$
$$p(\boldsymbol{x},t)=\boldsymbol{N}_p(\boldsymbol{x},t)\boldsymbol{X}_p\quad,\quad v(\boldsymbol{x},t)=\boldsymbol{N}_v(\boldsymbol{x},t)\boldsymbol{X}_v \tag{39b}$$

where $\boldsymbol{N}_h$ and $\boldsymbol{X}_h$ denote matrices of interpolation functions and vectors of governing variables, respectively, with h = t, u, p, v.

Let the above discretization be substituted into Eq. (18) and let the four BIEs in Eq. (18) be enforced in the Galerkin weighted-residual sense, i.e. using as weights the same shape matrices $\boldsymbol{N}_t,\boldsymbol{N}_u,\boldsymbol{N}_p$ and $\boldsymbol{N}_v$ on $\Gamma_u,\Gamma_t,\Gamma_v$ and Γ_p, respectively. Thus, Eq. (18) generates the system of linear algebraic equations:

$$\begin{bmatrix}\boldsymbol{A}_{uu} & \boldsymbol{A}_{ut} & \boldsymbol{A}_{uv} & \boldsymbol{A}_{up}\\ \boldsymbol{A}_{tu} & \boldsymbol{A}_{tt} & \boldsymbol{A}_{tv} & \boldsymbol{A}_{tp}\\ \boldsymbol{A}_{vu} & \boldsymbol{A}_{vt} & \boldsymbol{A}_{vv} & \boldsymbol{A}_{vp}\\ \boldsymbol{A}_{pu} & \boldsymbol{A}_{pt} & \boldsymbol{A}_{pv} & \boldsymbol{A}_{pp}\end{bmatrix}\begin{Bmatrix}\boldsymbol{X}_t\\ \boldsymbol{X}_u\\ \boldsymbol{X}_p\\ \boldsymbol{X}_v\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{D}_u\\ \boldsymbol{D}_t\\ \boldsymbol{D}_v\\ \boldsymbol{D}_p\end{Bmatrix} \tag{40}$$

having set e.g.:

$$\begin{aligned}A_{uu} = \int_0^T \int_{\Gamma_u} & N_u^t(x, T-t) \\ & \times \int_0^t \int_{\Gamma_u} G_{uu}(x, \xi; t-\tau; \oplus, \otimes) \\ & \times N_t(\xi, \tau) \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}x \mathrm{d}t \end{aligned} \tag{41a}$$

$$\begin{aligned}A_{ut} = -\int_0^T \int_{\Gamma_u} & N_u^t(x, T-t) \\ & \times \int_0^t \int_{\Gamma_t} G_{ut}(x, \xi; t-\tau; \oplus, \mathbf{v}) \\ & \times N_t(\xi, \tau) \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}x \mathrm{d}t \end{aligned} \tag{41b}$$

$$\begin{aligned}D_u = -\int_0^T \int_0^t \int_{\Gamma_u} & N_u^t(x, T-t) \\ & \times \left[\int_{\Gamma_t} G_{uu} \bar{t}(\xi, \tau) \mathrm{d}\xi \mathrm{d}\tau \right. \\ & - \int_{\Gamma_u} G_{ut} \bar{u}(\xi, \tau) \mathrm{d}\xi \mathrm{d}\tau \\ & - \int_{\Gamma_p} G_{uv} \bar{p}(\xi, \tau) \mathrm{d}\xi \mathrm{d}\tau \\ & \left. - \int_{\Gamma_v} G_{up} \bar{v}(\xi, \tau) \mathrm{d}\xi \mathrm{d}t - \bar{u}(x) \right] \mathrm{d}x \mathrm{d}t \end{aligned} \tag{42}$$

and similarly for the other submatrices A_{ij} of coefficients and for the other subvectors D_i of data, $(i = t, v, p;\ j = u, t, v, p)$.

Alternatively, the discretization (39) of the unknown boundary fields gathered in Y can be introduced into the functional F, Eq. (35), and the stationarity (38) of this functional can be enforced with respect to the nodal variables in vector X governing those fields through Eqs. (39). Such an alternate path leads to the linear algebraic equation system (40), which can be compactly rewritten as:

$$AX = D \tag{43}$$

It can be easily seen that the coefficient matrix A in Eq. (40) or (43) turns out to be symmetric, i.e., $A_{uu} = A_{uu}^t, A_{ut} = A_{tu}^t$ etc., so that $A = A^t$, as a consequence of the reciprocity relations between Green's functions for the poroelastic space pointed out in Sec. 4. It is worth noting that matrix A is not definite in sign, i.e. the stationarity corresponds to a saddle point (not to a minimum) of F, which is a nonconvex quadratic function (functional before modeling) of the boundary unknowns.

A noteworthy specialization of field modeling Eq. (39) consists of splitting the shape function matrices N_h into products of matrices of shape functions in space and time separately (the latter N_h'' being a block-diagonal matrix, where each diagonal block pertaining to a node in space consists of a row of shape functions for each one of the relevant nodes over the time interval T).

$$N_h(x, t) = N_h'(x) N_h''(t) \quad h = t, u, p, v \tag{44}$$

Unstructured BE meshes in space and time (over $\Gamma \times T$) imply laborious procedures of geometric data preparation. However, they have the advantage of refining only in the vicinity of locations x_k and instants t_k where and when detailed information on the system response is needed.

The variable separation (44) alleviates the mesh generation process by reducing its dimensionality, but may entail redundant output since refinement in space dictated by events in certain time subintervals are often unnecessary in others.

The time interval T can be regarded either as encompassing the whole phenomenon of interest, or as a time step in a step sequence. In the latter case, clearly, the size of each step in the sequence should be kept much smaller than that of the single problem in the former case. Also in the latter (time-stepping) approach re-initialization at the starting instant of each time step implies a burden possibly compensated for by the preservation of the coefficient matrix along a sequence of equal steps.

The above computational alternatives and relevant cost-benefit comparative assessments are beyond the present purposes. They are, however, similar to those available in the literature with reference to (uncoupled) BIV problems of, say, linear dynamics and transient heat conduction (see e.g.: Manolis and Beskos, 1989; Wiebe and Antes, 1991; Dominguez, 1992; Bonnet, 1995).

The integrations of coefficient matrices, like those in Eq. (41), are preferably to be carried out analytically for accuracy, in view of the singularities of the kernels involved.

The Green's functions in G_{tt}, see Eq. (A11), and G_{tp}, see Eq. (A14), are "hypersingular" (singular like r^{-3} in 3-D) and, hence, require special "ad hoc" provisions. The peculiar aspects and difficulties of the double integrations implied by the present formulations are common to all symmetric Galerkin BEMs and will not be examined here. In fact, they have been investigated in recent books (Kane, 1994; Bonnet, 1995) and in several publications surveyed there, though with reference to uncoupled BV and BIV problems.

7 Conclusions

A variational formulation has been developed herein for the transient linear poroelastic analysis centered on "direct" boundary integral equations ("direct" in the sense that the unknowns are actual physical quantities). These equations have been generated so that the integral operator (which transforms unknown boundary fields into data condensing external actions and initial values) turns out to be symmetric with respect to a bilinear form convolutive in time. Use was made of double-layer sources (discontinuities of solid displacements and of fluid pressure, concentrated in space and time) and the analytical expressions of all the employed time-dependent Green's functions for the two- and three-dimensional isotropic homogeneous poroelastic space have been gathered from the literature or generated "ad hoc". Various boundary element discretizations based on the above variational formulation have only briefly been envisaged herein and will be investigated elsewhere. However, it seems reasonable to expect that such discretizations of the present coupled (two-phase) time-dependent problem will provide the computational benefits of symmetry, pointed out by recent work for uncoupled single-phase problems (see e.g.: Sirtori et al., 1992; Maier et al., 1993; Balakrishna et al., 1994; Kane, 1994; Bonnet, 1995).

Appendix

A
Time-dependent Green's functions for the isotropic poroelastic space

It appears useful to provide below, with uniform notation, the analytical expressions of all the time-dependent Green's functions (or "fundamental solutions") for the "free isotropic poroelastic space" Ω_∞, which are employed in the present symmetric formulation (Section 3) of transient isotropic poroelastic analysis. In view of the reciprocity properties discussed in Section 4, only 10 of the 16 kernels $\boldsymbol{G}_{hk}$ (or $\boldsymbol{G}_{hk}$) in the matrix of Eq. (18) need to be specified, since the reciprocity relations of Sec. 4 confer symmetry to that matrix.

Most of these two-point functions are available in the literature (often in different formulations) and the relevant sources are specified in the reference list. The Green's functions which could not be found in the literature have been derived "ad hoc". Their detailed generation is presented in Appendix B.

Superscript 2 and 3 on symbol $\boldsymbol{G}$ will denote two-dimensional (2-D) plane-strain and three-dimensional (3-D) situations, respectively; whenever possible to cover both cases by a single expression, use will be made of superscript m, being understood that either $m = 2$ or $m = 3$. We represent by $r \equiv |\boldsymbol{x} - \xi|$ the distance between the field point $\boldsymbol{x}$ and the souce point ξ; by δ the Dirac distribution; by $\boldsymbol{x}$ an integration variable (length). The constant c is the generalized consolidation coefficient (Rice and Cleary, 1976) defined by:

$$c = \frac{2kGB^2(1-\nu)(1+\nu_u)^2}{9(1-\nu_u)(\nu_u-\nu)} \tag{A1}$$

A.1: $\boldsymbol{G}_{uu}(\boldsymbol{x}, \xi; t-\tau; \oplus, \otimes)$ = displacement at $(\boldsymbol{x},t)$ of the solid skeleton in coordinate direction i, due to a unit static discontinuity source at (ξ, τ) in the direction j (modified from Cheng and Predeleanu, 1987):

$$\begin{aligned}\boldsymbol{G}_{uu}^{2ij} = {} & \frac{1}{8\pi G(1-\nu_u)}\delta(t-\tau)[-(3-4\nu_u)\delta_{ij}\ln(r) + r_i r_j] \\ & + \frac{c(\nu_u-\nu)}{4\pi G(1-\nu_u)(1-\nu)r^2}[\delta_{ij}(1-e^{-z}) \\ & - 2r_i r_j(1-e^{-z}-ze^{-z})]\end{aligned} \tag{A2}$$

$$\begin{aligned}\boldsymbol{G}_{uu}^{3ij} = {} & \frac{1}{16\pi G(1-\nu_u)r}\delta(t-\tau)\Big[(3-4\nu_u)\delta_{ij} \\ & + r_i r_j\Big] + \frac{c(\nu_u-\nu)}{2G(1-\nu_u)(1-\nu)}\left(\frac{\sqrt{z}}{\sqrt{\pi}r}\right)^3 \\ & \times\left[r_i r_j e^{-z} + \frac{\delta_{ij}-3r_i r_j}{r^3}\int_0^r e^{-\frac{x^2}{4c(t-\tau)}}x^2\mathrm{d}x\right]\end{aligned} \tag{A3}$$

having set:

$$z = \frac{r^2}{4c(t-\tau)} \; ; \qquad r_i = \frac{\boldsymbol{x}_i - \xi_i}{r} \tag{A4}$$

A.2: $\boldsymbol{G}_{ut}(\boldsymbol{x}, \xi; t-\tau; \oplus, \boldsymbol{\nu})$ = displacement at $(\boldsymbol{x}, t)$ in direction i, due to a unit discontinuity of the k-th displacement component as source across a surface element of normal $\boldsymbol{\nu}$ at (ξ, τ) (Pan, 1991):

$$\begin{aligned}\boldsymbol{G}_{ut}^{mik} = {} & \frac{\delta(t-\tau)}{4\pi(2r)^{m-2}(1-\nu_u)r}\Big[(1-2\nu_u) \\ & \times(-r_i\nu_k + \delta_{ik}r_\nu + r_k\nu_i) + m r_i r_k r_\nu\Big] \\ & + \frac{c(\nu_u-\nu)f_m}{2\pi(2r)^{m-2}(1-\nu_u)(1-\nu)}\Big[r_i\nu_k + \delta_{ik}r_\nu + r_k\nu_i \\ & - (m+2)r_i r_k r_\nu g_m - \frac{ze^{-z}r_i(\nu_k - r_k r_\nu)}{cr(t-\tau)}\Big]\end{aligned} \tag{A5}$$

where $m = 2$ and $m = 3$ for 2-D plane-strain and 3-D cases, respectively; and where it has been set:

$$r_\nu \equiv r_i\nu_i \; ; \; f_2 = 1 \; ; \; f_3 = \frac{r}{\sqrt{\pi c(t-\tau)}} \tag{A6}$$

$$\begin{aligned}g_2 = {} & \frac{2(1-e^{-z}-ze^{-z})}{r^3} \; ; \; g_3 = \frac{-e^{-z}}{2cr(t-\tau)} \\ & + \frac{3}{2cr^4(t-\tau)}\int_0^r e^{-\frac{x^2}{4c(t-\tau)}}x^2\mathrm{d}x\end{aligned} \tag{A7}$$

A.3: $\boldsymbol{G}_{u\nu}$ $(\boldsymbol{x}, \xi; t-\tau; \oplus, \boldsymbol{\nu})$ = displacement in direction j at $(\boldsymbol{x},t)$, due to a unit pressure jump across a surface of normal $\boldsymbol{\nu}$, concentrated at (ξ, τ) (modified from Cheng and Predeleanu (1987), and obtained by making use of the symmetry property of the Green's functions):

$$\begin{aligned}\boldsymbol{G}_{u\nu}^{2j} = {} & \frac{3c(\nu_u-\nu)}{4\pi GB(1-\nu)(1+\nu_u)}\left[\frac{\nu_j - 2r_\nu r_j}{r^2}(e^{-z}-1)\right. \\ & \left. - \frac{r_\nu r_j e^{-z}}{2c(t-\tau)}\right]\end{aligned} \tag{A8}$$

$$\begin{aligned}\boldsymbol{G}_{u\nu}^{3j} = {} & \frac{-3c(\nu_u-\nu)}{16\pi GB(1-\nu)(1+\nu_u)c(t-\tau)\sqrt{c\pi(t-\tau)}} \\ & \times\left[r_\nu r_j e^{-z} + \frac{\nu_j - 3r_\nu r_j}{r^3}\int_0^r e^{-\frac{x^2}{4c(t-\tau)}}x^2\mathrm{d}x\right]\end{aligned} \tag{A9}$$

A.4: $\boldsymbol{G}_{up}(\boldsymbol{x}, \xi; t-\tau; \oplus, \otimes)$ = displacement in direction i at $(\boldsymbol{x},t)$, due to a time-integrated flux discontinuity as a unit source concentrated at (ξ, τ)(Pan, 1991):

$$G_{up}^{mi} = \frac{B(1+\nu_u)r_i}{3\pi(2r)^{m-1}(1-\nu_u)}\left[\delta(t-\tau) - \frac{f_m z\, e^{-z}}{t-\tau}\right] \tag{A10}$$

A.5: $\boldsymbol{G}_{tt}(\boldsymbol{x}, \xi; t-\tau; \boldsymbol{n}, \boldsymbol{\nu})$ = traction in direction i on a surface with normal $\boldsymbol{n}$ at $(\boldsymbol{x},t)$, due to a unit kinematic discontinuity source at (ξ, τ) across a surface of normal $\boldsymbol{\nu}$ (Pan, 1991):

$$\boldsymbol{G}_{tt}^{mik} = \frac{G}{2\pi(2r)^{m-2}(1-\nu_u)r^2}[-m(m+2)r_i r_n r_k r_v + m\nu_u(v_i r_n r_k + v_j n_j r_i r_k + \delta_{ik} r_n r_v + n_k r_i r_v) + m(1-2\nu_u)(n_i r_k r_v + v_k r_i r_n) + (1-2\nu_u) \times (v_i n_k + \delta_{ik} v_j n_j - (1-4\nu_u) n_i v_k]\delta(t-\tau) + \frac{cG(\nu_u - \nu)f_m}{\pi(1-\nu)(1-\nu_u)(2r)^{m-2}}\Big\{[n_i r_k r_v + v_k r_i r_n + \delta_{ik} r_v r_n + n_k r_i r_v + v_i r_n r_k + v_j n_j r_i r_k + (m-2) n_i v_k - (m+4) r_i r_n r_k r_v - 2z(n_i - r_i r_n)(v_k - r_k r_v)] \frac{e^{-z}}{4c^2(t-\tau)^2} - [(m+2)(n_i r_k r_v + v_k r_i r_n + \delta_{ik} r_n r_v + n_k r_i r_v + v_i r_n r_k + n_j v_j r_i r_k) - n_i v_k - \delta_{ik} v_j n_j - n_k v_i - (m+2)(m+4) r_i r_n r_k r_v] \frac{g_m}{r}\Big\} \tag{A11}$$

where

$$r_n \equiv r_i n_i \tag{A12}$$

A.6: $\boldsymbol{G}_{tv}$ ($\boldsymbol{x}$, ξ; $t-\tau$; $\boldsymbol{n}$, $\boldsymbol{v}$) = traction component k at ($\boldsymbol{x}$,t), due to a pressure jump concentrated at (ξ, τ) with normal $\boldsymbol{v}$ (from Pan, 1991, using the symmetric property of the Green's functions):

$$\boldsymbol{G}_{tv}^{mk} = \frac{-kGB(1+\nu_u)f_m}{3\pi(2r)^{m-2}(1-\nu_u)}\Big[(n_k r_v + v_k r_n + v_j n_j r_k - (m+2) r_v r_k r_n) g_m - \frac{z e^{-z} r_v (n_k - r_k r_n)}{cr(t-\tau)}\Big] \tag{A13}$$

A.7: $\boldsymbol{G}_{tp}$ ($\boldsymbol{x}$, ξ; $t-\tau$; $\boldsymbol{n}$, $\otimes$) = traction in direction $\boldsymbol{i}$ at ($\boldsymbol{x}$,t) with normal $\boldsymbol{n}$, due to a unit discontinuity source of time-integrated flux at (ξ, τ) (Pan, 1991):

$$\boldsymbol{G}_{tp}^{mi} = \frac{GB(1+\nu_u)}{3\pi(2r)^{m-2}(1-\nu_u)r^2}\Big\{(n_i - m\, r_i t_n)\delta(t-\tau) + [(m-1)n_i - 2z(n_i - r_i r_n)]\frac{f_m\, z e^{-z}}{t-\tau}\Big\} \tag{A14}$$

A.8: G_{vv} ($\boldsymbol{x}$, ξ; $t-\tau$; $\boldsymbol{n}$, $\boldsymbol{v}$) = time-integral of flux at ($\boldsymbol{x}$,t) across a surface of normal $\boldsymbol{n}$, due to a unit pressure jump source at (ξ, τ) across a surface of normal $\boldsymbol{v}$ (See Appendix B):

$$G_{vv}^2 = \frac{-9c(1-\nu_u)(\nu_u - \nu)}{4\pi GB^2(1-\nu)(1+\nu_u)^2}\Big[\frac{n_j v_j - 2 r_v r_n}{r^2} \times (e^{-z} - 1) - \frac{r_v r_n e^{-z}}{2c(t-\tau)}\Big] \tag{A15}$$

$$G_{vv}^3 = \frac{9c(1-\nu_u)(\nu_u-\nu)}{16\pi GB^2(1-\nu)(1+\nu_u)^2 c(t-\tau)\sqrt{c\pi(t-\tau)}} \times [r_v r_n e^{-z} + \frac{n_j v_j - 3 r_v r_n}{r^3}\int_0^r e^{-\frac{x^2}{4c(t-\tau)}} x^2 \mathrm{d}x] \tag{A16}$$

A.9: G_{vp} ($\boldsymbol{x}$, ξ; $t-\tau$; $\boldsymbol{n}$, $\otimes$) = time-integrated flux at ($\boldsymbol{x}$,t) with normal $\boldsymbol{n}$, due to a unit time-integrated flux source at (ξ, τ) (Pan, 1991):

$$G_{vp}^m = \frac{r_n f_m z e^{-z}}{\pi(2r)^{m-1}(t-\tau)} \tag{A17}$$

A.10: G_{pp} ($\boldsymbol{x}$, ξ; $t-\tau$; $\oplus, \otimes$) = pressure at ($\boldsymbol{x}$,t) due to a unit time-integrated flux jump concentrated at (ξ, τ) (Pan, 1991):

$$G_{pp}^m = -\frac{f_m e^{-z}(m-2z)}{8\pi k(2r)^{m-2}(t-\tau)^2} \tag{A18}$$

It is worth stressing that each of the above kernels can be interpreted as describing the response in terms of one kind of quantity (e.g. displacement) of Ω_∞ to an imposed discontinuity concentrated in space and time. As usual, such concentration can be conceived as a limiting process which leads to modelling a discontinuity by Dirac distributions in space $\delta(\boldsymbol{x}-\xi)$ and in time $\delta(t-\tau)$.

B
Generation of Green's function G_{VV} for the isotropic poroelastic space

As shown by Pan (1991), the j-th component u_j of the solid displacement at ($\boldsymbol{x}$,t) can be represented by the following integral:

$$u_j(\boldsymbol{x}, t) = \int_{\Gamma_\xi} \mathrm{d}\Gamma \int_0^t \mathrm{d}\tau \Big\{ \big[\sigma_{ik}(\xi,\tau) u_i^j(\boldsymbol{x}, \xi; t-\tau) + u_i(\xi,\tau)\sigma_{ik}^j(\boldsymbol{x}, \xi; t-\tau)\big] v_k - \big[v(\xi,\tau) p^j(\boldsymbol{x}, \xi; t-\tau) + p(\xi,\tau) v^j(\boldsymbol{x}, \xi; t-\tau)\big]\Big\} \tag{B1}$$

Here the quantities with superscript j denote effects at ($\boldsymbol{x}$,t) due to an instantaneous point force of unit impluse in the j-direction at (ξ, τ), and it has been set:

$$v^j(\boldsymbol{x}, \xi; t-\tau) \equiv v_i^j(\boldsymbol{x}, \xi; t-\tau) v_i(\xi) \tag{B2}$$

Let us assume that the static quantities are discontinuous while the kinematic quantities are continuous across Γ, namely:

$$\Delta \boldsymbol{t} = \boldsymbol{t}^+ + \boldsymbol{t}^- \neq 0; \qquad \Delta p = p^+ - p^- \neq 0 \tag{B3}$$

$$\Delta \boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^- = 0; \qquad \Delta v = -(v^+ + v^-) = 0 \tag{B4}$$

Integrating Eq. (B1) along the two sides of Γ with conditions (B3) and (B4) and assuming that the normal to Γ be the same as that to Γ^-, we arrive at the following integral expression for the solid displacements:

$$u_j(\boldsymbol{x}, t) = \int_{\Gamma_\xi} \mathrm{d}\Gamma \int_0^t \Big[\Delta t_i(\xi,\tau) u_i^j(\boldsymbol{x}, \xi; t-\tau). + \Delta p(\xi,\tau) v^j(\boldsymbol{x}, \xi; t-\tau)\Big] \mathrm{d}\tau \tag{B5}$$

This equation implies that:

$$u_j^{\Delta t_i}(\boldsymbol{x},\xi;t-\tau) = u_i^j(\boldsymbol{x},\xi;t-\tau);$$
$$u_j^{\Delta p}(\boldsymbol{x},\xi;t-\tau) = v^j(\boldsymbol{x},\xi;t-\tau) \qquad \text{(B6a, b)}$$

Equation (B6a) expresses the equivalence of a point force and a point traction in the sense that the solid displacement $u_j^{\Delta t_i}$ in the j-direction due to a point traction-jump in the i-direction is the same as the solid displacement u_i^j in the i-direction due to a point force in the j-direction. Equation (B6b) states that the solid displacement $u_j^{\Delta p}$ in the j-direction due to a point pore pressure jump is equivalent to the relative fluid displacement v^j in the outward normal direction due to a point force in the j-direction.

Since the point-force Green's function for the relative fluid displacement is available (Cheng and Predeleanu, 1987; see also Eqs. (A8) and (A9) in Appendix A), we can easily obtain the expression of the solid displacements due to a point pore pressure jump. These expressions read, for two- and three-dimensional situations, respectively:

$$\text{for 2-D}: u_j^{\Delta p}(\boldsymbol{x},\xi;t-\tau) = \frac{3c(v_u-v)}{4\pi GB(1-v)(1+v_u)}\left[\frac{\delta_{ij}-2r_ir_j}{r^2}(e^{-z}-1) - \frac{r_ir_je^{-z}}{2c(t-\tau)}\right]v_i \qquad \text{(B7a)}$$

$$\text{for 3-D}: u_j^{\Delta p}(\boldsymbol{x},\xi;t-\tau) = \frac{-3c(v_u-v)}{16\pi GB(1-v)(1+v_u)c(t-\tau)\sqrt{c\pi(t-\tau)}} \times\left[r_ir_je^{-z}+\frac{\delta_{ij}-3r_ir_j}{r^3}\int_0^r e^{-\frac{x^2}{4c(t-\tau)}}x^2\mathrm{d}x\right]v_i \qquad \text{(B7b)}$$

For a problem without fluid injection (i.e., Q = 0), the relative fluid displacement due to a pore pressure jump can be related to the above solid displacements; in fact (Pan, 1991):

$$v_j^{\Delta p}(\boldsymbol{x},\xi;t-\tau) = -\frac{3(1-v_u)}{B(1+v_u)}u_j^{\Delta p}(\mathrm{x},\xi;t-\tau) \qquad \text{(B8)}$$

Substituting Eqs (B7a,b) into Eq (B8) and projecting the result onto the normal direction, we arrive at the Green's function G_{vv} (for 2-D and 3-D, respectively):

$$\text{for 2-D}: v^{\Delta p}(\mathrm{x},\xi;t-\tau) = \frac{-9c(1-v_u)(v_u-v)}{4\pi GB^2(1-v)(1+v_u)^2}\left[\frac{\delta_{ij}-2r_ir_j}{r^2}(e^{-z}-1) - \frac{r_ir_je^{-z}}{2c(t-\tau)}\right]v_in_j \qquad \text{(B9a)}$$

$$\text{for 3-D}: v^{\Delta p}(\boldsymbol{x},\xi;t-\tau) = \frac{9c(1-v_u)(v_u-v)}{16\pi\, GB^2(1-v)(1+v_u)^2c(t-\tau)\sqrt{c\pi(t-\tau)}} \times\left[r_ir_je^{-z}+\frac{\delta_{ij}-3r_ir_j}{r^3}\int_0^r e^{-\frac{x^2}{4c(t-\tau)}}x^2\mathrm{d}x\right]v_in_j \qquad \text{(B9b)}$$

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