THREE-DIMENSIONAL FUNDAMENTAL SOLUTIONS IN MULTILAYERED PIEZOELECTRIC SOLIDS

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ABSTRACT

This article studies Green's functions in three-dimensional (3D), anisotropic piezoelectric, and multilayered media. The two-dimensional (2D) Fourier transforms are first applied to the two horizontal variables, thus changing the partial differential equation to an ordinary one. The general solution in the transformed domain is then expressed in terms of the 3D Stroh formalism. The propagator matrix for each layer is derived from the general solution so that the solution in the Fourier transformed domain at any vertical level can be simply expressed in terms of the propagator matrices. The physical domain solution is also studied, so is the numerical issue associated with the inverse Fourier transform. Finally, several special cases are discussed, including the 2D deformation in anisotropic piezoelectric and multilayered media, and 3D deformation in transversely isotropic piezoelectric and multilayered media.

1. INTRODUCTION

In the field of applied mechanics, material anisotropy is always an important topic. Response of an anisotropic material can be completely different to that of an isotropic one. In particular, anisotropic material may present unexpected features, which could be of special interests to the engineering design [1~3]. In recent years, study on anisotropic piezoelectric materials has become an active research area, where the piezoelectric material is an essential component in smart structures, health monitoring systems, etc.

In order to understand the basic properties, various analytical solutions have been also developed, including both the two-dimensional (2D) and three-dimensional (3D) fundamental solutions (i.e., the Green's functions) due to a point source. Under the assumption of a 2D deformation, Suo, *et al*. [4] studied the fracture problem in anisotropic piezoelectric media, while the corresponding fundamental solutions in an infinite-plane, half-plane, and bimaterial plane were studied by Barnett and Lothe [5], Chung and Ting [6], Pan [7], and Ru [8,9]. We mention that certain thin piezoelectric film structures under 2D deformation were also discussed; see for example, Ref. [10].

For a transversely isotropic piezoelectric solid under 3D deformation, Ding and co-workers [11,12] and Dunn and co-workers [13,14] derived the fundamental solutions in an infinite-space, half-space, and bimaterial space. When the material is general anisotropy, Pan and co-workers derived the corresponding fundamental solutions [15~18]. We also point out that Akamatsu and Tanuma [19] obtained the elastic displacement and electric potential in an anisotropic piezoelectric infinite-space using the extended Stroh formulism [5,20,21].

This article studies the Green's functions in 3D anisotropic piezoelectric and multilayered media. We first apply the 2D Fourier transforms to the two horizontal variables to change the partial differential equation to an ordinary one. The general solution in the Fourier transformed domain is then expressed in terms of the 3D Stroh formalism similar to Ting [1]. We also derive the propagator matrix for each layer from the general solution so that the solution in the Fourier transformed domain at any vertical level can be expressed in terms of the propagator matrices. The physical domain solution is expressed by the inverse Fourier transform where the involved numerical issues are discussed. We mention that the present solution contains several solutions as special cases, including the corresponding 2D solution and the 3D solution in the transversely isotropic piezoelectric and layered solid.

2. GOVERNING EQUATIONS OF ANISOTROPIC PIEZOELECTRIC SOLID

As is well known, for an anisotropic piezoelectric

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solid, the governing equations consist of (see, e.g., [1,5,7,22]):

Equilibrium equations:

$$\sigma_{ji,j} + f_i = 0\ ; \qquad D_{i,i} - q = 0 \tag{1}$$

Where σ_{ij} and D_i are the stress and electric displacement, respectively; f_i and q are the body force and electric charge density, which will be replaced later by a concentrated force and electric charge. In this paper, lowercase (uppercase) subscripts always range from 1 to 3 (1 to 4) and summation over repeated lowercase (uppercase) subscripts is implied. A subscript comma denotes the partial differentiation with respect to the coordinates (i.e., x_1, x_2, x_3 or x, y, z).

Constitutive relations:

$$\begin{aligned} \sigma_{ij} &= C_{ijlm}\,\gamma_{lm} - e_{kji}\,E_k \\ D_i &= e_{ijk}\,\gamma_{jk} + \varepsilon_{ij}\,E_j \end{aligned} \tag{2}$$

Where γ_{ij} is the strain and E_i is the electric field; C_{ijlm}, e_{ijk} and ε_{ij} are the elastic moduli, the piezoelectric coefficients, and the dielectric constants, respectively. We remark that the decoupled state (i.e., the purely elastic and purely electric deformations) can be obtained by simply setting $e_{ijk} = 0$.

Elastic strain-displacement and electric field-potential relations:

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\ ; \qquad E_i = -\phi_{,i} \tag{3}$$

Where u_i and ϕ are the elastic displacement and electric potential, respectively.

As before, we adopt the notation introduced first by Barnett and Lothe [5] and modified later by Pan [7], as

$$u_I = \begin{cases} u_i\,, & I = 1, 2, 3 \\ \phi\,, & I = 4 \end{cases} \tag{4}$$

$$\gamma_{Ij} = \begin{cases} \gamma_{ij}\,, & I = 1, 2, 3 \\ -E_j\,, & I = 4 \end{cases} \tag{5}$$

$$\sigma_{iJ} = \begin{cases} \sigma_{ij}\,, & J = 1, 2, 3 \\ D_i\,, & J = 4 \end{cases} \tag{6}$$

$$C_{iJKl} = \begin{cases} C_{ijkl}\,, & J, K = 1, 2, 3 \\ e_{lij}\,, & J = 1, 2, 3;\ K = 4 \\ e_{ikl}\,, & J = 4;\ K = 1, 2, 3 \\ -\varepsilon_{il}\,, & J = K = 4 \end{cases} \tag{7}$$

In terms of this shorthand notation, the constitutive relations (2) can be unified into the single equation:

$$\sigma_{iJ} = C_{iJKl}\,\gamma_{Kl} \tag{8}$$

Similarly, the equilibrium equations (1) in terms of the *extended stresses* can be recast into

$$\sigma_{iJ,i} + f_J = 0 \tag{9}$$

with f_J being the *extended* force, defined as

$$f_J = \begin{cases} f_j\,, & J = 1, 2, 3 \\ -q\,, & J = 4 \end{cases} \tag{10}$$

In the following sections, we use the *extended displacement* for the elastic displacement and electric potential as defined by (4), and the *extended stress* for the stress and electric displacement as defined by (6). Furthermore, we define

$$\boldsymbol{t} = (\sigma_{31}, \sigma_{32}, \sigma_{33}, D_3)^t \tag{11}$$

as the *extended traction* on the z = constant plane. In Eq. (11), the superscript t denotes the transpose of a matrix.

3. GENERAL SOLUTION IN A HOMOGENEOUS PLATE IN TRANSFORMED DOMAIN

Let us consider an anisotropic piezoelectric and layered half-space made of N parallel and homogeneous layers lying over a homogeneous half-space. We number the layers serially with the layer at the top being layer 1 and the half-space being layer N + 1. The origin of the Cartesian coordinates is at the surface of the layered half-space and the z-axis is drawn down into the medium. The k-th layer is bounded by the interfaces $z = z_{k-1}$, z_k. Therefore, we have $z_0 = 0$, and $z_N = H$, with H being the depth of the last interface. An extended point force at any z-level and the loading on the surface can all be considered, which will be discussed later.

We now apply the 2D Fourier transforms (i.e., for the two-point extended displacement)

$$\tilde{u}_K(y_1, y_2, z; \boldsymbol{d}) = \iint u_K(x_1, x_2, z; \boldsymbol{d})\, e^{i(y_1 x_1 + y_2 x_2)} dx_1\, dx_2 \tag{12}$$

to Eqs. (8) and (9), with the extended force in Eq. (9) being zero. Thus, in the Fourier transformed domain, we obtain the following ordinary differential equation which is similar to the purely elastic counterpart [1,23].

$$C_{I\alpha K\beta}\, y_\alpha\, y_\beta\, \tilde{u}_K + i(C_{I\alpha K3} + C_{I3K\alpha}) y_\alpha\, \tilde{u}_{K,3} - C_{I3K3}\, \tilde{u}_{K,33} = 0\,, \qquad \alpha, \beta = 1, 2 \tag{13}$$

A general solution of the extended displacement can thus be derived as [1,16]

$$\tilde{\boldsymbol{u}}(y_1, y_2, z; \boldsymbol{d}) = \boldsymbol{a}\, e^{-ip\eta z} \tag{14}$$

with p and $\boldsymbol{a}$ satisfying the following eigenrelation:

$$[\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^t) + p^2 \boldsymbol{T}]\ \boldsymbol{a} = 0 \tag{15}$$

where

$$Q_{IK} = C_{jIKs}\, n_j\, n_s\,,\ \ R_{IK} = C_{jIKs}\, n_j\, m_s\,,\ \ T_{IK} = C_{jIKs}\, m_j\, m_s \tag{16}$$

with

$$\begin{aligned}(n_1, n_2, n_3) &\equiv (\cos\theta, \sin\theta, 0)\\ (m_1, m_2, m_3) &\equiv (0, 0, 1)\end{aligned} \tag{17}$$

Note that a polar coordinate transform, defined below, has been used

$$y_1 = \eta\cos\theta\ ; \qquad y_2 = \eta\sin\theta \tag{18}$$

Equation (15) is the Stroh eigenrelation for the oblique plane spanned by $\boldsymbol{n}$ and $\boldsymbol{m}$ defined by (17). Such a connection between 3D and 2D Stroh eigenrelations was first made by Barnett and Lothe [5] and was described in detail by Ting [1] in a form suitable for the study of layered structures. We also mention that, by the positive requirement on the strain energy density, all the eigenvalues of Eq. (15) are either complex or purely imaginary (e.g., [1,4]).

Using now the Stroh eigenvalues and eigenvectors, the extended traction vector $\boldsymbol{t}$ on the z = constant plane, and the *extended* in-plane stress vector $\boldsymbol{s}$, defined as

$$\boldsymbol{s} = (\sigma_{11}, \sigma_{12}, \sigma_{22}, D_1, D_2)^t \tag{19}$$

can be expressed in the Fourier-transformed domain as [16]

$$\tilde{\boldsymbol{t}} = -i\eta\, \boldsymbol{b}\, e^{-ip\eta z} \tag{20}$$

$$\tilde{\boldsymbol{s}} = -i\eta\, \boldsymbol{c}\, e^{-ip\eta z} \tag{21}$$

with

$$\begin{aligned}\boldsymbol{b} &= (\boldsymbol{R}^t + p\boldsymbol{T})\ \boldsymbol{a} = -\frac{1}{p}(\boldsymbol{Q} + p\boldsymbol{R})\ \boldsymbol{a}\\ \boldsymbol{c} &= \boldsymbol{V}\,\boldsymbol{a}\end{aligned} \tag{22}$$

where the matrix $\boldsymbol{V}$ is defined by

$$\boldsymbol{V} = \begin{bmatrix} C_{111\alpha}n_\alpha + pC_{1113} & C_{112\alpha}n_\alpha + pC_{1123} & C_{113\alpha}n_\alpha + pC_{1133} & C_{114\alpha}n_\alpha + pC_{1143}\\ C_{121\alpha}n_\alpha + pC_{1213} & C_{122\alpha}n_\alpha + pC_{1223} & C_{123\alpha}n_\alpha + pC_{1233} & C_{124\alpha}n_\alpha + pC_{1243}\\ C_{221\alpha}n_\alpha + pC_{2213} & C_{222\alpha}n_\alpha + pC_{2223} & C_{223\alpha}n_\alpha + pC_{2233} & C_{224\alpha}n_\alpha + pC_{2243}\\ C_{141\alpha}n_\alpha + pC_{1413} & C_{142\alpha}n_\alpha + pC_{1423} & C_{143\alpha}n_\alpha + pC_{1433} & C_{144\alpha}n_\alpha + pC_{1443}\\ C_{241\alpha}n_\alpha + pC_{2413} & C_{242\alpha}n_\alpha + pC_{2423} & C_{243\alpha}n_\alpha + pC_{2433} & C_{244\alpha}n_\alpha + pC_{2443} \end{bmatrix} \tag{23}$$

with α taking the summation from 1 to 2.

We now denote by p_m, $\boldsymbol{a}_m$, and $\boldsymbol{b}_m$ (m = 1, 2, $\cdots$ 8) the eigenvalues and the associated eigenvectors of (15) and (22) and order them in such a way so that

$$\begin{aligned}&\mathrm{Im}p_J > 0,\ \ p_{J+4} = \bar{p}_J,\ \ \boldsymbol{a}_{J+4} = \bar{\boldsymbol{a}}_J,\ \ \boldsymbol{b}_{J+4} = \bar{\boldsymbol{b}}_J\ \ (J = 1, 2, 3, 4)\\ &\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4],\ \ \boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4],\\ &\boldsymbol{C} = [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4]\end{aligned} \tag{24}$$

where Im stands for the imaginary part and over bar for the complex conjugate. In the analysis followed, we assume that p_J are distinct and the eigenvectors $\boldsymbol{a}_J$, and $\boldsymbol{b}_J$ satisfy the normalization relation [1,5]

$$\boldsymbol{b}_I^t\, \boldsymbol{a}_J + \boldsymbol{a}_I^t\, \boldsymbol{b}_J = \delta_{IJ} \tag{25}$$

with δ_{IJ} being the 4 × 4 Kronecker delta, i.e., the 4 × 4 identity matrix. We also remark that repeated eigenvalues p_J can be avoided by using slightly perturbed material coefficients with negligible errors [7]. In doing so, the simple structure of the solution presented below can always be used.

It can be shown that Eqs. (15) and (22) can be recast into a 8 × 8 linear eigensystem

$$\boldsymbol{N}\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix} = s\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix} \tag{26}$$

where

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{T}^{-1}\boldsymbol{R}^t & \boldsymbol{T}^{-1}\\ -\boldsymbol{Q} + \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^t & -\boldsymbol{R}\boldsymbol{T}^{-1}\end{bmatrix} \tag{27}$$

Therefore, a general solution in the Fourier-transformed domain for the extended displacement and traction vectors in a homogeneous but general anisotropic piezoelectric medium can be expressed as

$$\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}\left\langle e^{-ip^*\eta z}\right\rangle & \bar{\boldsymbol{A}}\left\langle e^{-i\bar{p}^*\eta z}\right\rangle\\ \boldsymbol{B}\left\langle e^{-ip^*\eta z}\right\rangle & \bar{\boldsymbol{B}}\left\langle e^{-i\bar{p}^*\eta z}\right\rangle\end{bmatrix}\begin{bmatrix}\boldsymbol{K}_1\\ \boldsymbol{K}_2\end{bmatrix} \tag{28}$$

where

$$\langle e^{-ip^*\eta z}\rangle = \mathrm{diag}\,[e^{-ip_1\eta z},\ \ e^{-ip_2\eta z},\ \ e^{-ip_3\eta z},\ e^{-ip_4\eta z}] \tag{29}$$

and $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are two 4 × 1 constant column matrices to be determined.

4. PROPAGATOR MATRIX AND SOLUTION OF MULTILAYERED STRUCTURES

From Eq. (28), we can shown that the solution at any vertical level z in the homogeneous layer j (i.e., the extended displacement and traction vectors in the Fourier transformed domain) can be expressed by that at the top interface of the same layer, i.e., $z = z_{j-1}$, as

$$\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_z = \boldsymbol{P}(z - z_{j-1})\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_{z_{j-1}} \tag{30}$$

where

$$\boldsymbol{P}(z)=\begin{bmatrix}\boldsymbol{A}\langle e^{-ip^*\eta z}\rangle & \overline{\boldsymbol{A}}\langle e^{-i\overline{p}^*\eta z}\rangle\\ \boldsymbol{B}\langle e^{-ip^*\eta z}\rangle & \overline{\boldsymbol{B}}\langle e^{-i\overline{p}^*\eta z}\rangle\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \overline{\boldsymbol{A}}\\ \boldsymbol{B} & \overline{\boldsymbol{B}}\end{bmatrix}^{-1} \tag{31}$$

is called the propagator matrix of layer j [24,25]. Listed below are three important features of the propagator matrix, which can be easily proved.

$$\boldsymbol{P}(0)=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \tag{32}$$

$$\boldsymbol{P}(z_3-z_1)=\boldsymbol{P}(z_3-z_2)\,\boldsymbol{P}(z_2-z_1) \tag{33}$$

$$\boldsymbol{P}(z_3-z_1)=\boldsymbol{P}^{-1}(z_1-z_3) \tag{34}$$

It is noted that in order to construct the propagator matrix in Eq. (31), one needs to find the inverse of the compound matrix shown on the right-hand side of Eq. (31). Fortunately, this can be done easily by employing the following important orthogonality relation [1], extended to the piezoelectric case as

$$\begin{bmatrix}\boldsymbol{B}^t & \boldsymbol{A}^t\\ \overline{\boldsymbol{B}}^t & \overline{\boldsymbol{A}}^t\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \overline{\boldsymbol{A}}\\ \boldsymbol{B} & \overline{\boldsymbol{B}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \tag{35}$$

The propagating relation (30) can be used repeatedly so that one can propagate the solution from the top surface $z = 0$ to the bottom interface $z = H$ of the layered structure. Consequently, we have

$$\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_H=\boldsymbol{P}_N(h_N)\,\boldsymbol{P}_{N-1}(h_{N-1})\cdots\boldsymbol{P}_2(h_2)\,\boldsymbol{P}_1(h_1)\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_0 \tag{36}$$

where $h_j = z_j - z_{j-1}$ is the thickness of layer j and $\boldsymbol{P}_j$ the propagator matrix of the same layer.

It is observed that Eq. (36) is a surprisingly simple relation. For the given boundary conditions in the Fourier transformed domain on the top surface $z = 0$, along with the radiation condition for the solution in the homogeneous half-space $z > H$, the unknowns involved in Eq. (36) can be solved.

In order to obtain the extended displacement and traction vectors in the transformed domain at any depth z, say $z_{k-1} \le z \le z_k$ in layer k, we propagate the solution from the top surface to the z-level [25], i.e.,

$$\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_z=\boldsymbol{P}_k(z-z_{k-1})\boldsymbol{P}_{k-1}(h_{k-1})\cdots\boldsymbol{P}_2(h_2)\boldsymbol{P}_1(h_1)\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_0 \tag{37}$$

With the extended displacement and traction vectors at a given depth being solved, the corresponding in-plane quantities can be evaluated using Eqs. (21) and (23).

We further mention that in solving Eq. (36), the boundary condition given on the top surface can be first-, second-, or third-type. In the study of multi-layered structures using the boundary element method, one will need the Green's function solution, or the fundamental solution, in the layered structure. This is handled by the following approach.

If there is an internal source (force, charge, dislocation, etc.) located at $z = d_0$ level within layer j (z_j, z_{j-1}), we artificially divide this layer into two sub-layers $j1$(d_0, z_{j-1})(with $h_{j1} = d_0 - z_{j-1}$) and $j2$ (z_j, d_0)(with $h_{j2} = z_j - d_0$), and define the Fourier transformed discontinuities across the source level as

$$\begin{bmatrix}-i\eta\Delta\tilde{\boldsymbol{u}}\\ \Delta\tilde{\boldsymbol{t}}\end{bmatrix}\equiv\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}(d_0+0)\\ \tilde{\boldsymbol{t}}(d_0+0)\end{bmatrix}-\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}(d_0-0)\\ \tilde{\boldsymbol{t}}(d_0-0)\end{bmatrix} \tag{38}$$

We point out that for a given point source, its Fourier transform can usually be found in an exact closed-form. For example, for an extended point force applied at (0, 0, d_0), i.e.,

$$f_J(x,y,z)=f_J^0\,\delta(x)\,\delta(y)\,\delta(z-d_0) \tag{39}$$

where f_J^0 (J = 1, 2, 3, 4) are the amplitudes of the extended point force, the Fourier transformed discontinuities in Eq. (38) are found to be

$$\begin{bmatrix}-i\eta\Delta\tilde{\boldsymbol{u}}\\ \Delta\tilde{\boldsymbol{t}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}\\ -\boldsymbol{f}^0\end{bmatrix} \tag{40}$$

Again, propagating the propagator matrices from the top surface $z = 0$ to the bottom interface $z = H$ of the layered structure and making use of the discontinuity relation (39)[25], we arrive at the following important equation

$$\begin{aligned}&\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_H-\boldsymbol{P}_N(h_N)\boldsymbol{P}_{N-1}(h_{N-1})\cdots\boldsymbol{P}_2(h_2)\boldsymbol{P}_1(h_1)\begin{bmatrix}-i\eta\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{t}}\end{bmatrix}_0\\ &=\boldsymbol{P}_N(h_N)\boldsymbol{P}_{N-1}(h_{N-1})\cdots\boldsymbol{P}_{j+1}(h_{j+1})\boldsymbol{P}_{j2}(h_{j2})\begin{bmatrix}-i\eta\Delta\tilde{\boldsymbol{u}}\\ \Delta\tilde{\boldsymbol{t}}\end{bmatrix}\end{aligned} \tag{41}$$

Similarly, for the given boundary conditions in the Fourier transformed domain on the top surface $z = 0$, along with the radiation condition for the solution in the homogeneous half-space $z > H$, the unknowns involved in Eq. (41) can be solved. After that, the extended displacement and traction vectors in the transformed domain at any z-level can be found by using Eq. (37). We further mention that Eqs. (36) and (41) can be equally applied to multilayered plates where boundary conditions on both the top and bottoms surfaces can be imposed.

5. PHYSICAL-DOMAIN SOLUTION

Now, in order to find the physical domain solution, we need to apply the 2D inverse Fourier transforms to the transformed extended displacements and stresses. For example, for the extended displacement, we have

$$u_K(x_1,x_2,z;\boldsymbol{d})=\frac{1}{(2\pi)^2}\iint\tilde{u}_K(y_1,y_2,z;\boldsymbol{d})e^{i(y_1x_1+y_2x_2)}dy_1\,dy_2 \tag{42}$$

The integral limits of the variables y_1 and y_2 in Eq. (42) are from $-\infty$ to $+\infty$. Alternatively, the 2D inverse Fourier transforms can also be carried out in the polar coordinates as

$$u_K(x_1, x_2, z; \boldsymbol{d}) = \frac{1}{(2\pi)^2} \iint \tilde{u}_K(\eta, \theta, z; \boldsymbol{d}) e^{i(x_1 \cos\theta + x_2 \sin\theta)} d\eta d\theta \tag{43}$$

where the integral limit for η is from 0 to $+\infty$ and for θ from 0 to 2π.

The double integrals in Eq. (42) or (43) can be done efficiently using the adaptive integration algorithm developed by Yang and Pan [23,26]. However, a direct implementation of such an algorithm may not work for certain situations, in particular when the z-level of the source and field points are close to each other. In this case, the integrand is highly oscillating and even divergent. In such a case, the infinite-space Green's function [15], the bimaterial Green's function [16], and even the trimaterial Green's function [27] need to be incorporated into the multilayered Green's function to get rid of the numerical difficulty. This will be discussed in a future paper [28].

Another numerical difficulty may occur when the layer number is large, certain layers are very thick, or when the integral variable η in Eq. (43) is large. In such a situation, some elements in the propagator matrix (31) may overflow since they are exponential increasing functions of η and z. To overcome this problem, the forward and backward multiplication approach needs to be applied. That is, we propagate the propagator matrix from either the top surface to the bottom interface or from the bottom interface to the top surface, depending upon the relative z-level of the source and field points [25,29].

6. SPECIAL CASES

We point out that the present solution contains solutions for certain special cases. First, solution to the corresponding 2D deformation of the anisotropic piezoelectric and layered structure can be reduced from the 3D solution presented here. If we let one of the horizontal coordinates (x_1 and x_2) and one of the corresponding Fourier variables (y_1 or y_2) being zero, then the problem will be reduced to a generalized plane-strain one in the piezoelectric and multilayered structure. For this case, only 1D Fourier inverse transform is needed in order to get the physical-domain solution [30,31].

The second special solution is for 3D deformation but for material in each layer being transversely isotropic piezoelectric. For this case, the 2D Fourier transforms can be replaced by the Hankel transforms, thus only 1D semi-infinite integrals are required to be carried out numerically. Furthermore, for this case, the cylindrical and Cartesian systems of vector functions can be utilized [25,32,33]. One of the advantages of using such systems is that the axis-symmetric deformation and the 2D deformation can all be included as special cases of the general solutions [32,33]. Another advantage that is also associated with these systems is that the total solution can be expressed in terms of two parts: While one part is coupled with the electric quantities, the other one is purely elastic [32,33].

7. CONCLUSIONS

In this paper, we studied Green's functions in 3D anisotropic piezoelectric and multilayered media. We first apply the 2D Fourier transforms to the two horizontal variables, thus reducing the partial differential equation to an ordinary one. The general solution for the transformed displacements and stresses is then expressed in terms of the 3D Stroh formalism similar to that presented in Ting [1]. The propagator matrix for each layer is further derived from the general solution so that the displacements and stresses in the transformed domain at any vertical level can be expressed in terms of the propagator matrices. The physical-domain solution can be obtained by carrying out the inverse Fourier transforms. Several numerical issues associated with the inverse transforms are considered. Finally, two special cases of the present solution, i.e., the corresponding 2D deformation and 3D deformation in transversely isotropic piezoelectric and layered media, are briefly discussed.

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REFERENCES

1. Ting, T. C. T., *Anisotropic Elasticity*, Oxford University Press, Oxford (1996).
2. Ting, T. C. T., "Recent Developments in Anisotropic Elasticity," *Int. J. Solids Structures*, 37, pp. 401–409 (2000).
3. Ting, T. C. T., "The Wonderful World of Anisotropic Elasticity — An Exciting Theme Park to Visit," *Proc. 4th Pacific International Conference on Aerospace Science and Technology*, pp. 1–7 (2001).

4. Suo, Z., Kuo, C. M., Barnett, D. M. and Willis, J. R., "Fracture Mechanics for Piezoelectric Ceramics," *J. Mech. Phys. Solids*, 40, pp. 739–765 (1992).
5. Barnett, D. M. and Lothe, J., "Dislocations and Line Charges in Anisotropic Piezoelectric Insulators," *Phys. Stat. Sol*. (b), 67, pp. 105–111 (1975).
6. Chung, M. Y. and Ting, T. C. T., "Piezoelectric Solid with an Elliptic Inclusion or Hole," *Int. J. Solids Structures*, 33, pp. 3343–3361 (1996).
7. Pan, E., "A BEM Analysis of Fracture Mechanics in 2D Anisotropic Piezoelectric Solids," *Eng. Anal. Bound. Elements*, 23, pp. 67–76 (1999).
8. Ru, C. Q., "Eshelby's Problem for Two-Dimensional Piezoelectric Inclusions of Arbitrary Shape," *Proc. R. Soc. Lond. A*, 456, pp. 1051–1068 (2000).
9. Ru, C. Q., "Two-Dimensional Eshelby Problem for Two Bonded Piezoelectric Half-Planes," *Proc. R. Soc. Lond. A*, 457, pp. 865–883 (2001).
10. Liu, Y., Fan, H. and Yang, J., "Analysis of the Shear Stress Transferred from a Partially Electroded Piezoelectric Actuator to an Elastic Substrate," *Smart Mater. Struct*., 9, pp. 248–254 (2000).
11. Ding, H. J., Chen, B. and Liang, J., "On the Green's Functions for Two-Phase Transversely Isotropic Piezoelectric Media," *Int. J. Solids Struct*., 34, pp. 3041–3057 (1997).
12. Ding, H. J., Chi, Y. and Guo, F., "Solutions for Transversely Isotropic Piezoelectric Infinite Body, Semi-Finite Body and Bimaterial Infinite Body Subjected to Uniform Ring Loading and Charge," *Int. J. Solids Struct*., 36, pp. 2613–2631 (1999).
13. Dunn, M. L. and Wienecke, H. A., "Green's Functions for Transversely Isotropic Piezoelectric Solids," *Int. J. Solids Struct*., 33, pp. 4571–4581 (1996).
14. Dunn, M. L. and Wienecke, H. A., "Half-Space Green's Functions for Transversely Isotropic Piezoelectric Solids," *J. Appl. Mech*., 66, pp. 675–679 (1999).
15. Pan, E. and Tonon, F., "Three-Dimensional Green's Functions in Anisotropic Piezoelectric Solids," *Int. J. Solids Struct*., 37, pp. 943–958 (2000).
16. Pan, E. and Yuan, F. G., "Three-Dimensional Green's Functions in Anisotropic Piezoelectric Bimaterials," *Int. J. Eng. Sci*., 38, pp. 1939–1960 (2000).
17. Pan, E., "Mindlin's Problem for an Anisotropic Piezoelectric Half Space with General Boundary Conditions," *Proc. R. Soc. Lond. A*, 458, pp. 181–208 (2002).
18. Pan, E., "Some New Three-Dimensional Green's Functions in Anisotropic Piezoelectric Bimaterials," *Electronic Journal of Boundary Elements,* (in press, 2003).
19. Akamatsu, M. and Tanuma, K., "Green's Function of Anisotropic Piezoelectricity," *Proc. R. Soc. Lond. A*, 453, pp. 473–487 (1997).
20. Stroh, A. N., "Dislocations and Cracks in Anisotropic Elasticity," *Phil. Mag*., 3, pp. 625–646 (1958).
21. Stroh, A. N., "Steady State Problems in Anisotropic Elasticity," *J. Math. Phys*., 41, pp. 77–103 (1962).
22. Tiersten, H. F., *Linear Piezoelectric Plate Vibrations*, Plenum, New York (1969).
23. Yang, B. and Pan, E., "Three-Dimensional Green's Functions in Anisotropic Trimaterials," *Int. J. Solids Struct*., 39, pp. 2235–2255 (2002).
24. Gilbert, F. and Backus, G., "Propagator Matrices in Elastic Wave and Vibration Problems," *Geophys*., 31, pp. 326–332 (1966).
25. Pan, E., "Static Green's Functions in Multilayered Half Spaces," *Appl. Math. Modelling*, 21, pp. 509–521 (1997).
26. Yang, B. and Pan, E., "Efficient Evaluation of Three-Dimensional Green's Functions in Anisotropic Elastostatic Multilayered Composites," *Eng. Analy. Bound. Elements*, 26, pp. 355–366 (2002).
27. Yang, B., Pan, E. and Tewary, V. K., "Three-Dimensional Green's Functions in Anisotropic Piezoelectric Trimaterials," (to be submitted, 2003).
28. Yang, B., Pan, E. and Tewary, V. K., "Three-Dimensional Green's Functions in Anisotropic Piezoelectric Multilayered Structures," (submitted, 2003).
29. Ai, Z. Y., Yue, Z. Q., Tham, L. G. and Yang, M., "Extended Sneddon and Muki Solutions for Multilayered Elastic Materials," *Int. J. Eng. Sci*., 40, pp. 1453–1483 (2002).
30. Wu, K. C. and Chiu, Y. T., "The Elastic Fields of a Dislocation in an Anisotropic Strip," *Int. J. Solids Structures*, 32, pp. 543–552 (1995).
31. Chiu, Y. T. and Wu, K. C., "Analysis for Elastic Strips under Concentrated Loads," *J. Appl. Mech*., 65, pp. 626–634 (1998).
32. Pan, E., "Static Response of a Transversely Isotropic and Layered Half-Space to General Surface Loads," *Phys. Earth Planet. Inter*., 54, pp. 353–363 (1989).
33. Pan, E., "Static Response of a Transversely Isotropic and Layered Half-Space to General Dislocation Sources," *Phys. Earth Planet. Inter*., 58, pp. 103–117 (1989).