Chapter Five

Treatment of body forces in single-domain boundary integral equation method for anisotropic elasticity

E. Pan[1] & A.H.-D. Cheng[2]
[1] *Department of Civil Engineering. University of Akron, USA*
[2] *Department of Civil Engineering, University of Mississippi, USA*

Abstract

For an elastic domain with a body force and with cracks, direct application of the single-domain displacement and traction boundary integral equations will result in domain integrals. While the domain integral can be easily converted to the boundary integral exactly for the isotropic case, the anisotropic case needs special treatment. Utilizing the superposition method, this paper derives the exact particular solutions for the body force of gravity and the centrifugal force in their most general cases. Also discussed is the exact handling of the far-field stress/strain or initial stress/strain, for which the corresponding particular solutions are obtained. When both the body force and cracks are present in the problem domain, a new pair of single-domain boundary integral equations is derived. The related Green's functions in 2D and 3D anisotropic infinite spaces have been briefly reviewed and provided for the sake of completeness. Therefore, the general displacement, stress, and fracture problems in anisotropic 2D and 3D domains in the presence of far-field stress/strain or initial stress/strain, body force of gravity, and centrifugal force can all be analyzed using the proposed single-domain boundary integral equations with boundary discretization only.

1 Introduction

For elasticity with the presence of a body force, the conventional displacement boundary element method (BEM) involves a domain integral. Although the integration can be performed by dividing the domain into cells, this solution strategy is incongruent with the main attraction of BEM, namely a boundary-only discretization. For an isotropic solid, this issue was addressed by Cruse *et al.* [8] and Stippes and Rizzo [33]. However, only a special class was treated—the body force was given as the gradient of a potential, which in turn satisfied a Poisson's equation with a constant right hand side. Utilizing the divergent theorem, the domain integral was converted to a boundary one by an exact formulation. This class of body forces includes gravitational and centrifugal forces. Pape and Banerjee [26], and Henry *et al.* [12] proposed a different approach for gravitational and centrifugal forces that uses particular solutions to remove the domain integral.

To deal with an arbitrary body force in elasticity, Nardini and Brebbia [16] developed a procedure nowadays known as the dual reciprocity boundary element method (DRBEM). The DRBEM has been highly successful in a wide range of applications. The majority of implementations, however, focused on the Laplacian operator. Over a long period of time, the selection of interpolants in DRBEM was somewhat arbitrary. In 1994, Golberg and Chen called the attention of the BEM community to the existence of the radial basis functions (RBFs). In a survey article by Powell [27] the theoretical background of RBFs for interpolating scattered data has been well presented. The traditional basis function used in DRBEM was identified as a first order conical RBF. Other RBFs, such as spline, multiquadric, and Gaussian types were also introduced. Detailed discussions on this matter along with a complete review were presented recently by Cheng [6] and Cheng *et al.* [7].

Unlike the isotropic case, material anisotropy presents a great challenge to the BEM concerning the derivation of the related Green's functions as well as the exact treatment of the body force. While the treatment of body force has been investigated for certain special cases of material anisotropy and special cases of body forces [9, 20, 41,42], there is no treatment available for the generally anisotropic solid with the body force of gravity and centrifugal force in their most general cases.

In this paper, we first present the single-domain boundary integral equations using the superposition method. The particular solutions for the most general case of body force of gravity and centrifugal type are then derived in closed forms, as well as for the far-field stress/strain (or initial stress/strain) cases. The related Green's functions have also been briefly reviewed and presented in their most computationally advanced forms. Both 2D (generalized plane strain and plane stress) and 3D deformations have been included. With the single-domain BEMs, the particular solutions, and the Green's functions, various deformation, stress, and fracture problems can be analyzed using boundary discretization only.

2 Problem description for anisotropic elasticity

We assume an anisotropic domain V, which is either finite or infinite, bounded by an external boundary $S = \partial V$. Within the domain, we further assume that there are a finite number of cracks with their union defined as Γ. Their positive and negative sides are defined by Γ^+ and Γ^-, with their corresponding outward normal as $\mathbf{n}^+$ and $\mathbf{n}^-$, respectively. Finally, we define that $\mathbf{n} = \mathbf{n}^+ = -\mathbf{n}^-$.

The governing equation of elasticity with body force can be summarized as follows: the equilibrium equation

$$\sigma_{ij,j} + f_i = 0 \tag{1}$$

and the constitutive equation

$$\sigma_{ij} = C_{ijkl} e_{k,l} \tag{2}$$

where f_i is the body force vector, C_{ijkl} the elastic tensor, σ_{ij} the stress tensor, and e_{ij} the strain tensor related to the displacement vector u_i by the displacement-strain relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

For the isotropic case, the elastic tensor and its inverse, both being symmetric, can be expressed as

$$[C] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & sym. & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \tag{4}$$

$$[C]^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & sym. & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix} \tag{5}$$

where the conventional 3×3 matrix symbols for the elastic stiffness (and compliance) have been used. In eqns. (4) and (5), λ and μ are Lamé constants with μ also called shear modulus; and E and ν are Young's modulus and Poisson's ratio. The relations between those constants are

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\ ; \qquad \mu = \frac{E}{2(1+\nu)} \tag{6a,b}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\ ; \qquad \nu = \frac{\lambda}{2(\lambda + \mu)} \tag{7a,b}$$

For a well-posed boundary value problem, we require that on each part of the external boundary either the displacement or the traction boundary condition be prescribed:

$$\begin{aligned} u_i &= \bar{u}_i, \quad \text{on } S_u \\ t_i &= \bar{t}_i, \quad \text{on } S_t \end{aligned} \tag{8a,b}$$

where $S_u \cup S_t = S$, and

$$t_i = \sigma_{ij} n_j \tag{9}$$

is the boundary traction, again with n_j denoting the boundary outward normal.

On the crack surface union Γ, we assume that either the displacement discontinuity or the traction discontinuity is given. That is, one of the following two conditions must be satisfied on Γ

$$\begin{aligned} \Delta u_i &\equiv u_i^+ - u_i^- = \overline{\Delta u_i} \\ \Sigma t_i &\equiv t_i^+ + t_i^- = \overline{\Sigma t_i} \end{aligned} \tag{10a,b}$$

For a linear elastic medium, we can decompose displacements, stresses, and tractions into a homogeneous solution and a particular solution part, as follows:

$$\begin{aligned} u_i &= u_i^h + u_i^p \\ \sigma_{ij} &= \sigma_{ij}^h + \sigma_{ij}^p \\ t_i &= t_i^h + t_i^p \end{aligned} \tag{11a,b,c}$$

where the superscript h denotes the homogeneous solution, and p a particular solution corresponding to the body force (or a far-field stress field, without body force, i.e., $f_i = 0$). It is apparent that the particular solution must satisfy

$$C_{ijkl} u_{k,lj}^p + f_i = 0 \tag{12}$$

without restriction of the boundary condition. On the other hand, the homogeneous solution is governed by the homogeneous equation

$$C_{ijkl} u_{k,lj}^h = 0 \tag{13}$$

with the modified boundary conditions on the external boundary,

$$\begin{aligned} u_i^h &= \bar{u}_i - u_i^p, \quad \text{on } S_u \\ t_i^h &= \bar{t}_i - t_i^p, \quad \text{on } S_t \end{aligned} \tag{14a,b}$$

and on the crack surface union Γ,

$$\begin{aligned} \Delta u_i^h &= \overline{\Delta u_i} \\ \Sigma t_i^h &= \overline{\Sigma t_i} \end{aligned} \tag{15a,b}$$

where we have used the fact that, for the particular solution, both the displacement discontinuity and traction discontinuity across the crack surface are zero since they are continuous across the crack surface.

3 Single-domain boundary integral equations for anisotropic elasticity

Starting from Betti's reciprocity theorem, one can easily derive the following integral equation for the homogeneous boundary value problem described above by applying the displacement integral equation to the domain bounded externally by S and internally by the crack surface union Γ [17]

$$
\begin{aligned}
&U_i^h(\mathbf{x}_s) + \int_S T_{ij}^*(\mathbf{x}_s,\mathbf{x}_f)u_j^h(\mathbf{x}_f)\,dS(\mathbf{x}_f) + \int_\Gamma T_{ij}^*(\mathbf{x}_s,\mathbf{x}_{\Gamma+})\Delta u_j^h(\mathbf{x}_\Gamma)\,d\Gamma(\mathbf{x}_{\Gamma+}) \\
&= \int_S U_{ij}^*(\mathbf{x}_s,\mathbf{x}_f)t_j^h(\mathbf{x}_f)\,dS(\mathbf{x}_f) + \int_\Gamma U_{ij}^*(\mathbf{x}_s,\mathbf{x}_{\Gamma+})\Sigma t_j^h(\mathbf{x}_\Gamma)\,d\Gamma(\mathbf{x}_{\Gamma+})
\end{aligned}
\tag{16}
$$

where U_{ij}^* and T_{ij}^* are the Green's displacement and traction in the j-th direction at field point $\mathbf{x}_f$ or $\mathbf{x}_{\Gamma+}$ caused by a point source (force) in the i-th direction at source point $\mathbf{x}_s$. These Green's functions in the general anisotropic space (or plane) will be given in a later section. In eqn. (16) dS and $d\Gamma$ are the surface elements on the external boundary and crack surface, respectively. It is also noted that the term U_i^h in Eqn. (16) takes the following expression [40]:

$$
U_i^h(\mathbf{x}_s) = \begin{cases}
u_i^h(\mathbf{x}_s); & \mathbf{x}_s \in V \\
c_{ij}u_j^h(\mathbf{x}_s); & \mathbf{x}_s \in S \\
c_{ij}[u_j^h(\mathbf{x}_{\Gamma+}) + u_j^h(\mathbf{x}_{\Gamma-})]; & \mathbf{x}_s \in \Gamma \\
0; & \mathbf{x}_s \notin V + S + \Gamma
\end{cases}
\tag{17a,b,c,d}
$$

where the geometry matrix c_{ij} depends on the geometry of the boundary point $\mathbf{x}_s$. For a smooth point on the surface (external boundary and the crack surface), $c_{ij} = \delta_{ij}/2$ with δ_{ij} being the Kronecker delta.

Replacing the homogeneous solution in eqn. (16) by the total and particular solutions as given in eqns. (11a,b,c), we finally obtain the general displacement integral equation

$$
\begin{aligned}
&U_i(\mathbf{x}_s) - U_i^p(\mathbf{x}_s) + \int_S T_{ij}^*(\mathbf{x}_s,\mathbf{x}_f)[u_j(\mathbf{x}_f) - u_j^p(\mathbf{x}_f)]\,dS(\mathbf{x}_f) \\
&+ \int_\Gamma T_{ij}^*(\mathbf{x}_s,\mathbf{x}_{\Gamma-})\Delta u_j(\mathbf{x}_\Gamma)\,d\Gamma(\mathbf{x}_{\Gamma+}) = \int_S U_{ij}^*(\mathbf{x}_s,\mathbf{x}_f)[t_j(\mathbf{x}_f) - t_j^p(\mathbf{x}_f)]\,dS(\mathbf{x}_f) \\
&+ \int_\Gamma U_{ij}^*(\mathbf{x}_s,\mathbf{x}_{\Gamma+})\Sigma t_j(\mathbf{x}_\Gamma)\,d\Gamma(\mathbf{x}_{\Gamma+})
\end{aligned}
\tag{18}
$$

where the function $U_i(\mathbf{x}_s) - U_i^p(\mathbf{x}_s)$ has the following expression:

$$
U_i(\mathbf{x}_s) - U_i^p(\mathbf{x}_s) = \begin{cases}
u_i(\mathbf{x}_s) - u_i^p(\mathbf{x}_s); & \mathbf{x}_s \in V \\
c_{ij}[u_j(\mathbf{x}_s) - u_j^p(\mathbf{x}_s)]; & \mathbf{x}_s \in S \\
c_{ij}[u_i(\mathbf{x}_{\Gamma-}) + u_i(\mathbf{x}_{\Gamma-}) + 2u_j^p(\mathbf{x}_{\Gamma+})]; & \mathbf{x}_s \in \Gamma \\
0; & \mathbf{x}_s \notin V + S + \Gamma
\end{cases}
\tag{19a,b,c,d}
$$

It is noted that all the integral terms in eqn. (18) are integrable, with the exception of the term involving Green's traction where a Cauchy-type singularity exists. When the collocation point $\mathbf{x}_f$ is either on the external boundary S or on the crack union surface Γ, this Cauchy-type integral then needs to be treated by a special method, i.e., the rigid-body motion method. At the same time, the calculation of c_{ij}, which is geometry dependent, can also be avoided.

It is observed that for problems without cracks, the boundary integrals over the crack surface union Γ in eqn. (18) are discarded, and the boundary integral equation (18) thus reduces to the displacement integral equation with body force.

To handle problems with cracks, however, eqn. (18) is not enough for solving the unknowns. For this situation, the traction integral equation can be developed. Starting from eqn. (16), we take the derivative of both sides with respect to the internal field point $\mathbf{x}_f$, and multiply the result by the elastic constants to arrive at

$$\begin{aligned}&\Sigma^h_{lm}(\mathbf{x}_s) + \int_S C_{lmik} T^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_f) u^h_j(\mathbf{x}_f)\, dS(\mathbf{x}_f) + \int_\Gamma C_{lmik} T^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_{\Gamma+}) \Delta u^h_j(\mathbf{x}_\Gamma)\, d\Gamma(\mathbf{x}_{\Gamma+}) \\ &= \int_S C_{lmik} U^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_f) t^h_j(\mathbf{x}_f)\, dS(\mathbf{x}_f) + \int_\Gamma C_{lmik} U^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_{\Gamma+}) \Sigma t^h_j(\mathbf{x}_\Gamma)\, d\Gamma(\mathbf{x}_{\Gamma-})\end{aligned} \quad (20)$$

Again, replacing the homogeneous solution by the total and particular solutions, and also including the general situations for the field point $\mathbf{x}_f$, we obtain

$$\begin{aligned}&\Sigma_{lm}(\mathbf{x}_s) - \Sigma^p_{lm}(\mathbf{x}_s) + \int_S C_{lmik} T^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_f)[u_j(\mathbf{x}_f) - u^p_j(\mathbf{x}_f)]\, dS(\mathbf{x}_f) \\ &+ \int_\Gamma C_{lmik} T^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_{\Gamma+}) \Delta u_j(\mathbf{x}_\Gamma)\, d\Gamma(\mathbf{x}_{\Gamma-}) \\ &= \int_S C_{lmik} U^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_f)[t_j(\mathbf{x}_f) - t^p_j(\mathbf{x}_f)]\, dS(\mathbf{x}_f) \\ &+ \int_\Gamma C_{lmik} U^*_{ij,k}(\mathbf{x}_s, \mathbf{x}_{\Gamma+}) \Sigma t_j(\mathbf{x}_\Gamma)\, d\Gamma(\mathbf{x}_{\Gamma+})\end{aligned} \quad (21)$$

where $\Sigma_{lm}(\mathbf{x}_s) - \Sigma^p_{lm}(\mathbf{x}_s)$ is defined by

$$\Sigma_{lm}(\mathbf{x}_s) - \Sigma^p_{lm}(\mathbf{x}_s) = \begin{cases} \sigma_{lm}(\mathbf{x}_s) - \sigma^p_{lm}(\mathbf{x}_s); & \mathbf{x}_s \in V \\ [\sigma_{lm}(\mathbf{x}_s) - \sigma^p_{lm}(\mathbf{x}_s)]/2; & \mathbf{x}_s \in S \\ [\sigma_{lm}(\mathbf{x}_{\Gamma+}) + \sigma_{lm}(\mathbf{x}_{\Gamma-}) + 2\sigma^p_{lm}(\mathbf{x}_{\Gamma+})]/2; & \mathbf{x}_s \in \Gamma \\ 0; & \mathbf{x}_s \notin V + S + \Gamma \end{cases}$$

(22a,b,c,d)

A special case is the traction integral equation that can be obtained from eqn. (21) by further multiplying the equation on both sides by the normal n_m on the crack surface $\mathbf{y}_{\Gamma+}$. That is

$$
\begin{aligned}
&[t_l(\mathbf{y}_{\Gamma+}) - t_l(\mathbf{y}_{\Gamma-})]/2 + t_l^p(\mathbf{y}_{\Gamma+}) \\
&+ n_m(\mathbf{y}_{\Gamma+}) \int_S C_{lmik} T^*_{ij,k}(\mathbf{y}_{\Gamma+}, \mathbf{x}_f)[u_j(\mathbf{x}_f) - u_j^p(\mathbf{x}_f)]\, dS(\mathbf{x}_f) \\
&+ n_m(\mathbf{y}_{\Gamma-}) \int_\Gamma C_{lmik} T^*_{ij,k}(\mathbf{y}_{\Gamma+}, \mathbf{x}_{\Gamma+}) \Delta u_j(\mathbf{x}_\Gamma)\, d\Gamma(\mathbf{x}_{\Gamma+}) \\
&= n_m(\mathbf{y}_{\Gamma+}) \int_S C_{lmik} U^*_{ij,k}(\mathbf{y}_{\Gamma+}, \mathbf{x}_f)[t_j(\mathbf{x}_f) - t_j^p(\mathbf{x}_f)]\, dS(\mathbf{x}_f) \\
&+ n_m(\mathbf{y}_{\Gamma-}) \int_\Gamma C_{lmik} U^*_{ij,k}(\mathbf{y}_{\Gamma+}, \mathbf{x}_{\Gamma+}) \Sigma t_j(\mathbf{x}_\Gamma)\, d\Gamma(\mathbf{x}_{\Gamma+})
\end{aligned}
\tag{23}
$$

Similarly, for a problem including the crack surface only, the integral on the external surface S in eqn. (23) can be discarded, and then a simplified traction integral equation involving only the crack surface is obtained.

Displacement boundary integral eqn. (18) with eqn. (19b) (i.e., on the external boundary) and traction boundary integral eqn. (23) form a single-domain boundary integral equation set that can be used to solve general fracture problems in anisotropic solids under the effect of body force or far-field stresses (or far-field strains or initial stresses/strains). It should be noted that even to retrieve the individual crack displacements on both sides of the crack surface, discretization along both sides of the crack surface, as commonly used in the dual BEM method, is unnecessary. Actually, in order to get the individual crack displacement, one just needs to substitute the solved boundary data into eqn. (18) with eqn. (19c) (i.e., on the crack surface). Since this simple summation gives the sum of the crack surface displacements and the single-domain solution gives the difference of the crack surface displacements, the individual crack surface displacements can therefore be easily determined.

We emphasize that our new pair of single-domain boundary integral equations offers certain advantages over the previously proposed methods, and yet it can be applied to the most general cases, including material anisotropy, body force, and finite number of cracks. As mentioned above, simplified boundary integral equations can be reduced from our new pair of boundary integral equations. For example, for a problem domain without a crack, only equation (18) with relation (19b) is required, with the crack surface integral term being omitted. Similarly, for a problem containing crack surfaces only, such as cracks in an infinite or semi-infinite domain, only eqn. (23) is required with the external boundary integral terms being omitted.

The boundary integral equations (18) and (23) can be discretized and solved numerically for the unknown boundary displacements (or displacement discontinuities on the crack surface) and tractions. In solving these equations, the Cauchy-type integral and hypersingular integral term involved in eqns. (18) and (23) can be handled by several recently proposed techniques in the literature [1, 5, 17, 21]. Once the boundary problem is solved, eqns. (18) with (19a) and (21) with (22a) can be used to calculate the internal displacements and stresses.

In summary, we have presented a single-domain boundary integral equation set for the solution of displacement, stress and fracture analyses in general anisotropic solids. The effect of the body force and/or the far-field stresses has been successfully treated by decomposing the true solution into two parts: a particular solution corresponding to the body force and/or far-field stresses and a homogeneous solution. We have required that the particular solution satisfy the inhomogeneous equilibrium equation (or the homogeneous equation for the far-field stress case) without the restriction of boundary conditions; and the homogeneous solution is governed by the homogeneous equation with the modified boundary condition. In this single-domain set of boundary integral equations, no domain integral is involved. What we need are the particular solutions of displacements and stresses corresponding to the body force and/or the far-field stresses, and the Green's displacements and stresses (and their derivatives). These solutions will be presented in the next few sections after we first conduct a brief discussion on the difference of these boundary integral equations in 2D and 3D cases.

4 Difference between 2D and 3D boundary integral equations

For a 3D problem, the indices involved in the boundary integral equations vary from 1 to 3 with the boundary surface being generally a 2D surface in a 3D space. For a 2D problem in the (x, y)- or (x_1, x_2)-plane, displacements and stresses that depend on the z- or x_3-coordinate are omitted in the boundary integral equations. The boundary is now a 1D curve in a 2D plane with the third outward normal n_3 being zero. There are, however, certain differences between generalized plane strain and generalized plane stress for anisotropic solids. While for the generalized plane strain deformation, the displacement is still a 3-component vector and stress a 3×3 tensor, for the 2D plane stress, these are reduced to 2. Other differences are presented below.

For a generalized plane strain problem, the deformation in the (x, y)-plane (i.e., u_x and u_y) is generally coupled with u_z. If the material stiffness coefficient C_{ij}, obtained by the conventional reduction of indices from C_{ijkl}, satisfies the following conditions

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{46} = C_{56} = 0 \tag{24}$$

then the generalized plane strain deformation can be decoupled as plane strain deformation $(u_x, u_y \neq 0, \quad u_z = 0)$ and antiplane deformation $(u_x = u_y = 0, \quad u_z \neq 0)$ [34]. Monoclinic materials with the symmetric plane at $x_3 = 0$ (i.e., eqn. (24) plus conditions $C_{34} = C_{35} = 0$), orthotropic, transversely isotropic and isotropic materials all satisfy eqn. (24), and can therefore be decoupled as plane strain deformation and antiplane deformation.

For the 2D plane stress problem, $\sigma_{3i} = 0, \ i = 1, 2, 3$, and the solution is in the (x, y)-plane involving displacement (u_x, u_y) and the quantities derived from them. This is a true 2D case with eqns. (18) and (23) being taken only for the first two index values. When using those two boundary integral equations, however, the elastic coefficient C_{ijkl} needs to be replaced by C'_{ijkl}, defined as

$$C'_{ijkl} = C_{ijkl} - \frac{C_{ij33}C_{33kl}}{C_{3333}} \tag{25}$$

or in terms of C_{ij}, the elastic coefficient C_{ij} needs to be replaced by C'_{ij} defined as

$$C'_{ij} = C_{ij} - \frac{C_{i3}C_{3j}}{C_{33}} \tag{26}$$

Notice that in eqn. (25) the indices vary from 1 to 3, and those in eqn. (26) change from 1 to 6.

For the isotropic case, the plane strain and antiplane deformations satisfy, respectively, the following constitutive relations

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} u_{x,x} \\ u_{y,y} \\ u_{x,y} + u_{y,x} \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} u_{z,x} \\ u_{z,y} \end{bmatrix} \tag{28}$$

with σ_{zz} being obtained from the 3D constitutive relation under the condition that $u_{z,z}=0$.
The constitutive relation for the corresponding 2D plane stress deformation is obtained using eqns. (26) and (27)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{4\mu(\lambda + \mu)}{\lambda + 2\mu} & \dfrac{2\lambda\mu}{\lambda + 2\mu} & 0 \\ \dfrac{2\lambda\mu}{\lambda + 2\mu} & \dfrac{4\mu(\lambda + \mu)}{\lambda + 2\mu} & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} u_{x,x} \\ u_{y,y} \\ u_{x,y} + u_{y,x} \end{bmatrix} \tag{29}$$

From eqns. (27) and (29), the well-known correspondence between the plane strain and plane stress can be easily obtained. In terms of Young's modulus E and Poisson's ratio ν, these substitutions are as follows. To obtain the plane stress solution from a given plane strain one, we do the following substitutions simultaneously

$$\begin{aligned} E &\Leftarrow \frac{E(1+2\nu)}{(1+\nu)^2} \\ \nu &\Leftarrow \frac{\nu}{1+\nu} \end{aligned} \tag{30}$$

Similarly, to obtain the plane strain solution from a given plane stress one, the following substitutions are done simultaneously

$$\begin{aligned} E &\Leftarrow \frac{E}{1-\nu^2} \\ \nu &\Leftarrow \frac{\nu}{1-\nu} \end{aligned} \tag{31}$$

5 Particular solutions for 3D and 2D generalized plane strain problems

5.1 Far-field stresses/strains or initial stresses/strains

When the ratio of the outer boundary length scale to the inner boundary length scale are relatively large, the associated problems are usually treated as infinite domain problems where the outer boundary conditions are replaced by a far-field stress/strain or an initial stress/strain field. For an infinite domain problem, if we can derive the particular displacements corresponding to a far-field stress/strain or an initial stress/strain field, then the discretization on the artificially truncated outer boundary of the infinite domain, as commonly needed in FEM or even in BEM [14], can be avoided, saving computational time and increasing the accuracy of the solution.

We now seek the particular far-field displacement solution. First, we can always assume that the far-field or initial quantities are stresses. If they are strains, then, we get the corresponding stresses as

$$\sigma_{ij}^{p} = C_{ijkl} e_{kl}^{p} \tag{32}$$

Here we have used the superscript p for these far-field or initial quantities, since they are actually the particular stresses and strains.

We assume that the corresponding particular displacements are of the following form

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}/2 & a_{13}/2 \\ a_{12}/2 & a_{22} & a_{23}/2 \\ a_{13}/2 & a_{23}/2 & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{33}$$

for the 3D case, and

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{34}$$

for the 2D generalized plane strain deformation. Then, the constants can be found as (for $i = 1, 2, \ldots, 6$)

$$a_i = C_{ij}^{-1} \sigma_j^{p} \tag{35}$$

in which the following conventions have been adopted:

$$\begin{aligned} &[a_1, a_2, a_3, a_4, a_5, a_6]^T = [a_{11}, a_{22}, a_{33}, a_{23}, a_{13}, a_{12}]^T \\ &[\sigma_1^p, \sigma_2^p, \sigma_3^p, \sigma_4^p, \sigma_5^p, \sigma_6^p]^T = [\sigma_{xx}^p, \sigma_{yy}^p, \sigma_{zz}^p, \sigma_{yz}^p, \sigma_{xz}^p, \sigma_{xy}^p]^T \end{aligned} \tag{36a,b}$$

Also in eqn. (35), C_{ij}^{-1} is the inverse matrix of the stiffness matrix C_{ij} with the latter being the 6×6 matrix reduced from C_{ijkl}.

For the special case of isotropy, constants in eqn. (35) for the 3D and 2D generalized plane strain deformations can be expressed using the inverse of the stiffness matrix given in eqn. (5)

$$[a]=\frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & sym. & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma^p_{xx} \\ \sigma^p_{yy} \\ \sigma^p_{zz} \\ \sigma^p_{xz} \\ \sigma^p_{yz} \\ \sigma^p_{xy} \end{bmatrix} \quad (37)$$

5.2 General particular solutions

Before we consider the gravitational and centrifugal forces, we first discuss some general body-force cases for which closed-form solutions can be derived in the general anisotropic 3D and 2D (generalized plane strain deformation) domains.

If the body force in eqn. (12) depends upon only one of the three coordinates, then, we can seek the following three types of particular solutions:

$$\begin{aligned} C_{i1k1}u^p_{k,11} &= -f_i = b_{i1}F_1(x) \\ C_{i2k2}u^p_{k,22} &= -f_i = b_{i2}F_2(y) \\ C_{i3k3}u^p_{k,33} &= -f_i = b_{i3}F_3(z) \end{aligned} \quad (38)$$

It is obvious that the particular displacement solutions corresponding to these types of body forces can be assumed as

$$\begin{aligned} u^p_k &= d_{k1}X(x) \\ u^p_k &= d_{k2}Y(y) \\ u^p_k &= d_{k3}Z(z) \end{aligned} \quad (39)$$

with the coefficients and functions being determined by substituting these three types of solutions into eqn. (38) respectively. These particular displacement solutions are thus found to be

$$\begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} X(x) = \begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \int_0^x \int_0^r F_1(s)\, ds\, dr \quad (40a)$$

$$\begin{bmatrix} d_{12} \\ d_{22} \\ d_{32} \end{bmatrix} Y(y) = \begin{bmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix}^{-1} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} \int_0^y \int_0^r F_2(s)\, ds\, dr \quad (40b)$$

$$\begin{bmatrix} d_{13} \\ d_{23} \\ d_{33} \end{bmatrix} Z(z) = \begin{bmatrix} C_{55} & C_{45} & C_{35} \\ C_{45} & C_{44} & C_{34} \\ C_{35} & C_{34} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} \int_0^z \int_0^r F_3(s)\, ds\, dr \tag{40c}$$

The corresponding stresses are

$$\begin{aligned} \sigma_{ij}^{p} &= C_{ijk1} d_{k1} X'(x) \\ \sigma_{ij}^{p} &= C_{ijk2} d_{k2} Y'(y) \\ \sigma_{ij}^{p} &= C_{ijk3} d_{k3} Z'(z) \end{aligned} \tag{41}$$

where the prime denotes differentiation.

5.3 Gravitational force

Gravitational force is a universal force existing in any material. This force of gravity is particularly important when solving problems in geomechanics, such as the gravitational *in situ* stresses [18, 19, 24, 25], underground opening [23], landslides [31], and fracture [15, 17]. Assume that the gravity force is acting in the opposite direction of the x_i-coordinate, that is,

$$-f_i = \rho g_i \tag{42}$$

where ρ is the density of the material, and g_i the gravitational acceleration in the negative x_i-direction. One set of particular solutions can be obtained by assuming the particular solution corresponding to one of the following three types of forces and then superposing them together:

$$\begin{aligned} b_{i1} F_1(x) &= \rho g_i \delta_{i1} \\ b_{i2} F_2(y) &= \rho g_i \delta_{i2} \\ b_{i3} F_3(z) &= \rho g_i \delta_{i3} \end{aligned} \tag{43}$$

where δ_{ij} is the Kronecker delta. We mention that there is no summation over the index i on the right hand side of this equation. Using eqns. (39) and (40), the particular displacement solutions are found to be

$$\begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \end{bmatrix} = \begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} \rho g_x \\ 0 \\ 0 \end{bmatrix} \frac{x^2}{2} + \begin{bmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \rho g_y \\ 0 \end{bmatrix} \frac{y^2}{2} + \begin{bmatrix} C_{55} & C_{45} & C_{35} \\ C_{45} & C_{44} & C_{34} \\ C_{35} & C_{34} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \rho g_z \end{bmatrix} \frac{z^2}{2} \tag{44}$$

and the corresponding particular stresses are

$$\begin{bmatrix}\sigma^p_{xx}\\ \sigma^p_{yy}\\ \sigma^p_{zz}\\ \sigma^p_{yz}\\ \sigma^p_{xz}\\ \sigma^p_{xy}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{16} & C_{15}\\ C_{21} & C_{26} & C_{25}\\ C_{31} & C_{36} & C_{35}\\ C_{41} & C_{46} & C_{45}\\ C_{51} & C_{56} & C_{55}\\ C_{61} & C_{66} & C_{65}\end{bmatrix} \begin{bmatrix}C_{11} & C_{16} & C_{15}\\ C_{16} & C_{66} & C_{56}\\ C_{15} & C_{56} & C_{55}\end{bmatrix}^{-1} \begin{bmatrix}\rho g_x x\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}C_{16} & C_{12} & C_{14}\\ C_{26} & C_{22} & C_{24}\\ C_{36} & C_{32} & C_{34}\\ C_{46} & C_{42} & C_{44}\\ C_{56} & C_{52} & C_{54}\\ C_{66} & C_{62} & C_{64}\end{bmatrix} \begin{bmatrix}C_{66} & C_{26} & C_{46}\\ C_{26} & C_{22} & C_{24}\\ C_{46} & C_{24} & C_{44}\end{bmatrix}^{-1} \begin{bmatrix}0\\ \rho g_y y\\ 0\end{bmatrix}$$
$$+ \begin{bmatrix}C_{15} & C_{14} & C_{13}\\ C_{25} & C_{24} & C_{23}\\ C_{35} & C_{34} & C_{33}\\ C_{45} & C_{44} & C_{43}\\ C_{55} & C_{54} & C_{53}\\ C_{65} & C_{64} & C_{63}\end{bmatrix} \begin{bmatrix}C_{55} & C_{45} & C_{35}\\ C_{45} & C_{44} & C_{34}\\ C_{35} & C_{34} & C_{33}\end{bmatrix}^{-1} \begin{bmatrix}0\\ 0\\ \rho g_z z\end{bmatrix} \quad (45)$$

For the isotropic case, the particular displacements and stresses (for the non-zero components) are

$$\begin{bmatrix}u^p_x\\ u^p_y\\ u^p_z\end{bmatrix} = \frac{\rho}{2(\lambda+2\mu)} \begin{bmatrix}g_x x^2\\ g_y y^2\\ g_z z^2\end{bmatrix} \quad (46)$$

$$\begin{bmatrix}\sigma^p_{xx}\\ \sigma^p_{yy}\\ \sigma^p_{zz}\end{bmatrix} = \rho \begin{bmatrix}1 & \dfrac{\lambda}{\lambda+2\mu} & \dfrac{\lambda}{\lambda+2\mu}\\ \dfrac{\lambda}{\lambda+2\mu} & 1 & \dfrac{\lambda}{\lambda+2\mu}\\ \dfrac{\lambda}{\lambda+2\mu} & \dfrac{\lambda}{\lambda+2\mu} & 1\end{bmatrix} \begin{bmatrix}g_x x\\ g_y y\\ g_z z\end{bmatrix} \quad (47)$$

Some special features:

1. When the gravity is applied in the negative x-, y-, or z-direction only, three of the corresponding particular stress components have the special feature:

$$\begin{aligned}[\sigma^p_{xx}, \sigma^p_{xz}, \sigma^p_{xy}]^T &= [\rho g_x x, 0, 0]^T\\ [\sigma^p_{yy}, \sigma^p_{yz}, \sigma^p_{xy}]^T &= [\rho g_y y, 0, 0]^T\\ [\sigma^p_{zz}, \sigma^p_{yz}, \sigma^p_{xz}]^T &= [\rho g_z z, 0, 0]^T\end{aligned} \quad (48)$$

2. Also when the gravity is applied in the negative x-, y-, or z-direction only, the corresponding particular stress solutions are actually the gravitational stresses in a half space respectively with x, y, or z equal to zero as its flat surface, where the traction-free boundary conditions are satisfied and the gravitational stresses increase linearly with depth into the half space [2].

3. For the corresponding 2D generalized plane strain deformation (in the x-y plane), the particular solutions can be obtained simply by setting the z-variable equal to zero.
4. For the isotropic case and when the gravitational force is applied respectively in each of the negative axis directions, the present particular displacement and stress solutions do not reduce to those proposed before (i.e., by Sokolnikoff [32] and Pape and Banerjee [26]). This shows that the construction of the particular solution is not unique. We mention that the present solutions have been used for deformation, stress, and fracture analysis in rock mass for both the isotropic and anisotropic case, and the results solutions are consistent with those when analytical solutions are available [23]. Furthermore, the present particular stress solutions are consistent with the gravitational *in situ* stress observation that the gravitational stresses are not in a hydrostatic state, especially near the ground surface [3].

5.4 Centrifugal force

Centrifugal force exists in many machines, and study of its effect on the deformation, stress, and fracture is very important on grounds of reliability and safety [4, 10, 13, 28, 29, 37, 39].

For a body rotating about an arbitrary axis passing through the origin of the space-fixed Cartesian coordinates, with the angular velocity component being ω_i with respect to the x_i-axis, the corresponding centrifugal force, or pseudo body force, is

$$f_i = g_{ij} x_j \tag{49}$$

where

$$g_{ij} = \rho(\omega_k \omega_k \delta_{ij} - \omega_i \omega_j) \tag{50}$$

For this case, we have the following three types of body forces

$$\begin{aligned} -f_i &= -g_{i1} x \\ -f_i &= -g_{i2} y \\ -f_i &= -g_{i3} z \end{aligned} \tag{51}$$

Similar to the gravitational case, the particular displacement and stress solutions for this general centrifugal force can be obtained by superposing the corresponding three types of particular solutions:

$$\begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \end{bmatrix} = -\begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \end{bmatrix} \frac{x^3}{6} - \begin{bmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix}^{-1} \begin{bmatrix} g_{12} \\ g_{22} \\ g_{32} \end{bmatrix} \frac{y^3}{6} - \begin{bmatrix} C_{55} & C_{45} & C_{35} \\ C_{45} & C_{44} & C_{34} \\ C_{35} & C_{34} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} g_{13} \\ g_{23} \\ g_{33} \end{bmatrix} \frac{z^3}{6} \tag{52}$$

$$\begin{bmatrix}\sigma^p_{xx}\\ \sigma^p_{yy}\\ \sigma^p_{zz}\\ \sigma^p_{yz}\\ \sigma^p_{xz}\\ \sigma^p_{xy}\end{bmatrix}=-\begin{bmatrix}C_{11} & C_{16} & C_{15}\\ C_{21} & C_{26} & C_{25}\\ C_{31} & C_{36} & C_{35}\\ C_{41} & C_{46} & C_{45}\\ C_{51} & C_{56} & C_{55}\\ C_{61} & C_{66} & C_{65}\end{bmatrix}\begin{bmatrix}C_{11} & C_{16} & C_{15}\\ C_{16} & C_{66} & C_{56}\\ C_{15} & C_{56} & C_{55}\end{bmatrix}^{-1}\begin{bmatrix}g_{11}\\ g_{21}\\ g_{31}\end{bmatrix}\frac{x^2}{2}-\begin{bmatrix}C_{16} & C_{12} & C_{14}\\ C_{26} & C_{22} & C_{24}\\ C_{36} & C_{32} & C_{34}\\ C_{46} & C_{42} & C_{44}\\ C_{56} & C_{52} & C_{54}\\ C_{66} & C_{62} & C_{64}\end{bmatrix}\begin{bmatrix}C_{66} & C_{26} & C_{46}\\ C_{26} & C_{22} & C_{24}\\ C_{46} & C_{24} & C_{44}\end{bmatrix}^{-1}\begin{bmatrix}g_{12}\\ g_{22}\\ g_{32}\end{bmatrix}\frac{y^2}{2}$$
$$-\begin{bmatrix}C_{15} & C_{14} & C_{13}\\ C_{25} & C_{24} & C_{23}\\ C_{35} & C_{34} & C_{33}\\ C_{45} & C_{44} & C_{43}\\ C_{55} & C_{54} & C_{53}\\ C_{65} & C_{64} & C_{63}\end{bmatrix}\begin{bmatrix}C_{55} & C_{45} & C_{35}\\ C_{45} & C_{44} & C_{34}\\ C_{35} & C_{34} & C_{33}\end{bmatrix}^{-1}\begin{bmatrix}g_{13}\\ g_{23}\\ g_{33}\end{bmatrix}\frac{z^2}{2} \qquad (53)$$

For the isotropic case, these particular solutions are simplified to

$$\begin{bmatrix}u^p_x\\ u^p_y\\ u^p_z\end{bmatrix}=-\begin{bmatrix}\dfrac{g_{11}}{\lambda+2\mu}\\ \dfrac{g_{21}}{\mu}\\ \dfrac{g_{31}}{\mu}\end{bmatrix}\frac{x^3}{6}-\begin{bmatrix}\dfrac{g_{12}}{\mu}\\ \dfrac{g_{22}}{\lambda+2\mu}\\ \dfrac{g_{32}}{\mu}\end{bmatrix}\frac{y^3}{6}-\begin{bmatrix}\dfrac{g_{13}}{\mu}\\ \dfrac{g_{23}}{\mu}\\ \dfrac{g_{33}}{\lambda+2\mu}\end{bmatrix}\frac{z^3}{6} \qquad (54)$$

$$\begin{bmatrix}\sigma^p_{xx}\\ \sigma^p_{yy}\\ \sigma^p_{zz}\\ \sigma^p_{yz}\\ \sigma^p_{xz}\\ \sigma^p_{xy}\end{bmatrix}=-\begin{bmatrix}\lambda+2\mu & 0 & 0\\ \lambda & 0 & 0\\ \lambda & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \mu\\ 0 & \mu & 0\end{bmatrix}\begin{bmatrix}\dfrac{g_{11}}{\lambda+2\mu}\\ \dfrac{g_{21}}{\mu}\\ \dfrac{g_{31}}{\mu}\end{bmatrix}\frac{x^2}{2}-\begin{bmatrix}0 & \lambda & 0\\ 0 & \lambda+2\mu & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\\ 0 & 0 & 0\\ \mu & 0 & 0\end{bmatrix}\begin{bmatrix}\dfrac{g_{12}}{\mu}\\ \dfrac{g_{22}}{\lambda+2\mu}\\ \dfrac{g_{32}}{\mu}\end{bmatrix}\frac{y^2}{2}$$
$$-\begin{bmatrix}0 & 0 & \lambda\\ 0 & 0 & \lambda\\ 0 & 0 & \lambda+2\mu\\ 0 & \mu & 0\\ \mu & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dfrac{g_{13}}{\mu}\\ \dfrac{g_{23}}{\mu}\\ \dfrac{g_{33}}{\lambda+2\mu}\end{bmatrix}\frac{z^2}{2} \qquad (55)$$

Two observations:

1. For the corresponding 2D generalized plane strain deformation (in the x-y plane), the particular solutions can be obtained simply by setting the z-variable equal to zero.
2. For the isotropic case and when the rotating axis is coincident with one of the coordinate axes, the present particular displacement and stress solutions, which satisfy the equilibrium eqn. (12), do not reduce to the particular solutions proposed

previously (i.e., by Sokolnikoff [32] and Pape and Banerjee [26]). This again shows the non-uniqueness of constructing the particular solutions.

6 Particular solutions for 2D plane stress deformation

We now derive the particular displacements and stresses for a 2D plane stress deformation. First, for the 2D plane stress deformation in the x-y plane, the constitutive relation is expressed as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{16} \\ C'_{12} & C'_{22} & C'_{26} \\ C'_{16} & C'_{26} & C'_{66} \end{bmatrix} \begin{bmatrix} u_{x,x} \\ u_{y,y} \\ u_{x,y} + u_{y,x} \end{bmatrix} \tag{56}$$

where C'_{ij} is the reduced stiffness matrix defined by eqn. (26).

6.1 Far-field stresses

For the given far-field stresses (or the far-field strains or initial stresses/strain) denoted also as particular stresses $(\sigma^p_{xx}, \sigma^p_{yy}, \sigma^p_{xy})$, the corresponding particular displacements are assumed in the following form

$$\begin{bmatrix} u^p_x \\ u^p_y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{57}$$

Substituting this expression into constitutive relation (56) results in

$$\begin{bmatrix} a_{11} \\ a_{22} \\ a_{12} \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{16} \\ C'_{12} & C'_{22} & C'_{26} \\ C'_{16} & C'_{26} & C'_{66} \end{bmatrix}^{-1} \begin{bmatrix} \sigma^p_{xx} \\ \sigma^p_{yy} \\ \sigma^p_{xy} \end{bmatrix} \tag{58}$$

For the isotropic case, the constants are found to be

$$\begin{bmatrix} a_{11} \\ a_{22} \\ a_{12} \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda+\mu}{\mu(3\lambda+2\mu)} & \dfrac{-\lambda}{2\mu(3\lambda+2\mu)} & 0 \\ \dfrac{-\lambda}{2\mu(3\lambda+2\mu)} & \dfrac{\lambda+\mu}{\mu(3\lambda+2\mu)} & 0 \\ 0 & 0 & 1/\mu \end{bmatrix} \begin{bmatrix} \sigma^p_{xx} \\ \sigma^p_{yy} \\ \sigma^p_{xy} \end{bmatrix} \tag{59}$$

which is a result that can also be obtained from eqn. (37) using the substitution relation (30).

6.2 Gravitational force

Assuming that the gravitational force is acting along the negative direction of the x- and y-axis, the particular displacements and stresses can be found in the similar procedure used before:

$$\begin{bmatrix} u_x^p \\ u_y^p \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{16} \\ C'_{16} & C'_{66} \end{bmatrix}^{-1} \begin{bmatrix} \rho g_x \\ 0 \end{bmatrix} \frac{x^2}{2} + \begin{bmatrix} C'_{26} & C'_{66} \\ C'_{22} & C'_{26} \end{bmatrix}^{-1} \begin{bmatrix} \rho g_y \\ 0 \end{bmatrix} \frac{y^2}{2} \tag{60}$$

$$\begin{bmatrix} \sigma_{xx}^p \\ \sigma_{yy}^p \\ \sigma_{xy}^p \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{16} \\ C'_{12} & C'_{26} \\ C'_{16} & C'_{66} \end{bmatrix} \begin{bmatrix} C'_{11} & C'_{16} \\ C'_{16} & C'_{66} \end{bmatrix}^{-1} \begin{bmatrix} \rho g_x x \\ 0 \end{bmatrix} + \begin{bmatrix} C'_{12} & C'_{16} \\ C'_{22} & C'_{26} \\ C'_{26} & C'_{66} \end{bmatrix} \begin{bmatrix} C'_{26} & C'_{66} \\ C'_{22} & C'_{26} \end{bmatrix}^{-1} \begin{bmatrix} \rho g_y y \\ 0 \end{bmatrix} \tag{61}$$

The corresponding isotropic solutions are

$$\begin{bmatrix} u_x^p \\ u_y^p \end{bmatrix} = \frac{\rho(\lambda + 2\mu)}{8\mu(\lambda + \mu)} \begin{bmatrix} g_x x^2 \\ g_y y^2 \end{bmatrix} \tag{62}$$

$$\begin{bmatrix} \sigma_{xx}^p \\ \sigma_{yy}^p \end{bmatrix} = \rho \begin{bmatrix} 1 & \dfrac{\lambda}{2(\lambda+\mu)} \\ \dfrac{\lambda}{2(\lambda+\mu)} & 1 \end{bmatrix} \begin{bmatrix} g_x x \\ g_y y \end{bmatrix} \tag{63}$$

Again, eqns. (62) and (63) can be obtained from the corresponding 2D plane strain deformation solutions using the substitution relation (30).

6.3 Centrifugal force

For the 2D plane stress deformation, the rotational axis must be along the z-axis, and the corresponding body force is

$$-f_\alpha = -\rho\omega_3^2 x_\alpha\,; \qquad \alpha = 1,2 \tag{64}$$

For this centrifugal force, the particular displacements and stresses have structures very similar to the 2D gravitational force. That is

$$\begin{bmatrix} u_x^p \\ u_y^p \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{16} \\ C'_{16} & C'_{66} \end{bmatrix}^{-1} \begin{bmatrix} \rho\omega_3^2 \\ 0 \end{bmatrix} \frac{x^3}{6} + \begin{bmatrix} C'_{26} & C'_{66} \\ C'_{22} & C'_{26} \end{bmatrix}^{-1} \begin{bmatrix} \rho\omega_3^2 \\ 0 \end{bmatrix} \frac{y^3}{6} \tag{65}$$

$$\begin{bmatrix} \sigma_{xx}^p \\ \sigma_{yy}^p \\ \sigma_{xy}^p \end{bmatrix} = -\begin{bmatrix} C_{11} & C_{16} \\ C_{12} & C_{26} \\ C_{16} & C_{66} \end{bmatrix} \begin{bmatrix} C_{11} & C_{16} \\ C_{16} & C_{66} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\rho\omega_3^2 x^2}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} C_{12} & C_{16} \\ C_{22} & C_{26} \\ C_{26} & C_{66} \end{bmatrix} \begin{bmatrix} C_{26} & C_{66} \\ C_{22} & C_{26} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\rho\omega_3^2 y^2}{2} \\ 0 \end{bmatrix} \tag{66}$$

Similarly, for the isotropic case, these equations reduce to

$$\begin{bmatrix} u_x^p \\ u_y^p \end{bmatrix} = \frac{\rho\omega_3^2(\lambda+2\mu)}{24\mu(\lambda+\mu)}\begin{bmatrix} x^3 \\ y^3 \end{bmatrix} \tag{67}$$

$$\begin{bmatrix} \sigma_{xx}^p \\ \sigma_{yy}^p \end{bmatrix} = \rho\omega_3^2 \begin{bmatrix} 1 & \dfrac{\lambda}{4(\lambda+\mu)} \\ \dfrac{\lambda}{4(\lambda+\mu)} & 1 \end{bmatrix}\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \tag{68}$$

7 Green's functions in anisotropic 3D and 2D infinite space

Having presented the new pair of single-domain boundary integral equations and the particular solutions to the far-field (initial) stresses/strains, the body force of gravity, and the centrifugal force, we now discuss the related 3D and 2D Green's functions.

7.1 Green's functions in anisotropic 3D infinite space

Analytical studies of Green's functions in anisotropic elastic 3D infinite space have been conducted by Ting and Lee [35], Wang [38], Sales and Gray [30], and Tonon *et al.* [36]. Here the results from Tonon *et al.* [36] are briefly presented for the sake of completeness. These Green's functions include displacements, stresses, and their derivatives with respect to the source coordinates, due to a point force.

We assume that a general point force is located at the origin of the space-fixed Cartesian coordinates (O; x_1, x_2, x_3), and our purpose is to find the complete 3D Green's functions at the field point **x**. We first introduce a new, orthogonal, and normalized system (O; **e**, **p**, **q**), with their base (**e**, **p**, **q**) being chosen as the following

$$\mathbf{e} = \mathbf{x}/r; \quad r = |\mathbf{x}| \tag{69}$$

Now, let **v** be an arbitrary unit vector different from **e** ($\mathbf{v} \neq \mathbf{e}$), the two unit vectors orthogonal to **e** can then be selected as:

$$\mathbf{p} = \frac{\mathbf{e}\times\mathbf{v}}{|\mathbf{e}\times\mathbf{v}|}; \quad \mathbf{q} = \mathbf{e}\times\mathbf{p} \tag{70}$$

Let vector variable **n** be the 3D Fourier transform variable. Then, in the new reference system (O; **e**, **p**, **q**), it can be expressed as

$$\mathbf{n} = \xi\mathbf{p} + \zeta\mathbf{q} + \eta\mathbf{e} \tag{71}$$

It is clear then that

$$\mathbf{n}\cdot\mathbf{x} = \mathbf{p}\cdot\mathbf{x}\,\xi + \mathbf{q}\cdot\mathbf{x}\,\zeta + \mathbf{e}\cdot\mathbf{x}\,\eta = r\eta \tag{72}$$

In terms of the reference system (O; **e**, **p**, **q**), the 3D Fourier transform can be reduced and the Green's displacement tensor (3×3) can be expressed as

$$G_{jk}(\mathbf{x}) = \frac{1}{8\pi^2} \int_{\Omega} \frac{A_{jk}(\xi\,\mathbf{p} + \zeta\,\mathbf{q} + \eta\,\mathbf{e})}{D(\xi\,\mathbf{p} + \zeta\,\mathbf{q} + \eta\,\mathbf{e})} \delta(r\,\eta)\, d\Omega(\xi, \zeta, \eta) \tag{73}$$

where Ω is a closed surface enclosing the origin (ξ, ζ, η) = (0, 0, 0). In eqn. (73), A_{jk} and D are the co-factors and determinant, respectively, of the matrix Γ_{jk}, which is defined as

$$\Gamma_{jk}(\mathbf{n}) = C_{ijkl} n_i n_l \tag{74}$$

with C_{ijkl} being the elastic stiffness matrix. Carrying out the integration of (73) with respect to η yields [36]

$$G_{jk}(\mathbf{x}) = \frac{1}{4\pi^2 r} \int_{-\infty}^{\infty} \frac{A_{jk}(\mathbf{p} + \zeta\,\mathbf{q})}{D(\mathbf{p} + \zeta\,\mathbf{q})} d\zeta \tag{75}$$

It turns out that the matrix Γ_{jk} can be expressed by the Stroh formalism. That is,

$$\Gamma(\mathbf{p} + \zeta\,\mathbf{q}) \equiv \boldsymbol{Q} + \zeta(\boldsymbol{R} + \boldsymbol{R}^T) + \zeta^2 \boldsymbol{T} \tag{76}$$

where

$$Q_{ik} = C_{jikl} p_j q_l, \quad R_{ik} = C_{jikl} p_j q_l, \quad T_{ik} = C_{jikl} q_j q_l \tag{77}$$

The determinant $D(\mathbf{p}+\zeta\mathbf{q})$ is a sixth-degree polynomial of ζ and has six roots. For the materials studied in this paper, three of them are the conjugate of the others. These roots can be found either by expanding the determinant $D(\mathbf{p}+\zeta\mathbf{q})$ into a polynomial, or by finding the six eigenvalues of the following linear eigenequation [34]

$$\begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^T \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \zeta \begin{bmatrix} a \\ b \end{bmatrix} \tag{78}$$

where

$$N_1 = -\boldsymbol{T}^{-1}\boldsymbol{R}^T, \; N_2 = \boldsymbol{T}^{-1}, \; N_3 = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^T - \boldsymbol{Q} \tag{79}$$

and the eigenvectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are the coefficients of the displacement and traction vectors.

Assume that $\text{Im}\zeta_m > 0$ (m = 1, 2, 3), and $\zeta_m{}^*$ is the conjugate of ζ_m, the Green's displacement can be explicitly expressed as

$$G_{jk}(\mathbf{x}) = -\frac{\text{Im}}{2\pi r} \sum_{m=1}^{3} \frac{A_{jk}(\mathbf{p} + \zeta_m \mathbf{q})}{a_7(\zeta_m - \zeta_m^*) \prod_{\substack{k=1 \\ k \neq m}}^{3} (\zeta_m - \zeta_k)(\zeta_m - \zeta_m^*)} \tag{80}$$

where $a_7 = \det(T)$ is the coefficient of the ζ^6 term.

There are several features associated with this new expression. First of all, eqn. (80) is an explicit expression. It can therefore be evaluated very accurately and efficiently. For a given pair of field and source points, we need only to solve a sixth-order linear eigenequation, or a sixth-degree polynomial equation numerically once in order to obtain all the components of the Green's displacement. Secondly, in obtaining eqn. (80), we have assumed that all the poles are simple. Should the poles be multiple, a slight change in the material constants will result in single poles, with negligible errors in the computed Green's tensor [17, 21]. Thirdly, since Γ_{jk} is symmetric, so is its adjoint A_{jk}. Therefore, the Green's displacement G_{jk} is symmetric and one needs to calculate only 6 out of its 9 elements. Finally, although one can choose the vector $\mathbf{v}$ ($\neq \mathbf{e}$) arbitrarily, it should be one of the base vectors in the space-fixed Cartesian coordinates, i.e., (1, 0, 0), (0, 1, 0), or (0, 0, 1). The analytical expression for Green's displacement is much simpler using such a vector $\mathbf{v}$ than using any other vectors.

We have just derived an explicit expression for Green's displacement. In the application of the boundary integral equation and other related methods, one also needs the Green's stress, which can be obtained by taking the derivative of Green's displacement. However, an explicit expression for the derivative of Green's displacement is too complicated to be implemented efficiently. Here, the numerical method recently proposed by Pan and Tonon [22] and Tonon *et al.* [36] is used to evaluate these derivatives. It is based on the simple interpretation of the Lagrange polynomials, and yet it turns out to be very efficient and accurate.

Following Pan and Tonon [22] and Tonon *et al.* [36], for instance, the derivatives of the Green's tensor G_{jk} with respect to the coordinates at $\mathbf{x} = (x_1, x_2, x_3)$ are evaluated by

$$\frac{\partial G_{jk}}{\partial x_1} \approx \frac{1}{2h}\left[G_{jk}(x_1 + h, x_2, x_3) - G_{jk}(x_1 - h, x_2, x_3)\right] \tag{81}$$

$$\frac{\partial G_{jk}}{\partial x_2} \approx \frac{1}{2h}\left[G_{jk}(x_1, x_2 + h, x_3) - G_{jk}(x_1, x_2 - h, x_3)\right] \tag{82}$$

$$\frac{\partial G_{jk}}{\partial x_3} \approx \frac{1}{2h}\left[G_{jk}(x_1, x_2, x_3 + h) - G_{jk}(x_1, x_2, x_3 - h)\right] \tag{83}$$

where the interval h is chosen as [36]

$$h = r \cdot 10^{-6}$$

with r being the distance between the field and source points. From these relations, all the Green's stresses can be obtained.

The derivatives of Green's stresses are also required. These can also be carried out using the simple yet accurate formula as those given above [36].

7.2 Green's functions in anisotropic 2D infinite plane under generalized plane strain conditions

Green's displacements in an anisotropic infinite plane have been well known in the literature (e.g., Ting [34]). The following complete Green's functions (including displacements, stresses, and their derivatives with respect to the source point) are from Pan [17].

In general, the 2D Green's functions in an anisotropic, linear elastic medium in the (x, y) plane can be expressed in terms of three complex functions $f_j(z_j)$, $j = 1, 2, 3$, as

$$\begin{aligned} u_i &= 2\,\mathrm{Re}\left[\sum_{j=1}^{3} A_{ij} f_j(z_j)\right] \\ \sigma_{2i} &= 2\,\mathrm{Re}\left[\sum_{j=1}^{3} B_{ij} f'_j(z_j)\right] \\ \sigma_{1i} &= -2\,\mathrm{Re}\left[\sum_{j=1}^{3} B_{ij} \mu_j f'_j(z_j)\right] \end{aligned} \tag{84}$$

where $z_j = x + \mu_j y$; Re denotes the real part of a complex variable or function; a prime denotes the derivative; and μ_j $(j = 1, 2, 3)$ are the three distinct complex roots with a positive imaginary part of the following equation

$$\left|C_{1ij1} + \mu(C_{1ij2} + C_{1ji2}) + \mu^2 C_{2ij2}\right| = 0 \tag{85}$$

For each of the characteristic roots μ_k, each column of the matrix **A** in eqn. (84) is the eigenvector of the following equation:

$$\sum_{j=1}^{3}[C_{1ij1} + \mu_k(C_{1ij2} + C_{1ji2}) + \mu_k^2 C_{2ij2}]A_{jk} = 0 \tag{86}$$

Once the matrix **A** is found, the matrix **B** in eqn. (84) can be obtained as

$$B_{ik} = \sum_{j=1}^{3}(C_{1ji2} + \mu_k C_{2ij2})A_{jk} \tag{87}$$

Assume that there is a force $\mathbf{F} = \{F_j\}$ acting at the source point (x^0, y^0), the solution for the complex functions can be expressed as

$$f_j(z) = q_j \ln(z - s_j) \tag{88}$$

where $s_j = x^0 + \mu_j y^0$, and $\mathbf{q} = \{q_j\}$ is a complex coefficient vector, given by

$$\mathbf{q} = \frac{-1}{2\pi}\mathbf{A}^{-1}(\mathbf{Y}^{-1} + \overline{\mathbf{Y}}^{-1})^{-1}\mathbf{F} \tag{89}$$

where the overbar means complex conjugate; superscript -1 means matrix inverse, and **Y** is given by

$$\mathbf{Y} = i\mathbf{A}\mathbf{B}^{-1} \tag{90}$$

with $i = \sqrt{-1}$. The substitution of the complex function (88) in eqn. (84) gives the Green's displacement

$$U^{\bullet}_{kl} = \frac{-1}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{3} A_{lj} H_{jk} \ln(z_j - s_j)\right] \tag{91}$$

and traction

$$T^{\bullet}_{kl} = \frac{1}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{3} B_{lj} \frac{\mu_j n_x - n_y}{z_j - s_j} H_{jk}\right] \tag{92}$$

where n_x and n_y are the x and y components of the unit outward normal at the field point (x,y).

The derivatives of these Green's displacements and tractions, with respect to the source coordinates (x^0, y^0), are found to be

$$\begin{aligned} U^{\bullet}_{kl,x^0} &= \frac{1}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{3} \frac{A_{lj} H_{jk}}{z_j - s_j}\right]; \\ U^{\bullet}_{kl,y^0} &= \frac{1}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{3} \frac{A_{lj} \mu_j H_{jk}}{z_j - s_j}\right] \end{aligned} \tag{93}$$

$$\begin{aligned} T^{\bullet}_{kl,x^0} &= \frac{1}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{3} B_{lj} \frac{\mu_j n_x - n_y}{(z_j - s_j)^2} H_{jk}\right] \\ T^{\bullet}_{kl,y^0} &= \frac{1}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{3} B_{lj} \frac{(\mu_j n_x - n_y)\mu_j}{(z_j - s_j)^2} H_{jk}\right] \end{aligned} \tag{94}$$

where

$$\mathbf{H} = \mathbf{A}^{-1}(\mathbf{Y}^{-1} + \overline{\mathbf{Y}}^{-1})^{-1} \tag{95}$$

7.3 Green's functions in anisotropic 2D infinite plane under plane stress conditions

Green's functions for this case can be directly obtained from the generalized plane strain solution by replacing the elastic stiffness with the reduced elastic stiffness defined in eqn. (26). The resulting Green's functions include only two displacement components, three stress components (i.e., only the plane components), and their derivatives with respect to the source points, due to a line force in the x- and y-directions.

8 Conclusion

For elasticity with a body force, the regular boundary element method involves a domain integral. Special techniques have been devised to transform the domain integrals to boundary ones. While a recent paper has reviewed the more than two decades of development of treating body forces in BEM for isotropic media [7], the present paper concentrates on the exact, closed-form particular solutions for special body forces, including body forces of gravity and centrifugal type. A new pair of boundary integral equations have also been derived in their most general cases for cracked anisotropic solids under the action of body force. Particular solutions related to the far-field (initial) stresses/strains are derived, as are the Green's functions in anisotropic 3D and 2D infinite domains. Therefore, with the present single-domain boundary integral equations, along with the particular solutions and Green's functions, various deformation, stress, and fracture problems in cracked anisotropic solids, subjected to gravity force, centrifugal force, and far-field/initial stresses/strains, can be solved by discretizing along the external boundary and one side of the crack surface only.

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