

Displacements and Stresses Due to a Uniform Vertical Circular Load in an Inhomogeneous Cross-Anisotropic Half-Space

Cheng-Der Wang¹; Ernian Pan²; Chi-Shiang Tzeng³; Feng Han⁴; and Jyh-Jong Liao⁵

Abstract: In this work, we present the solutions for displacements and stresses along the centerline of a uniform vertical circular load in an inhomogeneous cross-anisotropic half-space with its Young's and shear moduli varying exponentially with depth. The planes of cross anisotropy are assumed to be parallel to the horizontal surface. The presented solutions can be directly integrated from the point load solution in a cylindrical coordinate system, which were derived by the writers. However, the resulting integrals of the circular solution for displacements and stresses cannot be given in closed form; hence, numerical integrations are required. For a homogeneous cross-anisotropic half-space, the numerical results agree very well with the exact solutions of Hanson and Puja, published in 1996. Two examples are given to elucidate the effect of inhomogeneity, and the type and degree of soil anisotropy on the vertical displacement and vertical normal stress in the inhomogeneous isotropic/cross-anisotropic soils subjected to a uniform vertical circular load acting on the surface. The proposed solutions can more realistically simulate the actual stratum of loading problem in many areas of engineering practice.

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Introduction

The solutions for displacements and stresses subjected to applied loads in the constituted materials are important in designing foundations. Traditionally, when one calculates the displacements and stresses in these foundations, the media are often assumed to be homogeneous, isotropic, and linearly elastic continua. However, there are many natural soils deposited through a geologic process of sedimentation over a period of time, such as flocculated clays, varved silts and sands, or rocks, such as foliated metamorphic, stratified sedimentary, regularly jointed rocks, or rock masses cut by discontinuities, the responses of displacements and stresses should account for anisotropy. For example, from the practical engineering point of view, anisotropy soils are often modeled as cross-anisotropic media. In addition, the mechanical response of anisotropic materials with spatial gradients in composition is of

considerable interest in soil/rock mechanics (Suresh 2001). The effects of deposition, overburden, desiccation, etc., can lead geological media, which exhibit both inhomogeneity and anisotropic deformability characteristics. However, theoretical understanding of both phenomena has not received much attention due to the mathematical difficulty associated with these media. Therefore, a uniform vertical circular load applied to a continuously inhomogeneous cross-anisotropic half-space with its Young's and shear moduli varying exponentially with depth is investigated in this work.

The load with circular shape is chosen because the solutions produced are of practical importance in foundation engineering. In particular, these solutions could have a direct application to problems associated with foundations under structures such as silos, chimneys, and tanks containing liquids (Gerrard and Wardle 1973). Numerous existing analytical/numerical solutions for inhomogeneous isotropic media due to a circular load are summarized in Table 1. Table 1 indicates the types of inhomogeneity, analytical or numerical solutions presented, and possible restrictions on the Poisson's ratio in these solutions. It is noted that a great majority of the literature was dedicated to the calculation of displacements/stresses in isotropic media with the Young's or shear modulus varies with depth according to the power law ($E=m_E Z^\alpha$ or $G=m_G Z^\alpha$), the linear law ($E=E_0+\lambda z$ or $G=G_0+\lambda z$), and the exponential law ($E=E_0+E_1 e^{\xi z}$ or $G=G_0+G_1 e^{\xi z}$), etc. The existing solutions for homogeneous and inhomogeneous cross-anisotropic media due to a circular load are given in Table 2. Table 2 interprets the type of material, the type of load, and the solutions developed. Compared with the inhomogeneous isotropic or homogeneous cross-anisotropic solutions, the literatures contributed to the inhomogeneous cross-anisotropic half-space are very limited. So far, to the best of the writers' knowledge, no analytical/numerical solution exists in the literatures for the circular load induced vertical displacement and vertical normal

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Table 1. Existing Analytical/Numerical Solutions for Inhomogeneous Isotropic Media Due to a Circular Load

Types of inhomogeneity	Author	Analytical or numerical solutions	Poisson's ratio
$E=m_E Z^\alpha$ or $G=m_G Z^\alpha$ ($0 \leq \alpha \leq 1$)	Rostovtsev (1961)	Settlement due to an elliptical, a circular, and a paraboloid of revolution load	$\nu=1/(2+\alpha)$
	Popov (1962)	Surface displacement due to a circular load	General
	Carrier and Christian (1973a)	Displacements and stresses due to a circular load by FEM	$\nu=1/2$
	Popov (1973)	Displacements due to vertical/horizontal circular punches	General
	Booker et al. (1985)	Surface displacement due to strip, ring, and circular loads	General
	Oner (1990)	Displacements due to vertical/horizontal point, circular, and rectangular loads	General
	Hemsley (1998)	Vertical displacement and contact pressure due to a rigid circular load	General
	Doherty and Deeks (2003b)	Circumferential displacement of a rigid circular footing subjected to vertical, horizontal, moment, and torsion loads by using the scaled boundary finite element method	General
	Doherty and Deeks (2003c)	Displacement response of rigid and flexible circular footings subjected to vertical load by using the scaled boundary finite element method	General
	Doherty and Deeks (2003a)	Load-displacement response of an embedded rigid circular footing subjected to vertical, horizontal, moment, and torsion loads by using the scaled boundary finite element method	General
$E=E_0(a+bz)^c$ or $G=G_0(a+bz)^c$	Chuaprasert and Kassir (1974)	Displacements and stresses due to a uniform circular load	$\nu=1/(2+c)$
	Rajakpase and Selvadurai (1989)	Stresses and displacement due to rigid circular and cylindrical foundations	General
$E=E_0+\lambda z$ or $G=G_0+\lambda z$	Gibson (1967)	Displacements and stresses due to strip and circular loads	$\nu=1/2$
	Brown and Gibson (1972)	Surface displacement due to a strip or circular load	General
	Carrier and Christian (1973b)	Settlement and stresses due to a rigid circular plate by FEM	General
	Gibson (1974)	Surface displacement of uniformly circular loads	$\nu=1/2$
	Alexander (1977)	Vertical displacement due to a circular load	$\nu=1/2$
	Rajakpase and Selvadurai (1991)	Axisymmetric elastic response of circular footings and anchor plates	$\nu=1/2$
	Yue et al. (1999)	Displacements and stresses due to a circular load for a layered half-space by backward transfer matrix method	$\nu=1/2$
$E=E_0+E_1 e^{\xi z}$ or $G=G_0+G_1 e^{\xi z}$	Ter-Mkrtych'ian (1961)	Stresses and displacements due to a circular load	General
	Selvadurai (1996)	Settlement due to a rigid circular foundation	General
$G=G_0^* h/(h-z)$ $G=\text{constant}$	Awojobi (1975)	Settlement of a circular foundation	General
	Gibson and Sills (1969)	Stresses and displacements due to point and circular loads	$\nu=f(z)$

stress in a cross-anisotropic half-space with Young's and shear moduli varying exponentially with depth ($Ee^{-kz}, E'e^{-kz}, G'e^{-kz}$). The lack of such a solution is mainly due to the fact that the general methods for solving problems involving inhomogeneous and anisotropy are very complicated (Kumar 1988).

It is well known that the point load solution forms the basis of solutions to complex loading problems. Utilizing the approach proposed by Liao and Wang (1998), Wang et al. (2003) derived the point load solution of displacements and stresses in the Hankel domain for the continuously inhomogeneous cross-anisotropic full/half-spaces subjected to a vertical point load. The physical domain displacements and stresses should be obtained by taking the numerical inversion of Hankel transforms since the resulting integrals in the point load solution involve products of Bessel functions of the first kind, an exponential function, and a polynomial, which cannot be given in the exact closed form. Hence, due to the complexity of the point load solution, the uniform circular load solution also needs to be obtained by numerical integration,

for which was achieved in this work. In order to check the accuracy of our numerical procedure, the presented solution is simplified to the homogeneous solution of Hanson and Puja (1996) by setting the inhomogeneity parameter k to zero. It is found that the numerical results agree with the exact solution very well. Further, two examples are given to elucidate the effect of inhomogeneity, and the type and degree of soil anisotropy on the induced vertical displacement and vertical normal stress along the axisymmetric axis (z axis), with the circular load being on the surface of the inhomogeneous isotropic/cross-anisotropic soils.

Displacements and Stresses in an Inhomogeneous Cross-Anisotropic Half-Space Due to a Vertical Point Load

In this paper, the solutions for displacements and stresses in an inhomogeneous cross-anisotropic half-space subjected to a uni-

Table 2. Existing Solutions for Homogeneous/Inhomogeneous Cross-Anisotropic Media Due to a Circular Load

Type of material	Author	Loaded type	Solutions	
Homogeneous	Quinlan (1949)	Uniform vertical stress Uniform vertical displacement	Vertical surface displacement, and vertical stress on load axis	
	Anon (1960)	Uniform vertical pressure Uniform vertical displacement	Vertical surface displacement at center and edge of load, and vertical stress on load axis	
	Barden (1963)	Uniform vertical pressure	Vertical stress along the vertical centerline	
	Lekhnitskii (1963)	Uniform vertical pressure	All stresses	
	Gerrard (1968)	Uniform vertical pressure Uniform vertical displacement Linear radial shear stress Torsional loads	All displacements, strains, and stresses	
	Gerrard and Harrison (1970)	Uniform vertical pressure Uniform vertical displacement Linear vertical pressure Linear vertical displacement Linear radial shear stress Linear radial shear stress displacement Linear torsional shear stress Linear torsional shear stress displacement Uniform unidirectional shear stress Uniform unidirectional shear displacement	All displacements, strains, and stresses	
	Gerrard and Wardle (1973)	Vertical force Horizontal force Moment about a horizontal axis Moment about a vertical axis Radial shear stresses	All displacements, strains, and stresses	
	Nayak (1973)	Uniform vertical pressure	Surface displacement at the center (eliminate one Poisson's ratio)	
	Hooper (1975)	Uniform vertical pressure Parabolic vertical pressure	Surface settlement	
	Misra and Sen (1975)	Uniform contact pressure A generalized parabolic contact pressure A generalized inverse parabolic contact pressure	Vertical stress along the vertical axis, and vertical surface displacement of the center and edge	
	Chowdhury (1987)	Buried, uniform vertical pressure	All surface displacements	
	Hanson and Puja (1996)	Uniform vertical pressure Uniform shear loading	All displacements and stresses	
	Inhomogeneous $E' = E'(0) + \lambda z$	Hooper (1975)	Uniform vertical pressure Parabolic vertical pressure	Surface settlement
		Rowe and Booker (1981)	Uniform vertical pressure	Parametric study of central displacement by finite layer method
	Inhomogeneous $(E' = E_0 - \rho_c z$ $z < z_c$; $E' = E_0 - \rho_c z_c +$ $\rho(z - z_c), z > z_c)$	Kumar (1988)	Uniform vertical pressure Uniform vertical rigid displacement	Surface settlement by finite/infinite element method
Pan (1989)		General vertical and shear loadings	All displacements and stresses	
Pan (1997)		General vertical and shear loadings and internal concentrated forces	All displacements and stresses	

form vertical circular load are directly integrated from the point load solutions in a cylindrical co-ordinate system. Elastic inhomogeneous cross-anisotropic half-space has planes of isotropy parallel to the boundary plane. The approaches for solving the displacements and stresses subjected to a static vertical point load P_z (force) in a cylindrical coordinate, which are expressed as the form of body forces, are shown in Fig. 1 (Wang et al. 2003). Fig. 1 depicts that a half-space is composed of two full spaces, one with a point load in its interior and the other with opposite traction of the first full space along $z=0$. The traction in the first full space along $z=0$ is due to the point load. The solutions for the

half-space are thus obtained by superposing the solutions of the two full spaces. That is, the solutions can be derived from the governing equations for a full space [including the general solutions (I) and homogeneous solutions (II)] by satisfying the traction-free boundary conditions on the surface of the half-space. The Hankel transforms with respect to r are employed for solving this problem. Hence, the solutions for the radial and vertical displacements, and vertical normal and shear stresses in Hankel domain subjected to a point load P_z acting at $z=h$ (measured from the surface, as shown in Fig. 1) in the interior of an inhomogeneous cross-anisotropic half-space are expressed as follows:

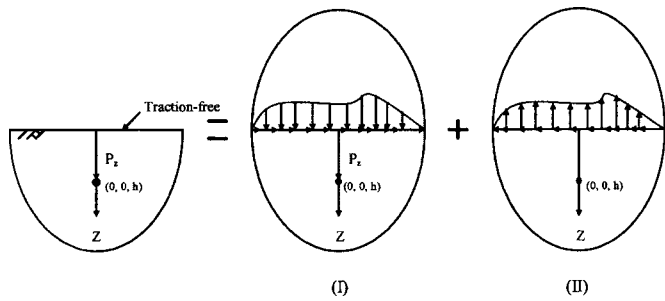


Fig. 1. Superposition approach to the point loading half-space problem

$$U_r^* = \frac{-P_z}{4\pi C_{33}C_{44}} \left[A_1 e^{-u_1 \xi |z-h|} + A_2 e^{(k-u_2 \xi)|z-h|} - A_1 \frac{\Delta_1}{\Delta} e^{-u_1 \xi(z+h)} - A_3 \frac{\Delta_2}{\Delta} e^{-u_1 \xi z} e^{(k-u_2 \xi)h} - A_4 \frac{\Delta_3}{\Delta} e^{(k-u_2 \xi)z} e^{-u_1 \xi h} - A_2 \frac{\Delta_4}{\Delta} e^{(k-u_2 \xi)(z+h)} \right] \quad (1)$$

$$U_z^* = \frac{P_z}{4\pi C_{33}C_{44}} \left[B_1 e^{-u_1 \xi |z-h|} + B_2 e^{(k-u_2 \xi)|z-h|} - B_1 \frac{\Delta_1}{\Delta} e^{-u_1 \xi(z+h)} - B_3 \frac{\Delta_2}{\Delta} e^{-u_1 \xi z} e^{(k-u_2 \xi)h} - B_4 \frac{\Delta_3}{\Delta} e^{(k-u_2 \xi)z} e^{-u_1 \xi h} - B_2 \frac{\Delta_4}{\Delta} e^{(k-u_2 \xi)(z+h)} \right] \quad (2)$$

$$\sigma_{zz}^* = \frac{P_z}{4\pi C_{33}C_{44}} \left[C_1 e^{-u_1 \xi |z-h|} + C_2 e^{(k-u_2 \xi)|z-h|} - C_1 \frac{\Delta_1}{\Delta} e^{-u_1 \xi(z+h)} - C_3 \frac{\Delta_2}{\Delta} e^{-u_1 \xi z} e^{(k-u_2 \xi)h} - C_4 \frac{\Delta_3}{\Delta} e^{(k-u_2 \xi)z} e^{-u_1 \xi h} - C_2 \frac{\Delta_4}{\Delta} e^{(k-u_2 \xi)(z+h)} \right] \quad (3)$$

$$\tau_{rz}^* = \frac{-P_z}{4\pi C_{33}} \left[D_1 e^{-u_1 \xi |z-h|} + D_2 e^{(k-u_2 \xi)|z-h|} + D_1 \frac{\Delta_1}{\Delta} e^{-u_1 \xi(z+h)} + D_3 \frac{\Delta_2}{\Delta} e^{-u_1 \xi z} e^{(k-u_2 \xi)h} + D_4 \frac{\Delta_3}{\Delta} e^{(k-u_2 \xi)z} e^{-u_1 \xi h} + D_2 \frac{\Delta_4}{\Delta} e^{(k-u_2 \xi)(z+h)} \right] \quad (4)$$

where the coefficients A_1 – A_4 , B_1 – B_4 , C_1 – C_4 , D_1 – D_4 , Δ , and Δ_1 – Δ_4 in Eqs. (1)–(4) are listed in the Appendix; whereas C_{ij} ($i, j=1$ – 6) are the elastic stiffness coefficients in a continuously inhomogeneous cross-anisotropic medium, which in a cylindrical co-ordinate system, are given by:

Table 3. Differences between the Homogeneous and Inhomogeneous Cross-Anisotropic Elastic Constants

Homogeneous ^a	Inhomogeneous ^b
E	Ee^{-kz}
E'	$E'e^{-kz}$
ν	ν
ν'	ν'
G'	$G'e^{-kz}$

^aLiao and Wang (1998).

^bWang et al. (2003).

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{11} - 2C_{66} & C_{13} & 0 & 0 & 0 \\ C_{11} - 2C_{66} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \times \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} e^{-kz} \quad (5)$$

where k is referred to as the inhomogeneity parameter. We further notice that C_{ij} can be expressed in terms of the five engineering elastic coefficients E , E' , ν , ν' , and G' as

$$C_{11} = \frac{E \left(1 - \frac{E}{E'} \nu'^2 \right)}{(1 + \nu) \left(1 - \nu - \frac{2E}{E'} \nu'^2 \right)}, \quad C_{13} = \frac{E \nu'}{1 - \nu - \frac{2E}{E'} \nu'^2}$$

$$C_{33} = \frac{E'(1 - \nu)}{1 - \nu - \frac{2E}{E'} \nu'^2}, \quad C_{44} = G', \quad C_{66} = \frac{E}{2(1 + \nu)} \quad (6)$$

where E and E' = Young's moduli in the plane of cross anisotropy and in a direction normal to it, respectively; ν and ν' = Poisson's ratios characterizing the lateral strain response in the plane of cross-anisotropy to a stress acting parallel or normal to it, respectively; and G' = shear modulus in planes normal to the plane of cross anisotropy.

The differences between the homogeneous cross-anisotropic elastic constants (Liao and Wang 1998) and inhomogeneous ones (Wang et al. 2003) adopted in this paper are expressed in Table 3. It is clear that, for the inhomogeneous cross-anisotropic medium described by Eq. (5), only three (E , E' , and G') of five engineering elastic constants exponentially depend on the inhomogeneity parameter k ; the two Poisson's ratios are constants. Besides, depending on the parameter k , we have the following three different situations:

1. $k > 0$, denotes a hardened surface, where E , E' , and G' decrease with increasing depth;
2. $k = 0$, is referred to as the conventional homogeneous condition (Liao and Wang 1998); and
3. $k < 0$, denotes a soft surface, where E , E' , and G' increase with increasing depth.

Finally, in Eqs. (1)–(4), we have defined:

$$u_1 = \sqrt{\frac{S + \sqrt{S^2 - 4Q}}{2}}, \quad u_2 = \sqrt{\frac{S - \sqrt{S^2 - 4Q}}{2}}$$

where

$$S = \frac{C_{11}C_{33} - C_{13}(C_{13} + 2C_{44})}{C_{33}C_{44}}, \quad Q = \frac{C_{11}}{C_{33}}$$

The displacements U_r, U_z , and stresses σ_{zz}, τ_{rz} in the physical domain for the inhomogeneous cross-anisotropic half-space can be obtained by taking the inversion of Hankel theorem for U_r^* [Eq. (1)], U_z^* [Eq. (2)], σ_{zz}^* [Eq. (3)], and τ_{rz}^* [Eq. (4)] with respect to ξ of order 1, 0, 0, and 1, respectively, in the following:

$$\begin{Bmatrix} U_r \\ U_z \\ \sigma_{zz} \\ \tau_{rz} \end{Bmatrix} = \int_0^\infty \xi \begin{Bmatrix} U_r^* J_1(\xi r) \\ U_z^* J_0(\xi r) \\ \sigma_{zz}^* J_0(\xi r) \\ \tau_{rz}^* J_1(\xi r) \end{Bmatrix} d\xi \quad (7)$$

From Eqs. (1)–(4), we found that the integrands under the infinite integrals in Eq. (7) involve products of Bessel function of the first kind of order n ($n=0, 1$), an exponential function, and a polynomial, which cannot be given in closed form so that numerical integrations are required. The numerical methods employed are to perform the integration over each of the first 20 half-cycles of Bessel functions. Additionally, Euler's transformation was applied to this series to speed up the convergence (Longman 1956). The first 20 terms of the Bessel function of the first kind of order n ($n=0, 1$) are quoted from Watson (1966). The method proposed by Longman (1956) can be expressed as follows:

$$\int_0^\infty J_0(x) dx \cong \sum_{n=0}^{20} \int_{x_n}^{x_{n+1}} J_0(x) dx \quad (8)$$

In each division, the 16 points of Gaussian quadrature are adopted, and x can be transferred by

$$x = \frac{(x_{n+1} - x_n)t + x_{n+1} + x_n}{2} \quad (9)$$

Then, revising Eq. (8) with Eq. (9) yields the following expression:

$$\begin{aligned} & \sum_{n=0}^{20} \int_{x_n}^{x_{n+1}} J_0(x) dx \\ & \cong \sum_{n=0}^{20} \frac{(x_{n+1} - x_n)}{2} \left[\sum_{i=1}^{16} w_i J_0 \left(\frac{(x_{n+1} - x_n)t_i + x_{n+1} + x_n}{2} \right) \right] \quad (10) \end{aligned}$$

In order to accelerate the convergence, we sum up the values of the first ten terms, and introduce the Euler's transformation to the other tens as

$$\int_0^\infty J_0(x) dx \cong \sum_{n=0}^{10} \int_{x_n}^{x_{n+1}} J_0(x) dx + \sum_{n=11}^{20} \frac{\Lambda^{n-10}}{2^{n-10}} = 0.99999992 \quad (11)$$

where Λ = first advancing row of differences (Longman 1956), and n is a constant ranging from 0 to 20. Thus, the approximate value (0.99999992) calculated by Eq. (11) is very close to the exact result (is equal to 1.0).

The singularities encountered can be solved by means of the Taylor's theorem expansion as (Davis and Rabinowitz 1984):

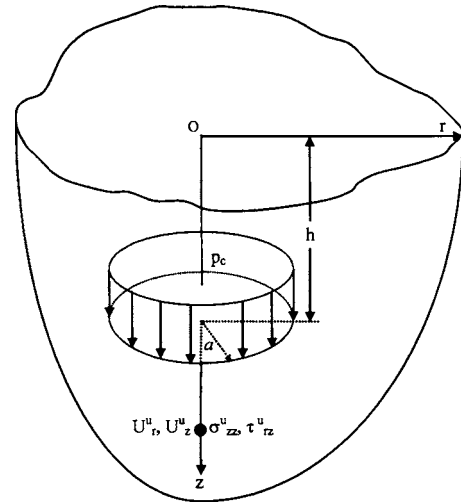


Fig. 2. A uniform vertical circular load p_c applied in the interior of an inhomogeneous cross-anisotropic half-space

$$\begin{aligned} f(x) &= \int_a^b \frac{f(t)}{t-x} dt \\ &= \int_a^b \frac{f(t) - f(x)}{t-x} dt + \int_a^b \frac{f(x)}{t-x} dt \\ &= \int_a^b \frac{f(t) - f(x)}{t-x} dt + f(x) \log \frac{b-x}{x-a} \\ &= \int_a^{x-\varepsilon} \frac{f(t) - f(x)}{t-x} dt + \int_{x-\varepsilon}^b \frac{f(t) - f(x)}{t-x} dt + f(x) \log \frac{b-x}{x-a} \\ &\quad + 2\varepsilon f'(x) + \frac{\varepsilon^3}{9} f'''(x) + \dots \quad (12) \end{aligned}$$

where x = singular point; ε = tiny parameter; and a, b = lower, upper limits, respectively.

Displacements and Stresses along the Axisymmetric Axis in an Inhomogeneous Cross-Anisotropic Half-Space Due to a Uniform Vertical Circular Load

Assume that a vertical pressure p_c (force per unit area) is uniformly distributed over the area of a circle in the interior ($z=h$) of an inhomogeneous cross-anisotropic half-space, which has a rotational symmetry about the center of the circle (Fig. 2). We take the center as the origin of a cylindrical coordinate system as depicted in Fig. 2. Consideration will be given to the displacement and stress fields along the axisymmetric axis (z axis) since these components are the most commonly quoted in axially symmetrical loading problems (Geddes 1975). The response can be obtained by double integration of the point load solution between the correct limits (Barden 1963). That is, we need to integrate Eq. (7) with respect to r from 0 to a (radius of a loaded circle), and with respect to θ from 0 to 2π , to get $U_r^u, U_z^u, \sigma_{zz}^u$, and τ_{rz}^u as given follows:

Table 4. Soil Types and their Elastic Properties

Soil type	E/E'	G'/E'	ν/ν'
Soil 1: isotropy	1	0.385	1
Soil 2: cross anisotropy	1	0.23	1
Soil 3: cross anisotropy	1.35	0.23	1

$$\begin{Bmatrix} U_r^u \\ U_z^u \\ \sigma_{zz}^u \\ \tau_{rz}^u \end{Bmatrix} = \int_0^{2\pi} \int_0^a \begin{Bmatrix} U_r \\ U_z \\ \sigma_{zz} \\ \tau_{rz} \end{Bmatrix} r dr d\theta \quad (13)$$

The values of displacements and stresses in Eq. (13) should be calculated numerically. The techniques include the integration over each of the first 20 half-cycles of Bessel functions. Moreover, the numerical integration technique known as Simpson's rule is also adopted in the analysis. This rule is a formula that gives a numerical approximation to the value of a definite integral. The interval of integration is divided into n subintervals. On each pair of subintervals the function is approximated by a quadratic shape function. The integral of the quadratic shape function over each pair of subintervals is the area under a parabolic arc. Adding those areas with the usual sign convention produces the desired approximation to the integral. It is well known that the approximation converges to the exact value of the integral as n becomes larger. In this paper, $n=10,000$ is selected to make the desired values very close to the exact ones.

The proposed solutions provide mathematical model for applications to the problems in soil/rock mechanics and foundation engineering where the media are of inhomogeneity and cross-anisotropy. In this paper, the most interesting results for vertical displacement (U_z^u), and vertical normal stress (σ_{zz}^u) are presented in the next section.

Illustrative Example

In this section, a parametric study is conducted to examine the effect of inhomogeneity, the type and degree of material anisotropy on the displacement and stress. The displacement and stress frequently of major interest are the vertical components (Geddes 1975), hence, the distributions of vertical displacement (U_z^u) and vertical normal stress (σ_{zz}^u) along the centerline of loaded circle acting on the surface ($h=0$) of inhomogeneous constituted materials are presented. The effect of the inhomogeneity parameter k , and the degree of anisotropy, specified by the ratios E/E' , G'/E' , and ν/ν' on the displacement and stress is studied. For typical ranges of cross-anisotropic parameters, Gazetas (1982) summarized several experimental data regarding deformational cross anisotropy of clays and sands. He concluded that the ratio E/E' for clays ranging from 0.6 to 4, and was as low as 0.2 for sands. However, for the heavily over consolidated London clay, the ratio for E/E' is in the range of 1.35–2.37, and for G'/E' is in 0.23–0.44 (Ward et al. 1965; Gibson 1974; Lee and Rowe 1989; Tarn and Lu 1991). Hence, the degree of anisotropy of London clay, including the ratios E/E' , G'/E' , and ν/ν' , is selected for investigating its effect on the displacement and stress. The elastic properties for several types of isotropic and cross-anisotropic soils are listed in Table 4. The values adopted in Table 4 for E and ν are 50 MPa and 0.3, respectively. The variation of the proposed so-

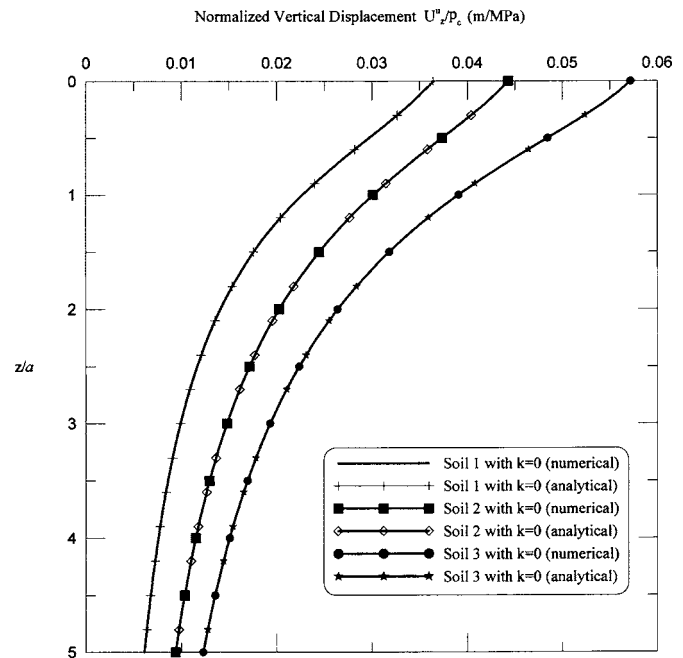


Fig. 3. Comparison of numerical and analytical results for the normalized vertical displacement along the z axis for Soils 1–3

lutions for the inhomogeneity parameter k varies between 0 (homogeneous) to -0.5 ($k < 0$ denotes a soft surface, where E , E' , and G' increase with increasing depth). The reason why the situation with $k > 0$ (i.e., the hardened surface) is not chosen because it corresponds to an underground with decreasing elastic moduli, which might not be the real case for earth structures. Based on Eqs. (1)–(4), (7), and (13), a *FORTRAN* program was written to calculate the displacements and stresses. In this program, the displacement and stress components under the axis of loading in the inhomogeneous cross-anisotropic half-space can be computed. The calculated results by aforementioned numerical techniques are demonstrated in Figs. 3–6.

First, the influence of inhomogeneity, and the type and degree of soil anisotropy on the vertical displacement is studied. Fig. 3 shows the variations of the normalized vertical displacement (U_z^u/p_c) with respect to the nondimensional factor z/a (a =radius of a loaded circle) for Soils 1–3 with $k=0$ (homogeneous case). It is clearly observed that the numerical results are nearly the same as the exact solutions of Hanson and Puja (1996). Figs. 4(a and b) show the effect of inhomogeneity parameter k (from 0 to -0.5) on the normalized vertical displacement for the cross-anisotropic Soil 2, and Soil 3, respectively. These figures reveal that as the degree of inhomogeneity of a soil increases (from $k=0$ to -0.5), the magnitude of the vertical displacement along the z axis decreases for Soil 2 [$E/E'=1$, $G'/E'=0.23$, and $\nu/\nu'=1$, Fig. 4(a)], and Soil 3 ($E/E'=1.35$, $G'/E'=0.23$, and $\nu/\nu'=1$, Fig. 4(b)).

Second, the effect of inhomogeneity, and soil anisotropy on the vertical normal stress is investigated. Fig. 5 depicts the non-dimensional vertical normal stress (σ_{zz}^u/p_c) along the z axis for Soils 1–3 with $k=0$, i.e., the relationship between the stress σ_{zz}^u/p_c and nondimensional factor z/a . Again, our numerical results are in strong agreement with the exact solutions (Hanson and Puja 1996). The effect of inhomogeneity parameter k on the vertical normal stress is plotted in Figs. 6(a and b) for the cross-anisotropic Soil 2, and Soil 3, respectively. It is found that with

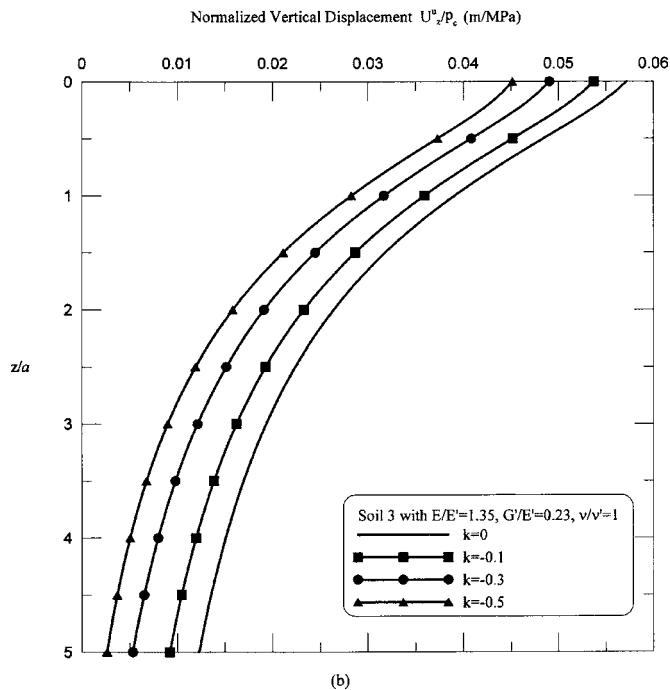
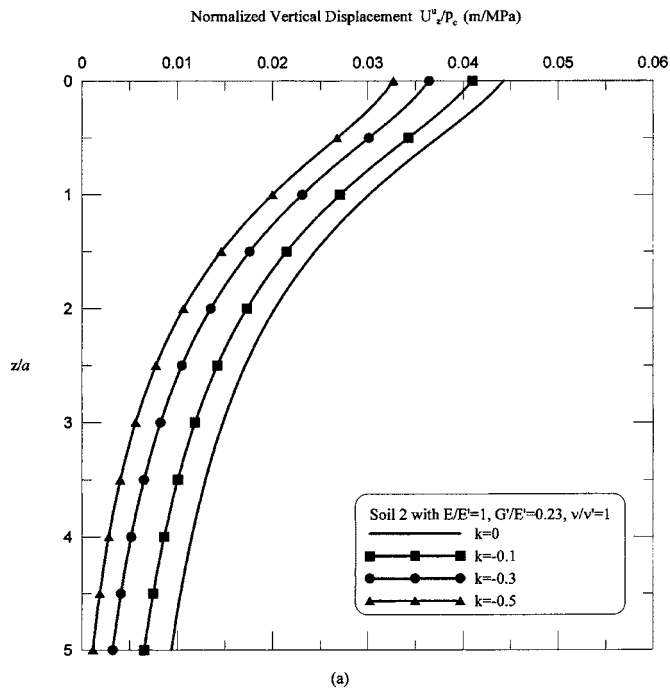


Fig. 4. Effect of inhomogeneity parameter k on the normalized vertical displacement along the z axis for (a) Soil 2 and (b) Soil 3

increasing the degree of inhomogeneity (from $k=0$ to -0.5), the magnitude of vertical normal stress will usually decrease for the hypothetical soils [Soil 2, Fig. 6(a) and Soil 3, Fig. 6(b)]. Notably, the nondimensional vertical normal stress in the regions of $k=-0.5$ for Soil 2 [Fig. 6(a)] might be transferred to tension.

According to Figs. 4(a and b) and Figs. 6(a and b), the magnitudes of vertical displacement and vertical normal stress strongly depend on the inhomogeneity of cross-anisotropic soils. This observation could be of particular interest to the design of foundation engineering.

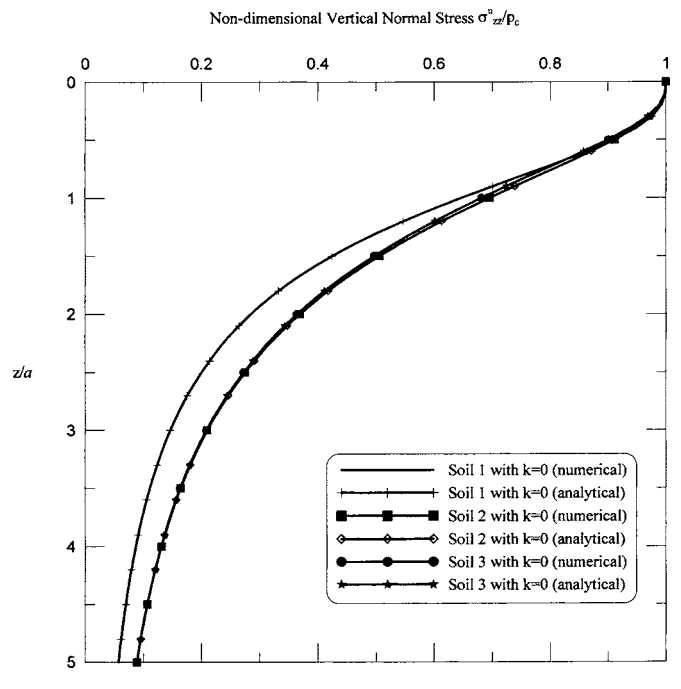


Fig. 5. Comparison of numerical and analytical results for the nondimensional vertical normal stress along the z axis for Soils 1–3

Practical Use of the Provided Figures

The displacements and stresses along the axisymmetric axis of a uniformly loaded circle in a continuously inhomogeneous cross-anisotropic half-space are presented. The planes of cross anisotropy are parallel to the horizontal surface of the half-space, and the Young's and shear moduli are assumed to vary exponentially with depth. The distributions and magnitudes of the normalized vertical displacement (U_z^u/p_c) and nondimensional vertical normal stress (σ_{zz}^u/p_c) under the footing center ($r=0$) in the inhomogeneous cross-anisotropic soils (Soils 2 and 3) with inhomogeneity parameter k (from 0 to -0.5) are presented in Figs. 4 and 6, respectively. From both figures, one can clearly observe the effect of inhomogeneity on the displacement and stress. Perhaps the most interesting feature is that for $k=-0.5$ in Soil 2, a tensile stress can be induced by the proposed applied load. These figures are useful not only theoretically but also practically in the fields of engineering for the determination of vertical displacement and normal stress on the axis of loading due to a uniform vertical circular load in the inhomogeneous cross-anisotropic geomaterials.

Conclusions

In this paper, solutions are presented for the displacements and stresses along the axisymmetric axis due to a uniform vertical circular load in an inhomogeneous cross-anisotropic half-space with its Young's and shear moduli varying exponentially with depth. The planes of cross isotropy are assumed to be parallel to the boundary surface. These solutions are obtained by directly integrating the point load solution developed by the authors. Since the polynomial, exponential function, and Bessel function of the first kind of order n ($n=0, 1$) are involved in the resulting integrals of the point load solution, the displacements and stresses

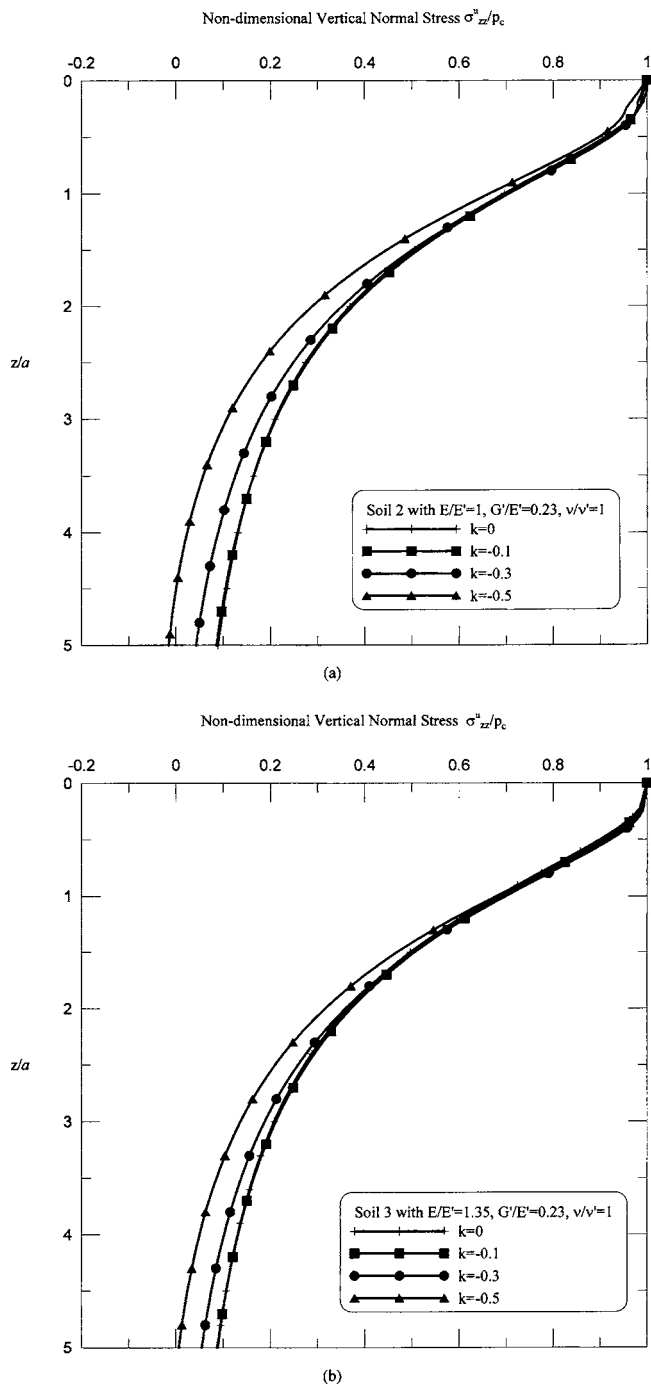


Fig. 6. Effect of inhomogeneity parameter k on the nondimensional vertical normal stress along the z axis for (a) Soil 2 and (b) Soil 3

due to the circular load cannot be obtained in the exact closed form. The numerical techniques we proposed include the integration over each of the first twenty half-cycles of Bessel functions, the Gauss quadrature formula, and the well-known Simpson's rule for solving this circular loading problem. When the inhomogeneity parameter k is equal to zero, our numerical result matches very well with the exact closed-form solution. A parametric study is carried out for London clay, and the following interesting conclusions are summarized:

1. As the degree of inhomogeneity increases (from $k=0$ to -0.5 ; $k < 0$ denotes a soft surface), the magnitude of the ver-

tical displacement along the z -axis decreases the cross-anisotropic soils [Soil 2, Fig. 4(a) and Soil 3, Fig. 4(b)].

2. Also with increasing the degree of inhomogeneity, the magnitude of vertical normal stress will usually decrease for the soils [Soil 2, Fig. 6(a) and Soil 3, Fig. 6(b)].
3. In some regions of the soils [Soil 2, Fig. 6(a)], an increase of inhomogeneity (i.e., $k=-0.5$) may induce a tensile normal stress.
4. In general, the inhomogeneity parameter k and the type and degree of soil anisotropy can all have a great influence on the vertical displacement and vertical normal stress.

The estimation of displacements and stresses subjected to a uniform vertical circular load in an inhomogeneous cross-anisotropic half-space is fast and accurate by the presented solutions. These solutions can more realistically simulate loading problems in soil/rock mechanics and foundation engineering where the stratum is of inhomogeneity and cross-anisotropy. Moreover, they can be extended to solve the displacements and stresses due to conical and parabolic vertical circular loads acting in the interior of a continuously inhomogeneous cross-anisotropic half-space. The interesting results will be presented in forthcoming papers.

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Appendix

The coefficients $A_1-A_4, B_1-B_4, C_1-C_4, D_1-D_4, \Delta$, and $\Delta_1-\Delta_4$ in Eqs. (1)–(4) are presented as follows:

$$A_1 = \frac{[(C_{13} + C_{44})u_1\xi + kC_{44}]}{u_1[(k + u_1\xi)^2 - u_2^2\xi^2]}$$

$$A_2 = \frac{[(C_{13} + C_{44})u_2\xi - kC_{13}]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$A_3 = \frac{[(C_{13} + C_{44})u_1\xi + kC_{44}]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$A_4 = \frac{[(C_{13} + C_{44})u_2\xi - kC_{13}]}{u_1[(k + u_1\xi)^2 - u_2^2\xi^2]}$$

$$B_1 = \frac{[(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]}{u_1[(k + u_1\xi)^2 - u_2^2\xi^2]}$$

$$B_2 = \frac{[(-C_{11} + C_{44}u_2^2)\xi - kC_{44}u_2]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$B_3 = \frac{[(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$B_4 = \frac{[(-C_{11} + C_{44}u_2^2)\xi - kC_{44}u_2]}{u_1[(k + u_1\xi)^2 - u_2^2\xi^2]}$$

$$C_1 = \frac{[(C_{33}C_{44}u_1^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_1\xi^2 + kC_{44}(C_{33}u_1^2 + C_{13})\xi]}{u_1[(k + u_1\xi)^2 - u_2^2\xi^2]}$$

$$C_2 = \frac{[(C_{33}C_{44}u_2^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_2\xi^2 - k(2C_{33}C_{44}u_2^2 + C_{13}^2 - C_{11}C_{33})\xi + k^2C_{33}C_{44}u_2]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$C_3 = \frac{[(C_{33}C_{44}u_1^2 + C_{13}^2 + C_{13}C_{14} - C_{11}C_{33})u_1\xi^2 + k(C_{13}C_{44} + C_{11}C_{33})\xi - k^2C_{33}C_{44}u_1]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$C_4 = \frac{[(C_{33}C_{44}u_2^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_2\xi^2 - k(2C_{33}C_{44}u_2^2 + C_{13}^2 - C_{11}C_{33})\xi + k^2C_{33}C_{44}u_2]}{u_1[(k - u_1\xi)^2 - u_2^2\xi^2]}$$

$$D_1 = \frac{(C_{11} + C_{13}u_1^2)\xi^2}{u_1[(k + u_1\xi)^2 - u_2^2\xi^2]}$$

$$D_2 = \frac{[(C_{11} + C_{13}u_2^2)\xi^2 - 2kC_{13}u_2\xi + k^2C_{13}]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$D_3 = \frac{[(C_{11} + C_{13}u_1^2)\xi^2 - k(C_{13} + C_{44})u_1\xi - k^2C_{13}]}{u_2[(k - u_2\xi)^2 - u_1^2\xi^2]}$$

$$D_4 = \frac{[(C_{11} + C_{13}u_2^2)\xi^2 - 2kC_{13}u_2\xi + k^2C_{13}]}{u_1[(k - u_1\xi)^2 - u_2^2\xi^2]}$$

$$\Delta = \{C_{13}[(C_{13} + C_{44})u_1\xi + kC_{44}] + C_{33}u_1[(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]\}^* \{(k - u_2\xi)[(C_{13} + C_{44})u_2\xi - kC_{13}] + [(-C_{11} + C_{44}u_2^2)\xi^2 - kC_{44}u_2\xi]\}$$

$$+ \{u_1[(C_{13} + C_{44})u_1\xi + kC_{44}] - [(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]\}^* \{C_{13}[(C_{13} + C_{44})u_2\xi^2 - kC_{13}\xi] - C_{33}(k - u_2\xi)[(-C_{11} + C_{44}u_2^2)\xi - kC_{44}u_2]\}$$

$$\Delta_1 = \{C_{13}[(C_{13} + C_{44})u_1\xi + kC_{44}] - C_{33}u_1[(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]\}^* \{(k - u_2\xi)[(C_{13} + C_{44})u_2\xi - kC_{13}] + [(-C_{11} + C_{44}u_2^2)\xi^2 - kC_{44}u_2\xi]\}$$

$$- \{u_1[(C_{13} + C_{44})u_1\xi + kC_{44}] + [(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]\}^* \{C_{13}[(C_{13} + C_{44})u_2\xi^2 - kC_{13}\xi] - C_{33}(k - u_2\xi)[(-C_{11} + C_{44}u_2^2)\xi - kC_{44}u_2]\}$$

$$\Delta_2 = 2(k - u_2\xi)\{C_{13}[(C_{13} + C_{44})u_2\xi - kC_{13}]^2 + C_{33}[(-C_{11} + C_{44}u_2^2)\xi - kC_{44}u_2]^2\}$$

$$\Delta_3 = 2u_1\xi\{C_{13}[(C_{13} + C_{44})u_1\xi + kC_{44}]^2 + C_{33}[(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]^2\}$$

$$\Delta_4 = -\{C_{13}[(C_{13} + C_{44})u_1\xi + kC_{44}] + C_{33}u_1[(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]\}^* \{(k - u_2\xi)[(C_{13} + C_{44})u_2\xi - kC_{13}] - [(-C_{11} + C_{44}u_2^2)\xi^2 - kC_{44}u_2\xi]\}$$

$$+ \{u_1[(C_{13} + C_{44})u_1\xi + kC_{44}] - [(-C_{11} + C_{44}u_1^2)\xi + kC_{44}u_1]\}^* \{C_{13}[(C_{13} + C_{44})u_2\xi^2 - kC_{13}\xi] + C_{33}(k - u_2\xi)[(-C_{11} + C_{44}u_2^2)\xi - kC_{44}u_2]\}$$

Notation

The following symbols are used in this paper:

- $C_{ij}(i, j=1-6)$
= the elastic moduli of the half-space;
 E, E', ν, ν', G' = engineering elastic coefficients of a cross-anisotropic half-space;
 h = buried depth, as seen in Figs. 1 and 2;
 $J_n(\cdot)$ = Bessel function of the first kind of order n ;
 k = the inhomogeneity parameter;
 P_z = a vertical point load (force);
 p_c = a uniform vertical circular load (force per unit area);
 U_r, U_z = displacements induced by a vertical point load;
 U_r^u, U_z^u = displacements induced by a uniform vertical circular load;

- σ_{zz}, τ_{rz} = stresses induced by a vertical point load; and
 $\sigma_{zz}^u, \tau_{rz}^u$ = stresses induced by a uniform vertical circular load.

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