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FREE VIBRATION OF FUNCTIONALLY GRADED, MAGNETO-ELECTRO-ELASTIC, AND MULTILAYERED PLATES ⋆

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ABSTRACT The state-space method is employed to evaluate the modal parameters of functionally graded, magneto-electro-elastic, and multilayered plates. Based on the assumption that the properties of the functionally graded material are exponential, the state equation of structural vibration which takes the displacement and stress of the structure as state variables is derived. The natural frequencies and modal shapes are calculated based on the general solutions of the state equation and boundary conditions given in this paper. The influence of the functionally graded exponential factor on the elastic displacement, electric, and magnetic fields of the structure are discussed by assuming a sandwich plate model with different stacking sequences.

KEY WORDS functionally graded material, magneto-electro-elastic plates, modal shapes, state-space method

I. INTRODUCTION

The magneto-electro-elastic and multilayered plate is becoming an important component in recent smart structures. The structure made of magneto-electro-elastic materials has the ability of converting one type of energy into another (among magnetic, electric, and mechanical fields). Owing to the three-phase coupling, the mechanical behavior of the magneto-electro-elastic plate is more complicated than that of a purely elastic, single-phase piezoelectric, or piezomagnetic plate[1,2]. However, the magneto-electro-dependent mechanical behavior could be very beneficial to the design of sensors and actuators. Therefore, it is necessary to investigate carefully the static and dynamic characteristics of the magneto-electro-elastic multilayered plates[3−−6], so that they can be applied correctly in design. Furthermore, the novel functionally graded material (FGM) magneto-electro-elastic structures are also important in the field of solid mechanics[7], and thus the magneto-electro-elastic multilayered structures with FGM are proposed to promote further development on the theories and applications of magneto-electro-elastic structures.

In order to study the static and dynamic characteristics of multilayered plates, many methods, including the state-space method[5,8], pseudo-Stroh formalism method[4,6], and finite element method[9], have been proposed in the literature. Because the state-space method has the advantage that the order of the global matrix does not depend on the number of layers, thus simplifying the formulation and solution

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procedure substantially, this method has been employed to analyze the purely elastic, piezoelectric or magneto-electro-elastic multilayered structure[5,8]. In this paper, the state-space method is extended to solve the modal parameters of the magneto-electro-elastic and multilayered plate where the material properties in each layer vary exponentially in the thickness direction. The influence of the exponential factor on the elastic displacement, electric and magnetic fields are also discussed.

II. FORMULATION

Consider an orthotropic, magneto-electro-elastic, N-layered rectangular plate with horizontal dimensions L_x and L_y, and a total thickness H (in the vertical z-direction). A Cartesian coordinate system is attached to the plate and its origin is placed at one of four corners on the bottom surface, with the plate in the positive z-axis region. The layer j is bonded by the upper interface z_j and the lower interface z_{j-1} with thickness $h_j = z_j - z_{j-1}$. In order to simplify the problem, the following assumptions are made: 1) The general out-of-plane displacements and stresses (to be defined later) are continuous along the interface. 2) The traction, the vertical components of the electric displacement and magnetic induction on the top and bottom surfaces of the plate satisfy the given boundary conditions. 3) The four edges of the plate are simply supported.

For a linear, orthotropic, and magneto-electro-elastic solid considered in this article, the coupled constitutive relation can be expressed as[6]

$$\begin{aligned} \sigma_i &= c_{ik}\gamma_k - e_{ki}E_k - q_{ki}H_k \\ D_i &= e_{ik}\gamma_k + \varepsilon_{ik}E_k + d_{ik}H_k \\ B_i &= q_{ik}\gamma_k + d_{ik}E_k + \mu_{ik}H_k \end{aligned} \tag{1}$$

where σ_i, D_i, and B_i are the stress, electric displacement, and magnetic induction, respectively; γ_k, E_k, and H_k are the strain, electric field, and magnetic field, respectively; c_{ik}, e_{ik}, and q_{ik} are the elastic, dielectric, and magnetic permeability coefficients, respectively; ε_{ik}, d_{ik}, and μ_{ik} are the piezoelectric, piezomagnetic, and magneto-electric coefficients respectively.

For a functionally graded material with exponential variation in the z-direction, the material coefficients in Eq.(1) can be described by

$$\begin{aligned} &c_{ik}(z) = c^0_{ik}e^{\xi z}, \quad e_{ik}(z) = e^0_{ik}e^{\xi z}, \quad \varepsilon_{ik}(z) = \varepsilon^0_{ik}e^{\xi z} \\ &q_{ik}(z) = q^0_{ik}e^{\xi z}, \quad \mu_{ik}(z) = \mu^0_{ik}e^{\xi z}, \quad d_{ik}(z) = d^0_{ik}e^{\xi z} \end{aligned} \tag{2}$$

where ξ is the exponential factor characterizing the degree of the material gradient in the z-direction, and the superscript '0' is attached to indicate the z-independent factors in the material coefficients.

The generalized relation between the strains and displacements can be given as

$$\gamma_{ij} = 0.5\left(u_{i,j} + u_{j,i}\right), \quad E_i = -\varphi_{,i}, \quad H_i = -\psi_{,i} \tag{3}$$

where u_i ($u_1 = u$, $u_2 = v$, $u_3 = w$) is the elastic displacement, and φ and ψ are the electric and magnetic potentials.

If the body force, electric charge density, and magnetic charge density are ignored, the equations of motion are written as

$$\sigma_{ij,j} = \rho\frac{\partial^2 u_i}{\partial t^2}, \quad D_{j,j} = 0, \quad B_{j,j} = 0 \tag{4}$$

with ρ as the density of the material and t is time. The variables in Eqs.(1) and (4) may be divided into two groups: One group is the out-of-plane and the other the in-plane. The former is actually the minimum set of variables (so-called state variables) required to describe the movement of the media exactly. The state variables in a vector form can be expressed as

$$\eta = \left\{ u \ v \ D_z \ B_z \ \sigma_z \ \tau_{zx} \ \tau_{zy} \ \varphi \ \psi \ w \right\}^{\mathrm{T}} \tag{5}$$

where the subscripts x, y, and z correspond to 1, 2, and 3 in Eqs.(1)~(4). For a simply supported and layered plate, the boundary conditions of the state vector along the four edges of the plate can be written as follows:

$$\boldsymbol{\eta} = \mathbf{0} \quad \text{at } x = 0 \text{ and } x = L_x,\ y = 0 \text{ and } y = L_y \tag{6}$$

By considering these boundary conditions and also assuming a harmonic vibration of the plate, the general solution of the state vector can be expressed as

$$\boldsymbol{\eta}(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\begin{array}{c}\tilde{u}_{mn}(z)\cos(px)\sin(qy)\\ \tilde{v}_{mn}(z)\sin(px)\cos(qy)\\ \tilde{D}_{zmn}(z)\sin(px)\sin(qy)\\ \tilde{B}_{zmn}(z)\sin(px)\sin(qy)\\ \tilde{\sigma}_{zmn}(z)\sin(px)\sin(qy)\\ \tilde{\tau}_{zxmn}(z)\cos(px)\sin(qy)\\ \tilde{\tau}_{zymn}(z)\sin(px)\cos(qy)\\ \tilde{\varphi}_{mn}(z)\sin(px)\sin(qy)\\ \tilde{\psi}_{mn}(z)\sin(px)\sin(qy)\\ \tilde{w}_{mn}(z)\sin(px)\sin(qy)\end{array}\right\}\mathrm{e}^{\mathrm{i}\omega t} \tag{7}$$

where ω is the angular frequency; $\mathrm{i}=\sqrt{-1}$; p and q are the wave numbers along the x- and y-axes, respectively, i.e.

$$p=m\pi/L_x,\quad q=n\pi/L_y \tag{8}$$

where m and n are two positive integers. Combining Eqs.(1), (3), (4) and (7) yields the following state equation

$$\frac{\mathrm{d}\tilde{\boldsymbol{\eta}}_{mn}(z)}{dz}=\tilde{\boldsymbol{A}}(z)\tilde{\boldsymbol{\eta}}_{mn}(z) \tag{9}$$

where $\tilde{\boldsymbol{\eta}}_{mn}(z)=\left\{\tilde{u}_{mn}\,\tilde{v}_{mn}\,\tilde{D}_{zmn}\,\tilde{B}_{zmn}\,\tilde{\sigma}_{zmn}\,\tilde{\tau}_{zxmn}\,\tilde{\tau}_{zymn}\,\tilde{\varphi}_{mn}\,\tilde{\psi}_{mn}\,\tilde{w}_{mn}\right\}^{\mathrm{T}}$, $\tilde{\boldsymbol{A}}(z)=\begin{bmatrix}0 & \tilde{\boldsymbol{A}}_1(z)\\ \tilde{\boldsymbol{A}}_2(z) & 0\end{bmatrix}$. All elements of the sub-matrix $\tilde{\boldsymbol{A}}_1(z)$ can be found in Ref.[5], except for $-\rho\omega^2$ which replaces 0 as element in the fifth row and the fifth column. And the sub-matrix $\tilde{\boldsymbol{A}}_2(z)$ is written as

$$\tilde{\boldsymbol{A}}_2(z)=\begin{bmatrix}\alpha_{11}p^2+c_{66}q^2-\rho\omega^2 & (\alpha_{12}+c_{66})pq & -\nu_{21}p & -\nu_{31}p & -\nu_{11}p\\ (\alpha_{21}+c_{66})pq & \alpha_{22}q^2+c_{66}p^2-\rho\omega^2 & -\nu_{22}q & -\nu_{32}q & -\nu_{12}q\\ \nu_{21}p & \nu_{22}q & \zeta_{22} & \zeta_{32} & \zeta_{12}\\ \nu_{31}p & \nu_{32}q & \zeta_{23} & \zeta_{33} & \zeta_{13}\\ \nu_{11}p & \nu_{12}q & \xi_{21} & \zeta_{31} & \zeta_{11}\end{bmatrix} \tag{10}$$

where $\alpha_{ij}=c_{ij}-c_{i3}\nu_{1j}-e_{3i}\nu_{2j}-q_{3i}\nu_{3j}$ $(i=1,2;\ j=1,2)$, $\nu_{ij}=\zeta_{i1}c_{j3}+\zeta_{i2}e_{3j}+\zeta_{i3}q_{3j}$ $(i=1,2,3;\ j=1,2)$, $\zeta_{ij}=\upsilon_{ji}/\det\boldsymbol{\kappa}\,(i=1,2,3;\ j=1,2,3)$ with $\boldsymbol{\kappa}=\begin{bmatrix}c_{33} & e_{33} & q_{33}\\ e_{33} & -\varepsilon_{33} & -d_{33}\\ q_{33} & -d_{33} & -\mu_{33}\end{bmatrix}$ and υ_{ij} as the corresponding algebraic cofactors of $\boldsymbol{\kappa}$.

By solving Eq.(9), the state vector at an arbitrary z-level of a given layer can be expressed as

$$\tilde{\boldsymbol{\eta}}_{mn}(z)=\boldsymbol{P}(z)\tilde{\boldsymbol{\eta}}_{mn}(0) \tag{11}$$

where $\boldsymbol{P}(z)$ is the propagator matrix which can be obtained by dividing the layer into many small sub-layers or employing the series expansion method for solving the state equations. From Eq.(11), the relation of the state vectors at the upper and lower interfaces of the j-th layer can be obtained using

$$\tilde{\boldsymbol{\eta}}_{mn}(z_j)=\boldsymbol{P}_j(h_j)\tilde{\boldsymbol{\eta}}_{mn}(z_{j-1}) \tag{12}$$

The propagating relation (12) can be used repeatedly so that we can propagate the physical quantities from the bottom surface $z=0$ to the top surface $z=H$ of the layered plate. Consequently, we have

$$\tilde{\boldsymbol{\eta}}_{mn}(H)=\boldsymbol{T}\tilde{\boldsymbol{\eta}}_{mn}(0) \tag{13}$$

where $\boldsymbol{T}$ is the multiplication of all the propagator matrices, i.e.

$$\boldsymbol{T}=\boldsymbol{P}_N(h_N)\boldsymbol{P}_{N-1}(h_{N-1})\cdots\boldsymbol{P}_2(h_2)\boldsymbol{P}_1(h_1) \tag{14}$$

Separating the variables of Eq.(13) into the general displacements and stresses can lead to a new equation

$$\begin{bmatrix} \tilde{\boldsymbol{S}}_{mn}(H) \\ \tilde{\boldsymbol{F}}_{mn}(H) \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{S}}_{mn}(0) \\ \tilde{\boldsymbol{F}}_{mn}(0) \end{bmatrix} \tag{15}$$

If we assume that the traction, the electric displacement, and the magnetic induction in z-direction are zero on the top and bottom surfaces (free vibration), the dispersion relation then corresponds to the simple condition so that the determination of the sub-matrix $\boldsymbol{T}_{21}$ vanishes. That is,

$$\det(\boldsymbol{T}_{21}) = 0 \tag{16}$$

If the wave numbers along x and y axes are given, the natural frequencies of the layered FGM plate can be calculated. The general modal shapes can also be obtained conveniently by substituting the natural frequencies into Eq.(15).

III. NUMERICAL ANALYSES

We now apply our solution to study the natural frequencies and modal shapes in a three layered, functionally graded, and magneto-electro-elastic plate which is composed of the piezoelectric material $BaTiO_3$ and magnetostrictive material $CoFe_2O_4$. We assume that the length and width of the plate are both equal to 1 m and the thickness of each layer is equal to 0.1 m with a total thickness of 0.3 m.

In order to verify the solution given in this paper, we use the same structure as the one given by Pan and Heyliger[6]. The property coefficients of the material with superscript 0 are identical to those in Tables 4 and 5 in Refs.[4,6]. To consider the effect of FGMs, we assume that the middle layer is homogeneous, and the top and bottom layers are functionally graded with a symmetric exponential variation shown in Fig.1 in which five different exponential factors, $\xi = -10, -5, 0, 5, 10$, are given.

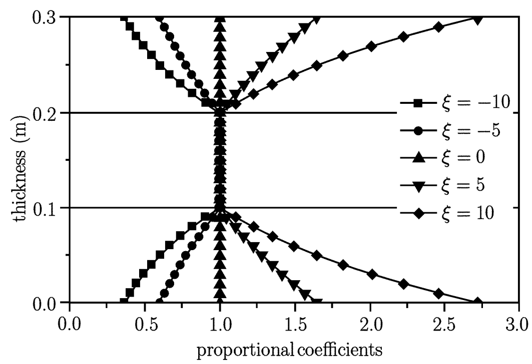

Fig. 1 Variation of FGM proportional coefficients.

Two different stacking sequences are also considered. The first one is B/F/B and the second is F/B/F. Here B represents $BaTiO_3$ and F represents $CoFe_2O_4$[4,6]. The m and n in Eq.(8) are equal to 1.

Table 1. **Dimensionless natural frequencies of FGM plates**

stacking sequence	Mode	ξ = −10	−5	0	5	10
B/F/B	1	0.7519	0.8440	0.9516	1.0747	1.2118
	2	1.6041	1.7035	1.8292	1.9887	2.1912
	3	2.8132	2.9822	3.1882	3.4410	3.7493
	4	4.0781	4.2849	4.4866	4.6928	4.9159
	5	4.8268	5.0877	5.3621	5.6716	6.041
F/B/F	1	0.8219	0.9212	1.0360	1.1662	1.3104
	2	1.6307	1.7516	1.9031	2.0931	2.3313
	3	2.8504	3.0521	3.2931	3.5814	3.9233
	4	4.1270	4.3370	4.5464	4.7667	5.0119
	5	4.9226	5.2786	5.6535	6.0655	6.5373

Table 1 lists the dimensionless natural frequencies of the first five modes for these two stacking sequences. The frequencies are normalized by $\Omega = \omega L_x/\sqrt{c_{\max}/\rho_{\max}}$ with $c_{\max}$ as the maximum of all elastic coefficients in the whole plate and $\rho_{\max}$ as the maximum density. Comparing the results in Table 1 with the results given by Pan and Heyliger[6], it can be seen that the present formulation predict the same nature frequencies for non-FGM (i.e., with exponential coefficient $\xi = 0$) as in Pan and Heyliger[6]. It also can be seen from Table 1 that the natural frequencies increase with increasing exponential factor ξ.

Modal shapes on the third mode for the B/F/B plate are shown in Fig.2. Because of the symmetry, the elastic displacement modal shapes in the x- and y-directions are the same (i.e., $u = v$); thus only the elastic displacement modal shapes in the x-direction are presented. Besides the modal shapes for u, the elastic displacement modal shapes in the z-direction and the modal shapes for the electric and magnetic potentials are also presented in Fig.2.

In Figs.2-4, the elastic displacement modal shapes are normalized by the maximum value in the whole thickness region of the three components, and the electric and magnetic potential modal shapes are normalized by their corresponding maximum value if the potentials are not equal to zero. In Figs.2(a)-(d), it is observed that for these symmetric modal shapes, the horizontal elastic displacement ($u = v$) and electric potential are relatively more sensitive to the exponential factor ξ than the other components. The enhanced effect of the exponential factor on the electric field (Fig.2(c)) is possibly due to the fact that the top and bottom layers in Figs.2 are the piezoelectric materials, and actually a similar trend can be observed (Fig.3(d)) for the magnetic field in the F/B/F stacking sequence to be discussed below.

Figures 3(a)-(d) show the modal shapes on the third mode for the corresponding F/B/F stacking sequence. The modal shapes are symmetric, as those for the B/F/B stacking sequence (Fig.2). However, while the elastic displacement modal shapes in F/B/F are nearly identical to those in B/F/B (Figs.2(a), (b) *vs.* Figs.3(a), (b), the features of the electric and magnetic modal shapes are switched (great influence of the exponential factor on the electric potential in Fig.2(c) for B/F/B and on the magnetic potential in Fig.3(d) for F/B/F).

Figures 4(a)-(b) show the anti-symmetric modal shapes in the B/F/B plate, which correspond to the fourth mode. It is observed that only the horizontal elastic displacement modal shapes vary slightly with

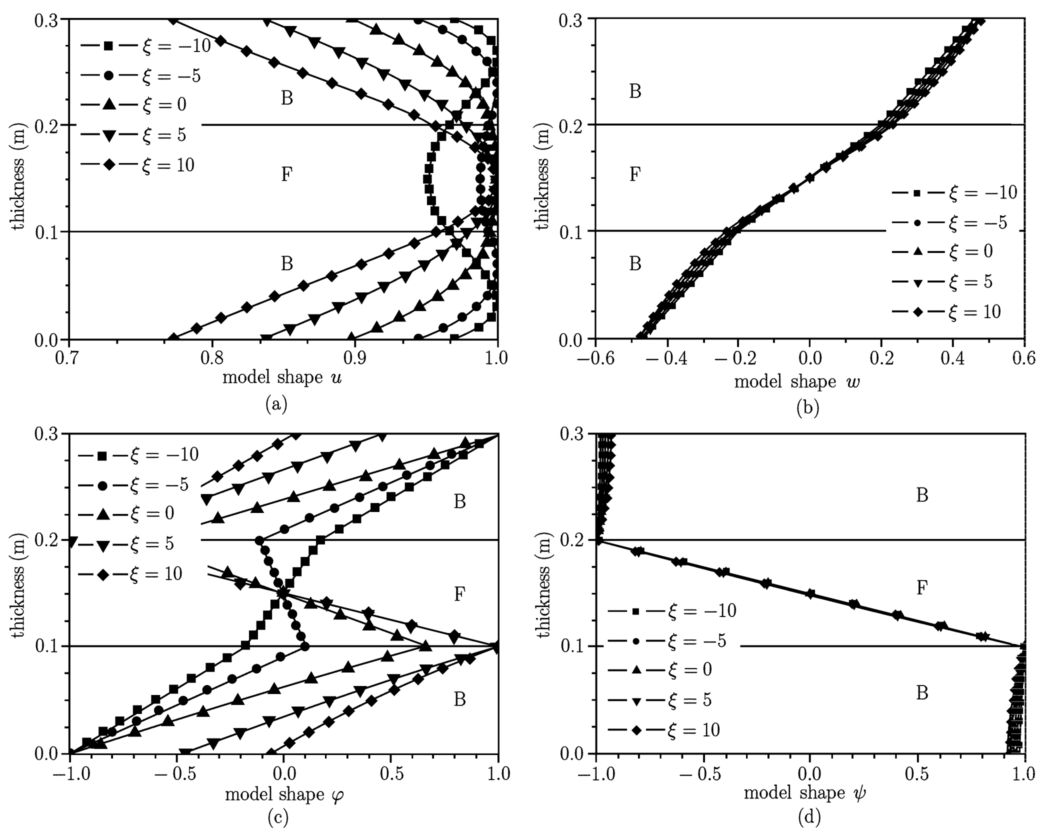

Fig. 2. Modal shapes on the 3rd mode of B/F/B.

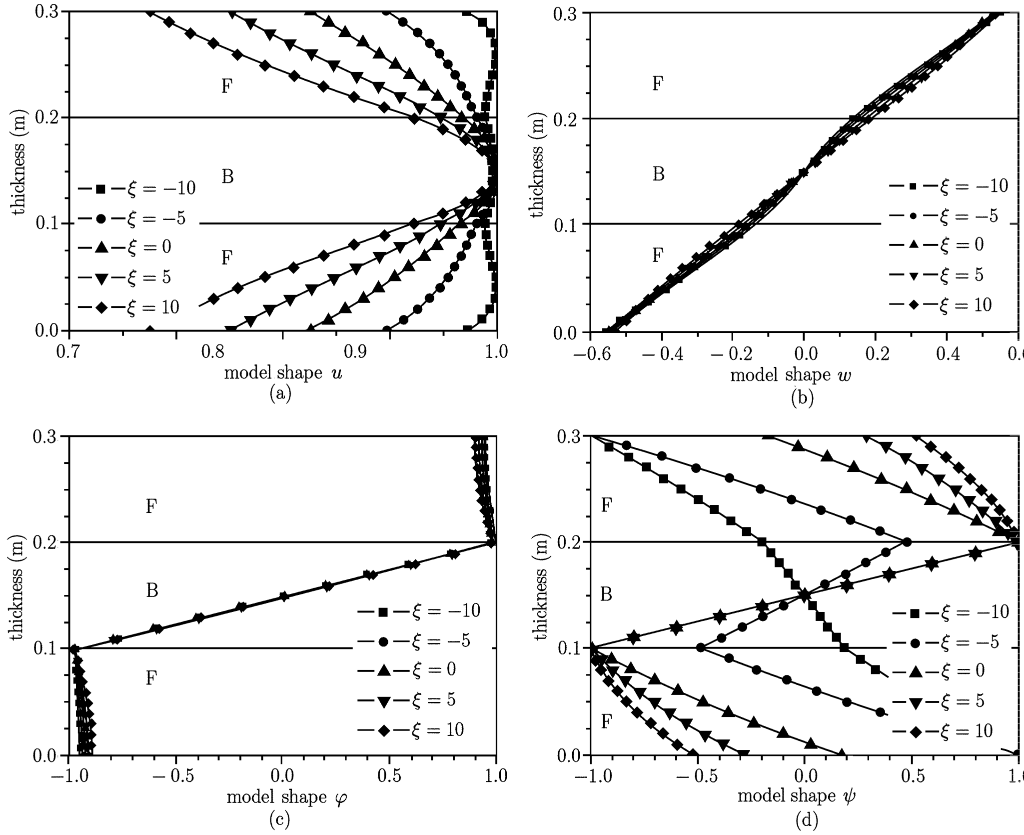

Fig. 3. Modal shapes on the 3^{rd} mode of F/B/F.

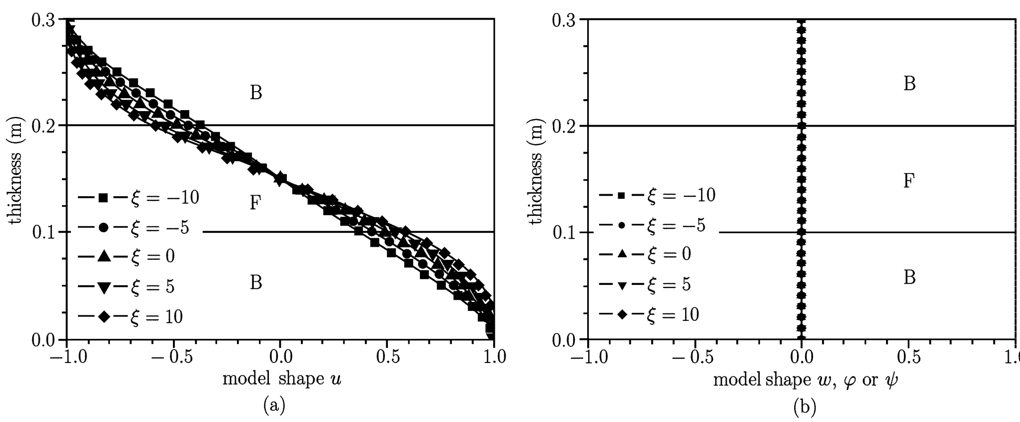

Fig. 4. Modal shapes on the 4^{th} mode of B/F/B.

the exponential factors; the other three physical quantities including w are identically zero. Furthermore, this mode is a purely elastic one, just as that in Pan and Heyliger[6].

IV. CONCLUSIONS

The state-space method is proposed to calculate the modal parameters of the FGM magneto-electro-elastic multilayered plates. In terms of formulation and calculation, the method proposed is concise and simple. The modal parameters are obtained and discussed for the B/F/B and F/B/F plates made of functionally graded, magneto-electro-elastic, and multilayered materials, with simply supported edges. For the special case where the functional exponential factor $\xi = 0$, the numerical results presented in this paper agree with those in Pan and Heyliger[6]. In addition, our numerical results show that the influence of the exponential factor on the modal parameters can be different on different modes: The modal parameters corresponding to the magneto-electric coupling mode are more sensitive to the exponential factor as compared to those corresponding to the pure elastic mode. On the coupling mode, the sensitivity of the electric and magnetic fields to the exponential factors varies with varying stacking sequences.

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