

# Vertical Vibration of a Flexible Plate with Rigid Core on Saturated Ground

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**Abstract:** In this paper, the vertical vibration of a flexible plate with rigid core resting on a semi-infinite saturated soil is studied analytically. The behavior of the soil is assumed to follow Biot's poroelastodynamic theory with compressible soil skeleton and pore water, and the response of the time-harmonic excited plate is governed by the classical thin-plate theory. By virtue of the Hankel transform technique, the fundamental solutions of the skeleton displacements, stresses, and pore pressure are derived, and a set of dual integral equations associated with the relaxed boundary and completely drained condition at the soil-foundation contact interface are also developed. These governing integral equations are further reduced to the standard Fredholm integral equations of the second kind and solved by numerical procedures. Comparison with existing solutions for a rigid permeable plate on saturated soil confirms the accuracy of the present solution. Selected numerical results are presented to show the influence of the permeability, the size of the rigid core, and the plate flexibility on the dynamic interaction between the elastic plate with rigid core and the underlying saturated soil.

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## Introduction

Dynamic response of a foundation on the soil is of fundamental importance in the field of soil-structure interaction, geomechanics, and foundation vibration. Since Reissner's pioneer work on the forced vertical translation of a rigid circular plate attached to an elastic half-space (Reissner 1936), the dynamic mixed boundary value problem has been a subject of extensive study. For example, Bycroft (1956), Awojobi and Grootenhuis (1965), Robertson (1966), and Luco and Westmann (1971) investigated various vibration responses of a rigid footing on an elastic half-space. Pak and Gobert (1991) studied vertical vibrations of an arbitrarily embedded rigid plate. The vibration of a flexible plate was also considered by a number of researchers. Lin (1978) presented an integral equation approach for the case of a flexible circular plate with a rigid perimeter. Iguchi and Luco (1982) treated the vibration problem of a flexible circular plate with a rigid core on a layered viscoelastic medium. Although Rajapakse (1989) presented a solution to the dynamic response of an elastic annular thin plate resting on a viscoelastic half-space, Mukherjee (2001) recently solved the forced vertical vibration problem of a flexible elliptical plate on an elastic half-space.

Soils in general are two-phase materials consisting of a solid skeleton with voids filled with water and thus should be more realistically regarded as poroelastic materials. The theory of wave propagation in a fluid-saturated porous medium was established by Biot (1956a,b, 1962) and has since been widely used in many engineering applications. Current development in this field can be found in Detournay and Cheng (1993), Cheng et al. (1998), and Chen et al. (2005). Based on Biot's poroelastic theory, various problems related to the foundation vibration on homogeneous poroelastic half-spaces have been solved in the last two decades. The first noteworthy publications were by Halpern and Christiano (1986a,b), who analyzed the time-harmonic responses of rigid permeable and impermeable plates in smooth contact with a saturated poroelastic half-space. Consequently, Kassir and Xu (1988) and Kassir et al. (1989) examined the vibration of rigid rectangular strip and circular foundations on a poroelastic half-space. Philippacopoulos (1989) investigated a similar problem by considering the supporting medium as a partially saturated poroelastic half-space. A conceptually similar approach was followed by Jin and Liu (1999) for the vertical vibration of a circular plate on a poroelastic half-space. Zeng and Rajapakse (1999) extended the problem to include the influence of the embedded depth on the vibration of a vertically loaded rigid plate. On the other hand, Bougachia et al. (1993) and Senjuntichai and Rajapakse (1996) obtained the dynamic solutions for a rigid footing on a multilayered poroelastic medium using the finite element method and the dynamic Green's function approach, respectively. Senjuntichai and Sapsathiam (2003) even solved the forced vertical vibration of a circular plate embedded in a multilayered poroelastic medium. It is worth mentioning that, although many problems involving foundation vibrations on poroelastic media have been studied, the dynamic responses of a circular plate with a rigid core supported by the saturated soil have not yet been reported in the literature. Such solutions are important in geomechanics in general and are necessary in evaluating the influences of the foundation flexibility on the dynamic response in particular.

The objective of this paper is therefore to present an analytical

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solution to the vertical vibration of a flexible plate with rigid core on saturated ground. We start with the general field equations in cylindrical coordinates under the framework of Biot's poroelasto-dynamic theory, with the compressibility of both the soil skeleton and the pore water being taken into account. General solutions for the displacement and stress components of the saturated soil are obtained by using Hankel integral transform technique. These general solutions, in combination with the boundary conditions, then lead to a set of dual integral equations which correspond to the mixed boundary value problem for the vibration of the flexible plate. The dual integral equations can be further reduced to a Fredholm integral equation of the second kind and be finally solved using the standard numerical procedures. The dynamic impedance functions are finally derived. As numerical examples, the influences of the permeability, the size of the rigid core, and the plate flexibility on the dynamic impedance function are presented and analyzed.

## Governing Equations

Consider a massless flexible circular plate of radius  $r_0$  with a rigid core of radius  $r_b$  resting on the surface of a semi-infinite saturated soil, with the cylindrical coordinates being used (Fig. 1). The flexible rigidity of the plate is denoted by  $D$  and Poisson's ratio by  $\nu_f$ . A harmonic vertical force  $Pe^{i\omega t}$  with a circular frequency  $\omega$  is applied at the center of the massless plate.

Assuming that the constitutive behavior of the soil follows Biot's two-phase linear theory, then due to the axis-symmetry of the problem, the governing differential equations for the saturated soil in the cylindrical coordinates  $(r, \theta, z)$ , in terms of displacements, can be written as (Biot 1962; Halpern and Christiano 1986a,b; Senjuntichai and Rajapakse 1996)

$$G\left(\nabla^2 u_r - \frac{1}{r^2} u_r\right) + (\lambda + \alpha^2 M + G) \frac{\partial e}{\partial r} + \alpha M \frac{\partial \varepsilon}{\partial r} = \rho \ddot{u}_r + \rho_w \ddot{v}_r \quad (1a)$$

$$G\nabla^2 u_z + (\lambda + \alpha^2 M + G) \frac{\partial e}{\partial z} + \alpha M \frac{\partial \varepsilon}{\partial z} = \rho \ddot{u}_z + \rho_w \ddot{v}_z \quad (1b)$$

$$\alpha M \frac{\partial e}{\partial r} + M \frac{\partial \varepsilon}{\partial r} = \rho_w \ddot{u}_r + m \ddot{v}_r + b \dot{v}_r \quad (1c)$$

$$\alpha M \frac{\partial e}{\partial z} + M \frac{\partial \varepsilon}{\partial z} = \rho_w \ddot{u}_z + m \ddot{v}_z + b \dot{v}_z \quad (1d)$$

where  $u_r$  and  $u_z$ =radial and vertical displacements of the solid matrix, respectively;  $v_r$  and  $v_z$ =average fluid displacements relative to the solid matrix in the  $r$  and  $z$  directions, respectively;  $e$  and  $\varepsilon$ =matrix dilation and the fluid dilation relative to the solid, respectively, which are expressed as  $e = \partial u_r / \partial r + u_r / r + \partial u_z / \partial z$ ,  $\varepsilon = \partial v_r / \partial r + v_r / r + \partial v_z / \partial z$ ;  $\lambda$ ,  $G$ =Lame's constants of the solid matrix;  $\alpha$ ,  $M$ =Biot's compressibility parameters of the soil skeleton and water, respectively;  $\rho_w$ =mass density of the water;  $\rho$ =mass density of the bulk material [ $\rho = n\rho_w + (1-n)\rho_s$ ,  $n$ =porosity and  $\rho_s$ =mass density of grains];  $m$ =density-like parameter that depends on  $\rho_w$  and the geometry of the pores;  $b$ =parameter accounting for the internal friction due to the relative motion between the solid matrix and pore water, and is equal to the ratio between the fluid viscosity and the intrinsic permeability of the medium; and

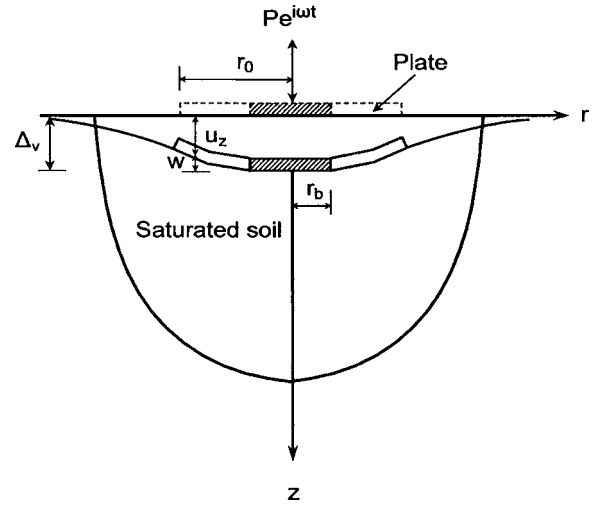


Fig. 1. Excited circular plate with a rigid core bearing on saturated soils

$\nabla^2$  denotes the (symmetric) Laplacian operator which is given by  $\nabla^2 = \partial^2 / \partial r^2 + (1/r) \partial / \partial r + \partial^2 / \partial z^2$ .

The constitutive relations for the  $z$ -direction traction components and pore pressure in the saturated soil can be expressed as

$$\sigma_z = 2G \frac{\partial u_z}{\partial z} + \lambda e \quad (2a)$$

$$\tau_{zr} = G \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \quad (2b)$$

$$\sigma_f = -\alpha M e - M \varepsilon \quad (2c)$$

where  $\sigma_z$ =effective normal stress component in the vertical ( $z$ ) direction;  $\tau_{zr}$ =shear stress; and  $\sigma_f$  denotes the excess pore pressure.

## Solutions of the Governing Equations

The motion under consideration is assumed to be time-harmonic proportional to  $e^{i\omega t}$ . With the introduction of the following dimensionless variables  $\bar{r} = r/r_0$ ,  $\bar{z} = z/r_0$ ,  $\bar{u}_r = u_r/r_0$ ,  $\bar{u}_z = u_z/r_0$ ,  $\bar{v}_r = v_r/r_0$ ,  $\bar{v}_z = v_z/r_0$ ,  $\bar{\rho}_w = \rho_w/\rho$ ,  $\bar{m} = m/\rho$ ,  $\bar{\lambda} = \lambda/G$ ,  $\bar{M} = M/G$ ,  $\bar{\sigma}_z = \sigma_z/G$ ,  $\bar{\tau}_{zr} = \tau_{zr}/G$ ,  $\bar{\sigma}_f = \sigma_f/G$ ,  $a_0 = (\sqrt{\rho/G})\omega r_0$ ,  $\bar{t} = (\sqrt{G/\rho})(t/r_0)$ , and  $\bar{b} = r_0 b / \sqrt{\rho G}$ , Eqs. (1a)–(1d) and (2a)–(2c) are reduced to

$$\nabla^2 \bar{u}_r - \frac{1}{\bar{r}^2} \bar{u}_r + (\bar{\lambda} + \alpha^2 \bar{M} + 1) \frac{\partial e}{\partial \bar{r}} + \alpha \bar{M} \frac{\partial \varepsilon}{\partial \bar{r}} = -a_0^2 \bar{u}_r - \bar{\rho}_w a_0^2 \bar{v}_r \quad (3a)$$

$$\nabla^2 \bar{u}_z + (\bar{\lambda} + \alpha^2 \bar{M} + 1) \frac{\partial e}{\partial \bar{z}} + \alpha \bar{M} \frac{\partial \varepsilon}{\partial \bar{z}} = -a_0^2 \bar{u}_z - \bar{\rho}_w a_0^2 \bar{v}_z \quad (3b)$$

$$\alpha \bar{M} \frac{\partial e}{\partial \bar{r}} + \bar{M} \frac{\partial \varepsilon}{\partial \bar{r}} = -\bar{\rho}_w a_0^2 \bar{u}_r - \bar{m} a_0^2 \bar{v}_r + i \bar{b} a_0 \bar{v}_r \quad (3c)$$

$$\alpha \bar{M} \frac{\partial e}{\partial \bar{z}} + \bar{M} \frac{\partial \varepsilon}{\partial \bar{z}} = -\bar{\rho}_w a_0^2 \bar{u}_z - \bar{m} a_0^2 \bar{v}_z + i \bar{b} a_0 \bar{v}_z \quad (3d)$$

$$\bar{\sigma}_z = 2 \frac{\partial \bar{u}_z}{\partial \bar{z}} + \bar{\lambda} e \quad (4a)$$

$$\bar{\tau}_{zr} = \frac{\partial \bar{u}_r}{\partial \bar{z}} + \frac{\partial \bar{u}_z}{\partial \bar{r}} \quad (4b)$$

$$\bar{\sigma}_f = -\alpha \bar{M} e - \bar{M} \varepsilon \quad (4c)$$

where the time factor  $e^{i\omega t}$  or  $e^{ia_0 \bar{t}}$  has been suppressed in all expressions and also in the sequel for convenience.

Performing  $\partial/\partial \bar{r}$  [Eq. (3a)]+ $1/\bar{r}$  [Eq. (3a)]+ $\partial/\partial \bar{z}$  [Eq. (3b)], and  $\partial/\partial \bar{r}$  [Eq. (3c)]+ $1/\bar{r}$  [Eq. (3c)]+ $\partial/\partial \bar{z}$  [Eq. (3d)], we obtain

$$(\bar{\lambda} + \alpha^2 \bar{M} + 2) \nabla^2 e + \alpha \bar{M} \nabla^2 \varepsilon = -a_0^2 e - \bar{\rho}_w a_0^2 \varepsilon \quad (5)$$

$$\alpha \bar{M} \nabla^2 e + \bar{M} \nabla^2 \varepsilon = -\bar{\rho}_w a_0^2 e + (i b a_0 - \bar{m} a_0^2) \varepsilon \quad (6)$$

For the problem considered, it is convenient to introduce the  $\mu$ th Hankel transform with respect to  $r$  defined as (Sneddon 1970)

$$\tilde{f}^\mu(p) = H^\mu[f] = \int_0^\infty r f(r) J_\mu(rp) dr \quad (7a)$$

$$f(r) = \int_0^\infty p \tilde{f}^\mu(p) J_\mu(rp) dp \quad (7b)$$

where  $p$ =parameter for the Hankel transform; and  $J_\mu$  denotes the first-kind Bessel function of order  $\mu$ .

Application of the zero-order Hankel transform to Eqs. (5) and (6) then results in

$$\frac{d^4 \bar{e}^0}{d\bar{z}^4} + \chi_1 \frac{d^2 \bar{e}^0}{d\bar{z}^2} + \chi_2 \bar{e}^0 = 0 \quad (8)$$

where

$$\chi_1 = \frac{-2\bar{M}(\bar{\lambda} + 2)p^2 + (\bar{m}a_0^2 - i\bar{b}a_0)(\bar{\lambda} + \alpha^2\bar{M} + 2) - 2\alpha\bar{M}\bar{\rho}_w a_0^2 + \bar{M}a_0^2}{\bar{M}(\bar{\lambda} + 2)}$$

$$\chi_2 = \frac{\bar{M}(\bar{\lambda} + 2)p^4 - p^2[(\bar{m}a_0^2 - i\bar{b}a_0)(\bar{\lambda} + \alpha^2\bar{M} + 2) - 2\alpha\bar{M}\bar{\rho}_w a_0^2 + \bar{M}a_0^2] + [(\bar{m}a_0^2 - i\bar{b}a_0)a_0^2 - \bar{\rho}_w a_0^4]}{\bar{M}(\bar{\lambda} + 2)}$$

The general solution of Eq. (8) can be written as

$$\bar{e}^0 = -p_1^2 A_1 e^{-c\bar{z}} - p_2^2 A_2 e^{-d\bar{z}} \quad (9)$$

in which  $c = \sqrt{p^2 - p_1^2}$  and  $d = \sqrt{p^2 - p_2^2}$ ,  $p_1, p_2$ =complex wave numbers associated with the dilatational waves of the first and second kind, respectively, given by

$$p_1^2 = \frac{\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2}, \quad p_2^2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\beta_2}}{2}$$

$$\beta_1 = \frac{(\bar{\lambda} + \alpha^2 \bar{M} + 2)(\bar{m}a_0^2 - i\bar{b}a_0) - 2\alpha\bar{M}\bar{\rho}_w a_0^2 + \bar{M}a_0^2}{(\bar{\lambda} + 2)\bar{M}}$$

$$\beta_2 = \frac{(\bar{m}a_0^2 - i\bar{b}a_0)a_0^2 - \bar{\rho}_w a_0^4}{(\bar{\lambda} + 2)\bar{M}}$$

and  $A_1$  and  $A_2$  are arbitrary functions of  $p$ . It is noted that the wave numbers  $c$  and  $d$  are selected in such a way that  $\text{Re}[c] \geq 0$ ,  $\text{Re}[d] \geq 0$ .

Substituting Eq. (9) into the zero-order Hankel transform of Eq. (5) or (6), we find that

$$\bar{\varepsilon}^0 = -p_1^2 \delta_1 A_1 e^{-c\bar{z}} - p_2^2 \delta_2 A_2 e^{-d\bar{z}} \quad (10)$$

where

$$\delta_1 = \frac{(\bar{\lambda} + \alpha^2 \bar{M} + 2)p_1^2 - a_0^2}{\bar{\rho}_w a_0^2 - \alpha \bar{M} p_1^2}$$

$$\delta_2 = \frac{(\lambda + \alpha^2 \bar{M} + 2)p_2^2 - a_0^2}{\bar{\rho}_w a_0^2 - \alpha \bar{M} p_2^2}$$

Applying further the Hankel transforms to Eqs. (3a)–(3d) and substituting Eqs. (9) and (10) into the result, we obtain the following general solutions for the displacements:

$$\tilde{u}_z^0(p, \bar{z}) = -c A_1 e^{-c\bar{z}} - d A_2 e^{-d\bar{z}} + p^2 A_3 e^{-j\bar{z}} \quad (11)$$

$$\tilde{v}_z^0(p, \bar{z}) = -c \delta_1 A_1 e^{-c\bar{z}} - d \delta_2 A_2 e^{-d\bar{z}} + p^2 \delta_3 A_3 e^{-j\bar{z}} \quad (12)$$

$$\tilde{u}_r^1(p, \bar{z}) = -p A_1 e^{-c\bar{z}} - p A_2 e^{-d\bar{z}} + p j A_3 e^{-j\bar{z}} \quad (13)$$

$$\tilde{v}_r^1(p, \bar{z}) = -p \delta_1 A_1 e^{-c\bar{z}} - p \delta_2 A_2 e^{-d\bar{z}} + p j \delta_3 A_3 e^{-j\bar{z}} \quad (14)$$

where  $j = \sqrt{p^2 - s^2}$

$$s^2 = \frac{a_0^2(i\bar{b}a_0 - \bar{m}a_0^2) + \bar{\rho}_w a_0^4}{i\bar{b}a_0 - \bar{m}a_0^2}$$

$s$ =complex wave number associated with the rotational wave and is again selected in such a way so that  $\text{Re}[j] \geq 0$ ;  $\delta_3 = \bar{\rho}_w a_0^2 / (i\bar{b}a_0 - \bar{m}a_0^2)$ ; and  $A_3$ =another arbitrary function of  $p$ .

Now making use of Eqs. (4a)–(4c), the expressions for the stresses of the solid matrix and pore water are obtained as follows:

$$\tilde{\sigma}_z^0 = k_1 A_1 e^{-c\bar{z}} + k_2 A_2 e^{-d\bar{z}} - 2p^2 j A_3 e^{-j\bar{z}} \quad (15)$$

$$\tilde{\tau}_{zr}^1 = 2pc A_1 e^{-c\bar{z}} + 2pd A_2 e^{-d\bar{z}} - p(p^2 + j^2) A_3 e^{-j\bar{z}} \quad (16)$$

$$\tilde{\sigma}_z^0 = a_1 A_1 e^{-c\bar{z}} + a_2 A_2 e^{-d\bar{z}} \quad (17)$$

where  $k_1 = (\bar{\lambda} + 2)c^2 - \bar{\lambda}p^2$ ;  $k_2 = (\bar{\lambda} + 2)d^2 - \bar{\lambda}p^2$ ;  $a_1 = (\alpha + \delta_1)\bar{M}p_1^2$ ; and  $a_2 = (\alpha + \delta_2)\bar{M}p_2^2$ .

Eqs. (11)–(17) are the Hankel-domain solutions for the displacements and stresses of the solid matrix as well as the pore water pressure. The three unknown functions  $A_1$ ,  $A_2$ , and  $A_3$  appearing in these equations can be determined from the mixed boundary condition at the ground surface, which is discussed in the following.

### Mixed Boundary Value Problem

Let us assume that the flexible plate is in frictionless contact with the underlying soil half-space and that the ground surface is fully permeable, either within or exterior to the contact area. The relevant boundary conditions at  $\bar{z}=0$  can thus be expressed as

$$\bar{\tau}_{z,r}(\bar{r}, 0) = 0 \quad (0 \leq \bar{r} \leq \infty) \quad (18a)$$

$$\bar{\sigma}_z(\bar{r}, 0) = 0 \quad (1 \leq \bar{r} \leq \infty) \quad (18b)$$

$$\bar{\sigma}_f(\bar{r}, 0) = 0 \quad (0 \leq \bar{r} \leq \infty) \quad (18c)$$

$$\bar{u}_z(\bar{r}, 0) = \bar{\Delta}_v - H(\bar{r} - \bar{r}_b)\bar{w}(\bar{r}) \quad (0 \leq \bar{r} \leq 1) \quad (18d)$$

where  $\bar{w} = w/r_0$ ,  $\bar{\Delta}_v = \Delta_v/r_0$  with  $\Delta_v$  denoting the vertical displacement of the rigid portion of the plate and  $w$  the plate deflection relative to its rigid core, respectively;  $\bar{r}_b = r_b/r_0$ ; and  $H(\bar{r} - \bar{r}_b)$  is the Heaviside function.

Making use of the boundary conditions (18a)–(18c), the three unknown functions  $A_1$ ,  $A_2$ , and  $A_3$  in Eqs. (15)–(17) can be easily determined. Subsequent substitution of the result into Eq. (11) leads to the following expression:

$$\bar{u}_z(\bar{r}, 0) = \int_0^\infty p \tilde{u}_z^0(p, 0) J_0(p\bar{r}) dp = \int_0^\infty p f(p) \tilde{\sigma}_z^0(p, 0) J_0(p\bar{r}) dp \quad (19)$$

in which

$$f(p) = \frac{s^2(a_1 d - a_2 c)}{a_1 k_2 (2p^2 - s^2) - a_2 k_1 (2p^2 - s^2) - 4p^2 j(a_1 d - a_2 c)}$$

Substituting Eq. (19) into Eq. (18d) yields

$$\int_0^\infty p^{-1} [1 + h(p)] B(p) J_0(p\bar{r}) dp = -\frac{\bar{\Delta}_v}{1 - \nu} + H(\bar{r} - \bar{r}_b) \frac{\bar{w}(\bar{r})}{1 - \nu} \quad (20)$$

where  $B(p) = p \tilde{\sigma}_z^0(p, 0)$ ;  $h(p) = -pf(p)/(1 - \nu) - 1$ ; and  $\nu$  is Poisson's ratio of soil. It is important to note that  $\lim_{p \rightarrow \infty} pf(p) = -(1 - \nu)$ , i.e.,  $\lim_{p \rightarrow \infty} h(p) = 0$ .

The relative deflection of the flexible part of the plate,  $\bar{w}(\bar{r})$  in Eq. (20), must satisfy the following differential equation (Szilard 1974):

$$\left( \frac{d}{d\bar{r}} + \frac{1}{\bar{r}} \frac{d}{d\bar{r}} \right)^2 \bar{w}(\bar{r}) = -\delta \bar{\sigma}_z(\bar{r}, 0), \quad \bar{r}_b \leq \bar{r} \leq 1 \quad (21)$$

where the dimensionless flexural rigidity  $\delta$  is defined as  $\delta = Gr_0^3/D$ .

Introducing the integral representation

$$\bar{w}(\bar{r}) = -\delta \int_0^\infty p \tilde{\sigma}_z^0(p, 0) \bar{w}(\bar{r}, p) dp \quad (22)$$

Eq. (21) then becomes

$$\left( \frac{d}{d\bar{r}} + \frac{1}{\bar{r}} \frac{d}{d\bar{r}} \right)^2 \bar{w}(\bar{r}, p) = J_0(p\bar{r}) \quad (23)$$

The solution for  $\bar{w}(\bar{r}, p)$  can be obtained as

$$\bar{w}(\bar{r}, p) = A_0 + B_0 \bar{r}^2 + C_0 \ln \bar{r} + D_0 \bar{r}^2 \ln \bar{r} + \frac{J_0(p\bar{r})}{p^4} \quad (24)$$

where  $A_0$ ,  $B_0$ ,  $C_0$ , and  $D_0$  can be determined from the plate boundary conditions, i.e.,

$$\left[ \bar{w} \right]_{\bar{r}=\bar{r}_b} = \left[ \frac{d\bar{w}}{d\bar{r}} \right]_{\bar{r}=\bar{r}_b} = 0, \quad \left[ \frac{d^3 \bar{w}}{d\bar{r}^3} + \frac{1}{\bar{r}} \frac{d^2 \bar{w}}{d\bar{r}^2} \right]_{\bar{r}=1} = 0 \quad (25)$$

$$\left[ \frac{d^2 \bar{w}}{d\bar{r}^2} + \frac{v_f d\bar{w}}{\bar{r} d\bar{r}} \right]_{\bar{r}=1} = 0$$

Therefore, the final expression for  $\bar{w}(\bar{r}, p)$  is

$$\begin{aligned} \bar{w}(\bar{r}, p) = & \frac{1}{p^4} [J_0(p\bar{r}) - J_0(p\bar{r}_b)] + \frac{\bar{r}^2 - \bar{r}_b^2 J_1(p\bar{r}_b)}{2\bar{r}_b p^3} \\ & + e_0 \left( \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2 - \bar{r}_b^2}{2\bar{r}_b^2} \right) \cdot \left\{ (1 + \nu_f) \left[ \frac{J_1(p\bar{r}_b)}{\bar{r}_b p^3} - \frac{J_1(p)}{p^3} \right] \right. \\ & + \left. \frac{J_2(p)}{p^2} - \frac{J_1(p)}{2p} \right\} - \left[ \bar{r}^2 \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2 - \bar{r}_b^2}{2} \right. \\ & \left. - 2e_0(1 + \nu_f) \left( \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2 - \bar{r}_b^2}{2\bar{r}_b^2} \right) \ln \bar{r}_b \right] \frac{J_1(p)}{4p} \quad (26) \end{aligned}$$

where

$$e_0 = \frac{\bar{r}_b^2}{1 + \bar{r}_b^2 + \nu_f(1 - \bar{r}_b^2)}$$

Substituting Eq. (22) into Eq. (20), and with the aid of the relation

$$\bar{\sigma}_z(\bar{r}, 0) = \int_0^\infty p \tilde{\sigma}_z^0(p, 0) J_0(p\bar{r}) dp \quad (27)$$

we finally obtain

$$\begin{aligned} & \int_0^\infty p^{-1} B(p) J_0(p\bar{r}) dp \\ & = -\frac{\bar{\Delta}_v}{(1 - \nu)} - \int_0^\infty p^{-1} h(p) B(p) J_0(p\bar{r}) dp \\ & \quad - \int_0^\infty \frac{\delta}{(1 - \nu)} H(\bar{r} - \bar{r}_b) B(p) \bar{w}(\bar{r}, p) dp \quad (0 \leq \bar{r} \leq 1) \quad (28a) \end{aligned}$$

$$\int_0^\infty B(p) J_0(p\bar{r}) dp = 0 \quad (\bar{r} > 1) \quad (28b)$$

Eqs. (28a) and (28b) constitute a set of dual integral equations where the mixed boundary value problem is formulated in terms of the transformed unknown stresses under the circular plate. These equations can be treated by employing the following integral representation

$$B(p) = -\frac{2}{\pi} \frac{\bar{\Delta}_v}{1-\nu} p \int_0^1 \theta(x) \cos px dx \quad (29)$$

so that the determination of the normal contact stress is reduced to the evaluation of  $\theta(x)$ . It can be shown that the representation (29) for  $B(p)$  satisfies Eq. (28b) identically. Further, substitution of Eq. (29) in Eq. (28a) leads to the Fredholm integral equation

$$\theta(x) + \int_0^1 K(x,y) \theta(y) dy = 1 \quad (30)$$

where the kernel

$$K(x,y) = \int_0^\infty \frac{2}{\pi} h(p) \cos(px) \cos(py) dp + \frac{2\delta}{(1-\nu)\pi} \int_0^\infty p \cos(py) \times \left\{ \frac{d}{dx} \int_0^\infty \frac{\bar{r} H(\bar{r} - \bar{r}_b) \bar{w}(\bar{r}, p)}{\sqrt{x^2 - \bar{r}^2}} d\bar{r} \right\} dp \quad (31)$$

can be further reduced to (see the Appendix)

$$K(x,y) = \frac{2}{\pi} \int_0^\infty h(p) \cos px \cos py dp, \quad 0 < x < \bar{r}_b \text{ or } 0 < y < \bar{r}_b \quad (32a)$$

$$K(x,y) = \frac{2}{\pi} \int_0^\infty h(p) \cos px \cos py dp + \frac{\delta}{\pi(1-\nu)} \times \left\{ (x^2 + y^2) \ln \frac{\bar{r}_b \sqrt{|x^2 - y^2|}}{x \sqrt{y^2 - \bar{r}_b^2} + y \sqrt{x^2 - \bar{r}_b^2}} + 2xy \ln \frac{\sqrt{y^2 - \bar{r}_b^2} + \sqrt{x^2 - \bar{r}_b^2}}{\sqrt{|x^2 - y^2|}} + \frac{xy}{\bar{r}_b^2} \sqrt{x^2 - \bar{r}_b^2} \sqrt{y^2 - \bar{r}_b^2} - e_0(1 + \nu_f) \cdot \left[ \ln \frac{x + \sqrt{x^2 - \bar{r}_b^2}}{\bar{r}_b^2} - \frac{x \sqrt{y^2 - \bar{r}_b^2}}{\bar{r}_b^2} \right] \times \left[ \ln \frac{y + \sqrt{y^2 - \bar{r}_b^2}}{\bar{r}_b^2} - \frac{y \sqrt{x^2 - \bar{r}_b^2}}{\bar{r}_b^2} \right] \right\} \quad (32b)$$

In addition, the force equilibrium for the massless circular plate requires

$$P = - \int_0^{r_0} \sigma_z(r,0) 2\pi r dr = -2\pi Gr_0^2 \bar{\sigma}_z^0(0,0) \quad (33)$$

In view of Eq. (29) and by considering the typical complex stiffness representation, the force-displacement relationship for the flexible plate can be obtained as

$$P = \frac{4Gr_0\Delta_v}{1-\nu} \int_0^1 \theta(x) dx = \frac{4Gr_0\Delta_v}{1-\nu} [K_{vv}(a_0) + ia_0 C_{vv}(a_0)] \quad (34)$$

where  $4Gr_0/(1-\nu)$  corresponds to the static vertical impedance for a rigid plate with radius  $r_0$  on an elastic half-space

characterized by shear modulus  $G$  and Poisson's ratio  $\nu$ ;  $K_{vv} = \int_0^1 \text{Re}[\theta(x)] dx$  and  $C_{vv} = (1/a_0) \int_0^1 \text{Im}[\theta(x)] dx$  can be interpreted, respectively, as the vertical dynamic stiffness and damping coefficients. The function  $\theta(x)$  is solved from the Fredholm integral equation (30) using standard numerical techniques.

To find the contact stress, we substitute  $B(p) = p \bar{\sigma}_z^0(p,0)$  into Eq. (29) and then carry out the inverse Hankel transform of the result. In so doing, the contact stresses between the plate and the underlying saturated soil can be expressed in terms of  $\theta(x)$  as

$$\bar{\sigma}_z(\bar{r},0) = \frac{P}{2\pi Gr_0^2} \frac{\frac{1}{\bar{r}} \frac{d}{d\bar{r}} \int_{\bar{r}}^1 \frac{x}{\sqrt{x^2 - \bar{r}^2}} \theta(x) dx}{\int_0^1 \theta(x) dx} \quad (35)$$

Similarly, the bending moment of the outer flexible plate,  $M_r$ , can be obtained by substituting Eq. (22) into the relationship

$$M_r = D \left[ \frac{d^2 w}{dr^2} + \frac{\nu_f}{r} \frac{dw}{dr} \right]$$

(Szilard 1974). The resulting expression is

$$\bar{M}_r(\bar{r}) = \frac{M_r}{Gr_0^2} = \frac{P}{2\pi Gr_0^2} \frac{\int_0^1 \theta(x) \int_0^\infty p \left[ \frac{d^2 \bar{w}}{d\bar{r}^2} + \frac{\nu_f}{\bar{r}} \frac{d\bar{w}}{d\bar{r}} \right] \cos(px) dp dx}{\int_0^1 \theta(x) dx} \quad (36)$$

$$\bar{r}_b \leq \bar{r} \leq 1$$

We remark that the present solution contains the vertical vibration of a rigid or flexible plate as two special cases. For a completely rigid plate,  $\delta=0$ , Eqs. (32a) and (32b) are then simplified to

$$K(x,y) = \int_0^\infty h(p) \cos px \cos py dp, \quad 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \quad (37)$$

This kernel function is identical to the one derived in Chen (2000). On the other hand, if the plate is fully flexible,  $\bar{r}_b$  in Eq. (26) then approaches zero, which gives us

$$\lim_{\bar{r}_b \rightarrow 0} \bar{w}(\bar{r},p) = \frac{J_0(p\bar{r})}{p^4} - \frac{1}{p^4} + \frac{\bar{r}^2}{4p^2} + \frac{\bar{r}^2}{2(1+\nu_f)} \times \left\{ -\frac{1+\nu_f}{2p^2} + \frac{J_1'(p)}{p^2} + \nu_f \frac{J_1(p)}{p^3} + \frac{J_1(p)}{2p} \right\} + \frac{\bar{r}^2}{8p} J_1(p) - \frac{\bar{r}^2}{4p} \ln \bar{r} J_1(p) \quad (38)$$

Substitution of this expression in Eq. (28a) reduces the integral equation exactly to that for a fully flexible plate (Chen 2000).

Finally, it should be noted that with some modifications, the preceding approach for fully pervious plate can be used to analyze the case of an impermeable plate. In that case the boundary conditions Eqs. (18a), (18b), and (18d) remain unchanged, whereas the hydraulic boundary condition Eq. (18c) at the plate-soil interface becomes

$$\bar{v}_z(\bar{r}, 0) = \left[ \frac{d\bar{\sigma}_r(\bar{r}, \bar{z})}{d\bar{z}} \right]_{\bar{z}=0} = 0 \quad (0 \leq \bar{r} \leq 1) \quad (39a)$$

$$\bar{\sigma}_r(\bar{r}, 0) = 0 \quad (1 \leq \bar{r} \leq \infty) \quad (39b)$$

The corresponding mixed boundary value problem should be formulated in terms of the transformed contact stress and the surface pore pressure which results in a coupled system of dual integral equations. These dual integral equations can again be reduced to two coupled Fredholm integral equations of the second kind by introducing two auxiliary functions and subsequently solved by numerical procedure. The dynamic response of the vertically loaded impermeable plate can thus be obtained. Details can be found in Zeng and Rajapakse (1999) and Jin and Liu (1999).

## Numerical Results

### Comparison with Existing Solutions for Purely Elastic Soil

The accuracy of the present solution is verified by comparing the compliance functions for a vertically loaded rigid circular footing on a homogeneous saturated semi-infinite soil with the corresponding results presented by Zeng and Rajapakse (1999). In order to make the comparison, Eq. (34) is now given in an alternative form as

$$\Delta_v = \frac{(1-\nu)P}{4Gr_0} [F_1(a_0) + iF_2(a_0)] \quad (40)$$

where  $F_1(a_0) + iF_2(a_0)$  denotes the dynamic compliance function that is given by

$$F_1(a_0) = \text{Re} \left[ \frac{1}{K_{vv}(a_0) + ia_0 C_{vv}(a_0)} \right]$$

$$F_2(a_0) = \text{Im} \left[ \frac{1}{K_{vv}(a_0) + ia_0 C_{vv}(a_0)} \right]$$

The parameters used for this reduced problem are as follows:  $\alpha=0.97$ ,  $\bar{M}=M/G=12.2$ ,  $\bar{m}=m/\rho=1.1$ ,  $\bar{\rho}_w=\rho_w/\rho=0.53$ ,  $\bar{b}=ab/\sqrt{\rho G}=2.3$ ,  $\delta=0$ , and  $\nu=1/4$ . The computed compliance coefficients,  $F_1$  and  $F_2$ , from the present study for the rigid permeable plate case as well as those from Zeng and Rajapakse (1999) are shown in Fig. 2. It is evident that the two results are in excellent agreement.

### Solution for Saturated Soil

After validating our formulation for the reduced case, we now consider the dynamic response of a flexible plate with rigid core resting on the saturated soil and investigate in detail the influence of the soil and plate properties. For illustration purposes, numerical results are obtained for the following nondimensional material parameters:  $\nu_f=0.167$ ,  $\nu=0.4$ ,  $\bar{M}=12.2$ ,  $\bar{m}=1.1$ ,  $\alpha=0.97$ , and  $\bar{\rho}_w=0.53$ . In addition, different values of  $\bar{b}=1, 100, 10,000$ ,  $\bar{r}_b=0.25, 0.5, 0.75$ , and  $\delta=0, 1, 10, 100, 1,000$  are adopted to examine the influence of the soil permeability, rigid core size, and plate flexibility on the dynamic impedance function. Evidently a value of  $\delta=0$  corresponds to a fully rigid foundation.

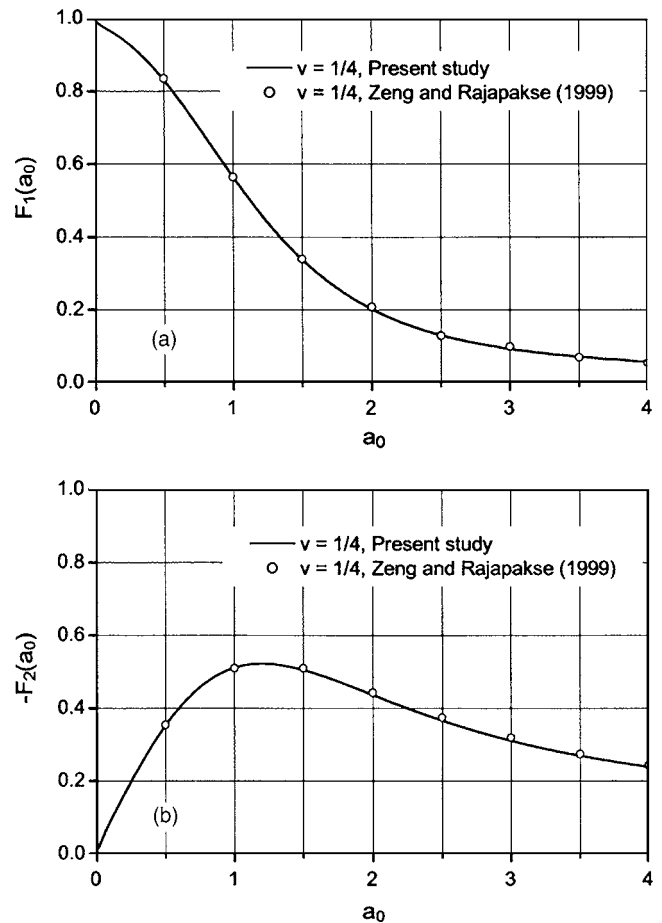
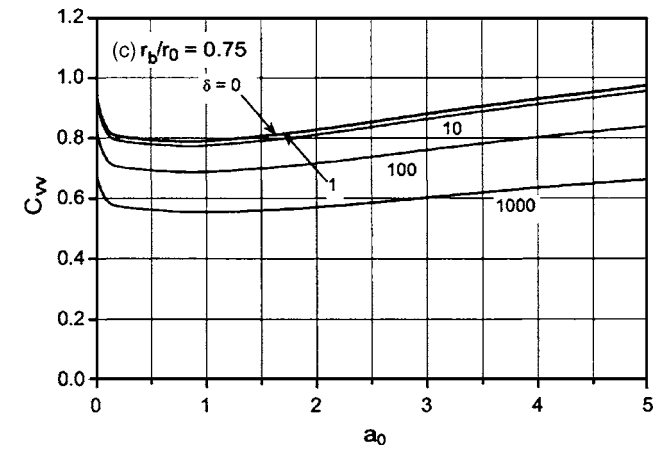
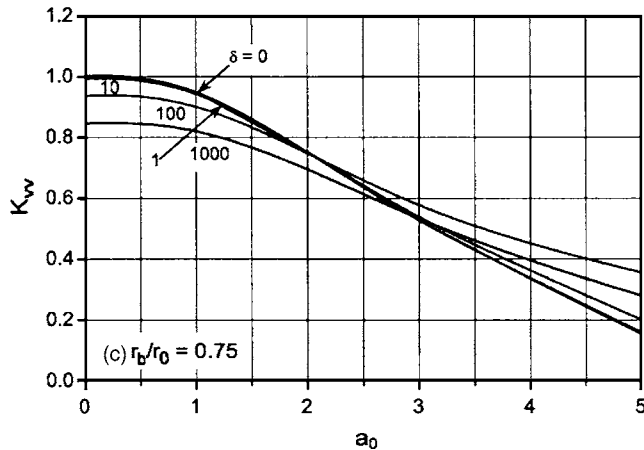
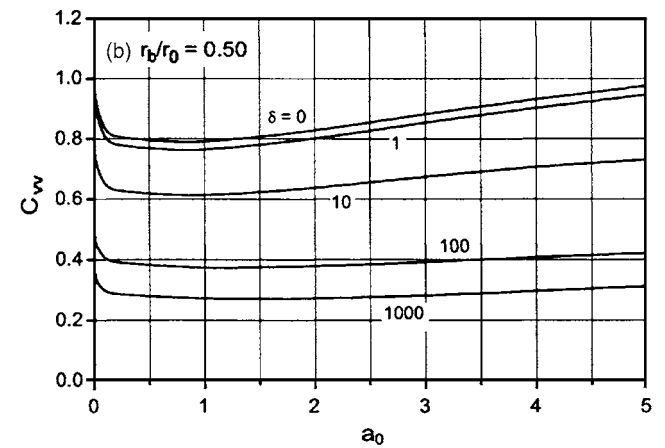
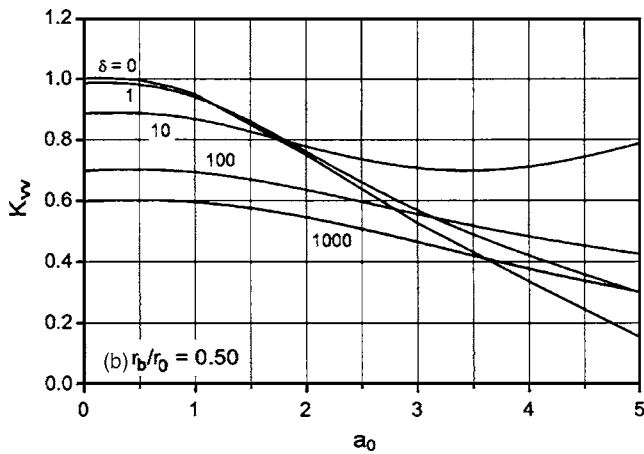
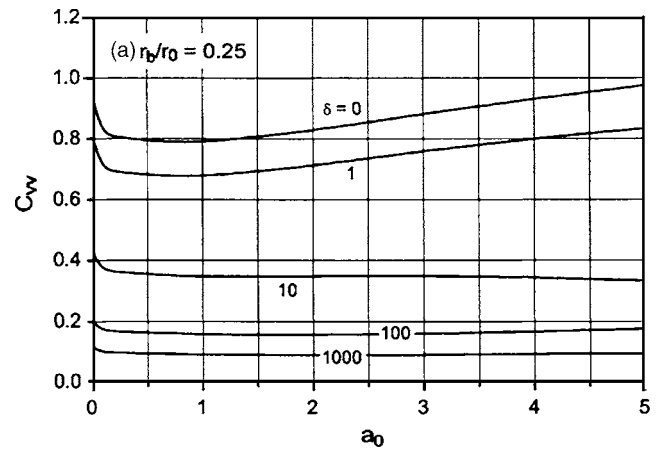
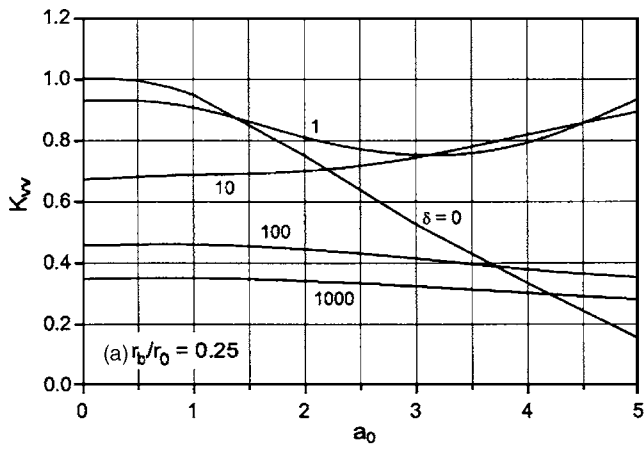


Fig. 2. Comparison of compliance function for a rigid plate on saturated soil: (a) real part; (b) imaginary part

Figs. 3–8 show the variation of  $K_{vv}$  and  $C_{vv}$  with the normalized frequency  $a_0$  for three different values of internal friction parameter  $\bar{b}$ . It is observed that for  $\bar{b}=10,000$  (low-permeability soil), the dynamic stiffness coefficient  $K_{vv}$  decreases monotonically with increasing frequency  $a_0$  except for the very low frequency range (Fig. 7). However, for  $\bar{b}=1$  and 100, the dynamic stiffness coefficient  $K_{vv}$  could be oscillatory with respect to  $a_0$  for certain combinations of  $r_b/r_0$  and  $\delta$  [e.g., Figs. 3(a) and 5(a)]. It is further shown that at low frequencies the dynamic stiffness coefficients for flexible plates are considerably smaller than those for rigid foundations. Generally the stiffness coefficient decreases as the plate becomes more flexible. At higher frequencies, the influence of the flexural rigidity on the stiffness coefficient is rather complicated, although for large values of  $\bar{b}$  and  $\bar{r}_b$  the stiffness  $K_{vv}$  may increase with increasing  $\delta$ . On the other hand, for all the three values of  $\bar{b}$  and the frequency range covered in this paper, the dynamic damping coefficient  $C_{vv}$  decreases as  $\delta$  increases. Compared with the stiffness coefficient, the damping coefficient is relatively insensitive to the variation of the internal friction parameter.

There are also a couple of interesting features in Figs. 3–8 that are worthy to discuss: First, with increasing flexural rigidity of the plate ( $\delta$ ), the dynamic stiffness and damping coefficients become less frequency-dependent. In addition, the influence of the plate flexibility on the stiffness and damping coefficients decreases as



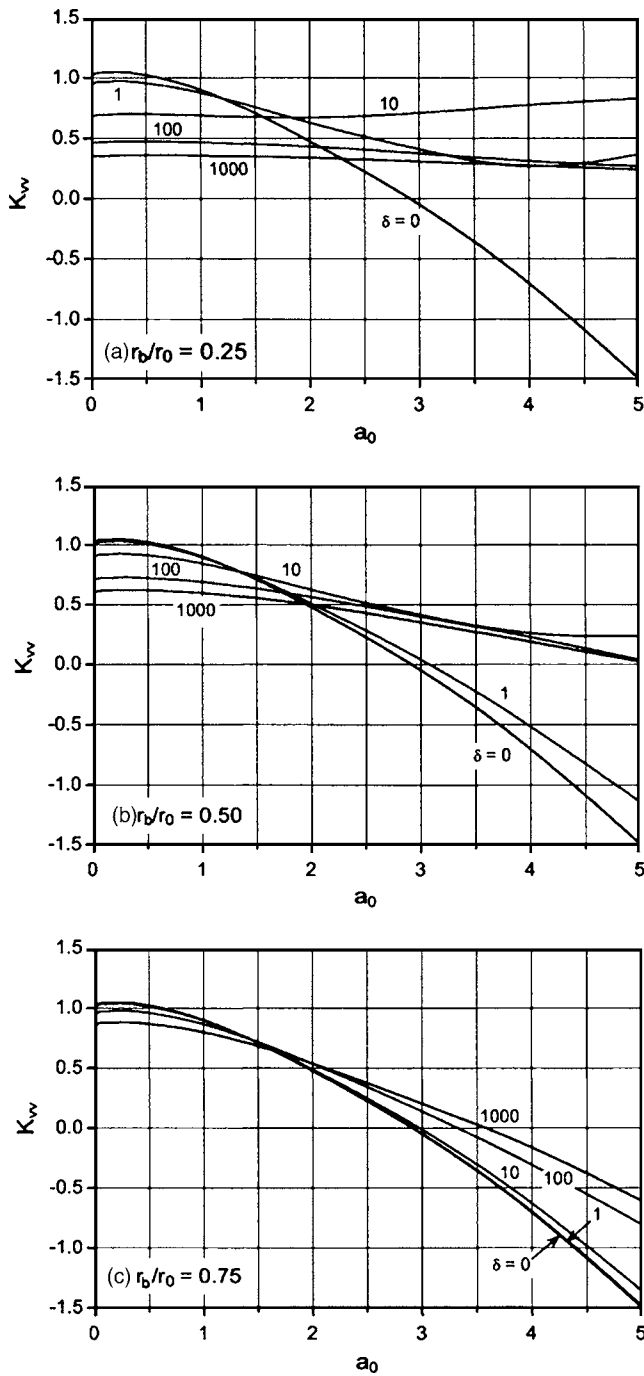
**Fig. 3.** Influence of plate flexibility on dynamic stiffness coefficient for  $\bar{b}=1$ : (a)  $r_b/r_0=0.25$ ; (b)  $r_b/r_0=0.50$ ; and (c)  $r_b/r_0=0.75$

**Fig. 4.** Influence of plate flexibility on dynamic damping coefficient for  $\bar{b}=1$ : (a)  $r_b/r_0=0.25$ ; (b)  $r_b/r_0=0.50$ ; and (c)  $r_b/r_0=0.75$

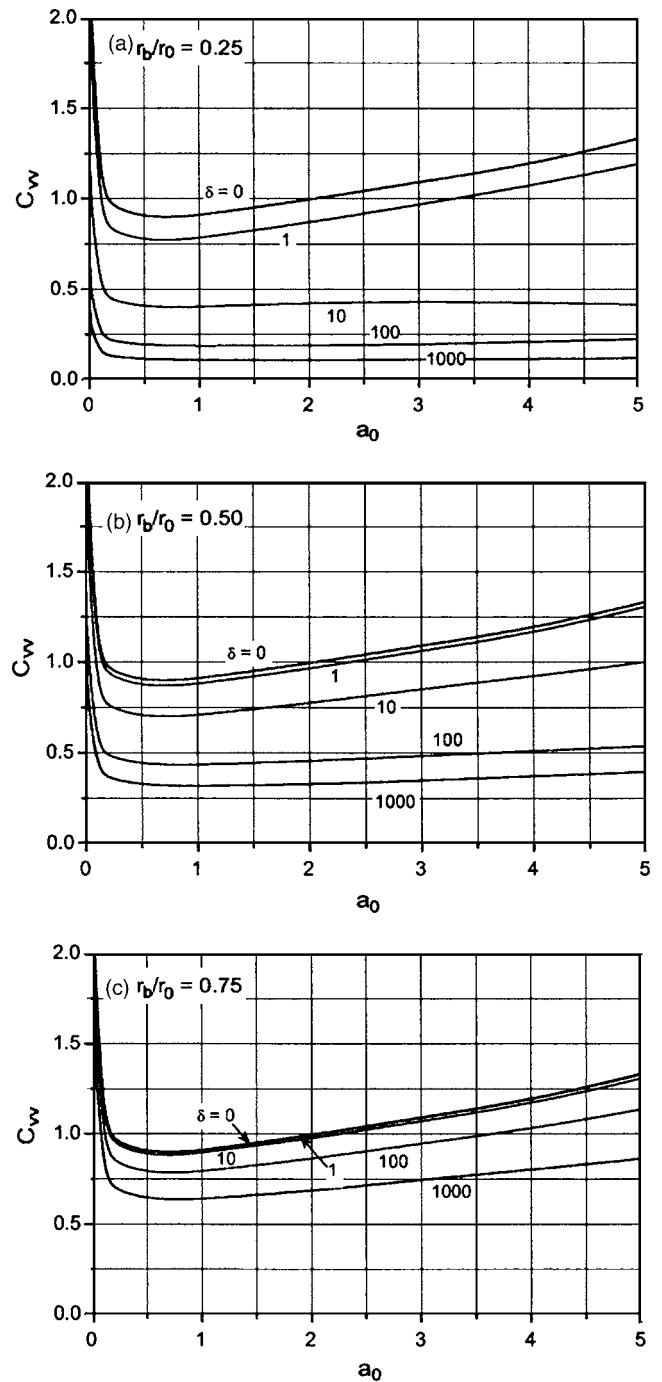
the radius of the rigid core increases. The effects of the plate rigidity and rigid core size are essentially similar to those described by Iguchi and Luco (1982) for the corresponding viscoelastic case.

Of particular interest are the vibration responses of a circular foundation supported on a highly permeable soil or a nearly impermeable soil. In these two limiting cases, the dynamic stiffness and damping coefficients of the foundation can be determined in a straightforward way from Eqs. (30) and (34), where the nondimensional parameter  $\bar{b}$  is taken respectively to be arbitrarily small

( $\bar{b} \rightarrow 0$ ) and arbitrarily large ( $\bar{b} \rightarrow \infty$ ). In the limit of  $\bar{b}$  approaching zero, all excess pore pressure is dissipated and the elastic skeleton will alone carry the stress. Therefore the saturated soil can be adequately represented by using an elastic model with the shear modulus and Poisson's ratio corresponding to the drained values, i.e.,  $G_0=G$ ,  $\nu_0=\nu=\lambda/2(\lambda+G)$ . On the other hand, if  $\bar{b}$  approaches infinity, no pore water would have yet been drained so that the variation of the fluid volume per unit reference volume  $\varepsilon$  is zero. This implies that the constitutive relations, e.g., Eqs. (2a)–(2c), should be modified to



**Fig. 5.** Influence of plate flexibility on dynamic stiffness coefficient for  $\bar{b}=100$ : (a)  $r_b/r_0=0.25$ ; (b)  $r_b/r_0=0.50$ ; and (c)  $r_b/r_0=0.75$



**Fig. 6.** Influence of plate flexibility on dynamic damping coefficient for  $\bar{b}=100$ : (a)  $r_b/r_0=0.25$ ; (b)  $r_b/r_0=0.50$ ; and (c)  $r_b/r_0=0.75$

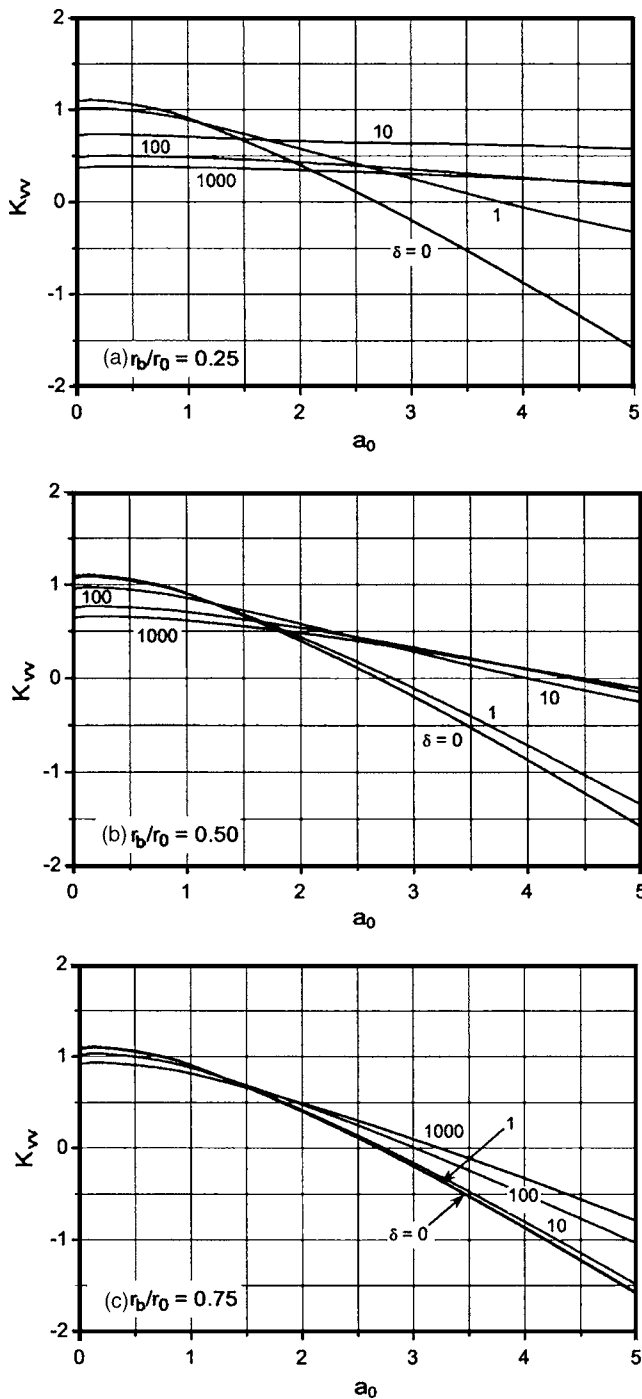
$$\sigma_z - \alpha\sigma_f = 2G \frac{\partial u_z}{\partial z} + (\lambda + \alpha^2 M)e, \quad \tau_{zr} = G \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \quad (41)$$

$$\sigma_f = -\alpha M e$$

where  $\sigma_z - \alpha\sigma_f$  essentially presents the total normal stress component including the contribution of the pore water. Thus the saturated soil behaves as if it were an equivalent elastic medium with shear modulus  $G_\infty = G$  and Poisson's ratio  $\nu_\infty = (\lambda + \alpha^2 M) / 2(\lambda + \alpha^2 M + G)$ . Again, the dynamic impedance functions can be obtained by resorting to the ideal elastic case. The above-presented statement is indeed graphically confirmed

by Fig. 9, which depicts the variation of the impedance coefficients with  $a_0$  for saturated soils with  $\bar{b}=10^{-2}$  and  $\bar{b}=10^6$  as well as those corresponding to the elastic media with Poisson's ratio  $\nu_0 = \nu = \lambda / 2(\lambda + G) = 0.4$  and  $\nu_\infty = (\lambda + \alpha^2 M) / 2(\lambda + \alpha^2 M + G) = 0.47$ . In preparing Fig. 9, the soil parameters utilized are the same as those used in Figs. 3–8, but the relative flexibility of the plate,  $\delta$ , and the size of the rigid core,  $\bar{r}_b$ , are fixed to be 10 and 0.5, respectively. It should also be pointed out that the results for the elastic case of  $\nu_\infty = 0.47$  have been multiplied by  $(1 - \nu) / (1 - \nu_\infty)$ . The necessity for this change stems from the fact that the stiffness and damping coefficients for the elastic case are in fact normalized by  $\nu_\infty = 0.47$ , and obviously not consistent with the Poisson's



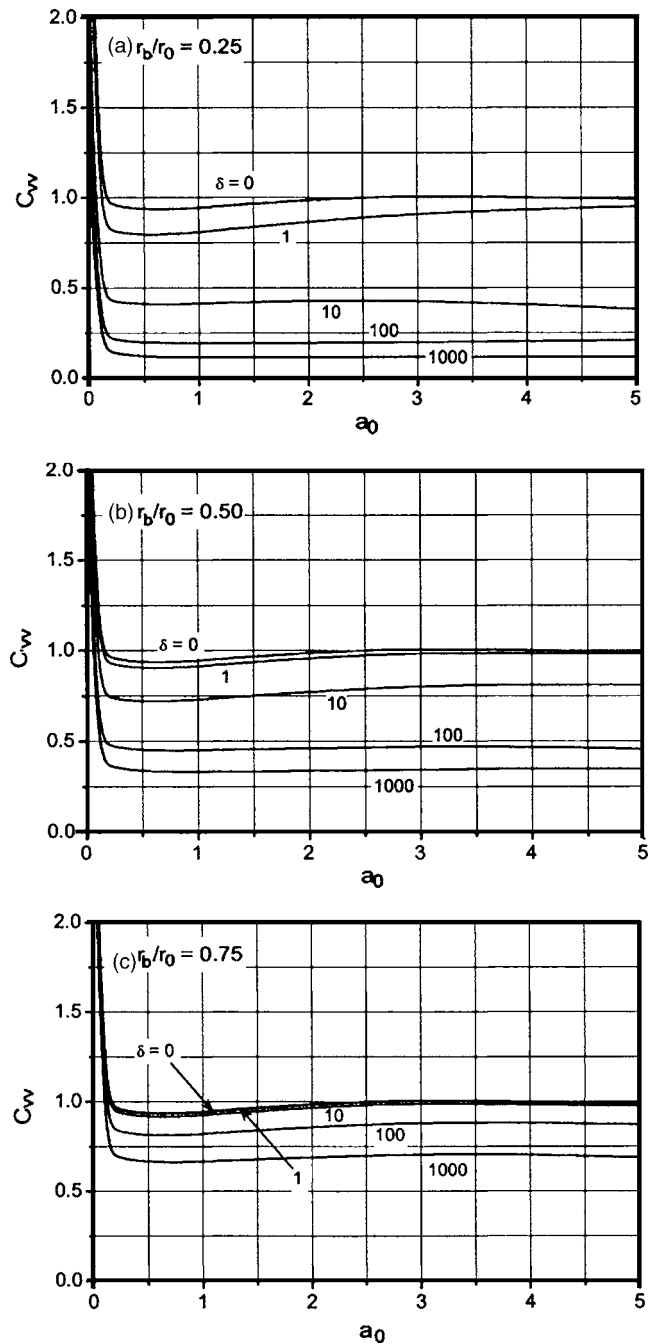


**Fig. 7.** Influence of plate flexibility on dynamic stiffness coefficient for  $\bar{b}=10,000$ : (a)  $r_b/r_0=0.25$ ; (b)  $r_b/r_0=0.50$ ; and (c)  $r_b/r_0=0.75$

ratio of  $\nu=0.4$  adopted for normalizing the saturated soil. Similarly, for comparison, the impedance functions of elastic medium with  $\nu_0=0.4$  are plotted against an adjusted nondimensional frequency defined by

$$a'_0 = \sqrt{\frac{(1-n)\rho_s}{\rho}} \sqrt{\frac{\rho}{G}} r_0 \omega = \sqrt{1 - \frac{\bar{\rho}_w^2}{\bar{m}}} a_0$$

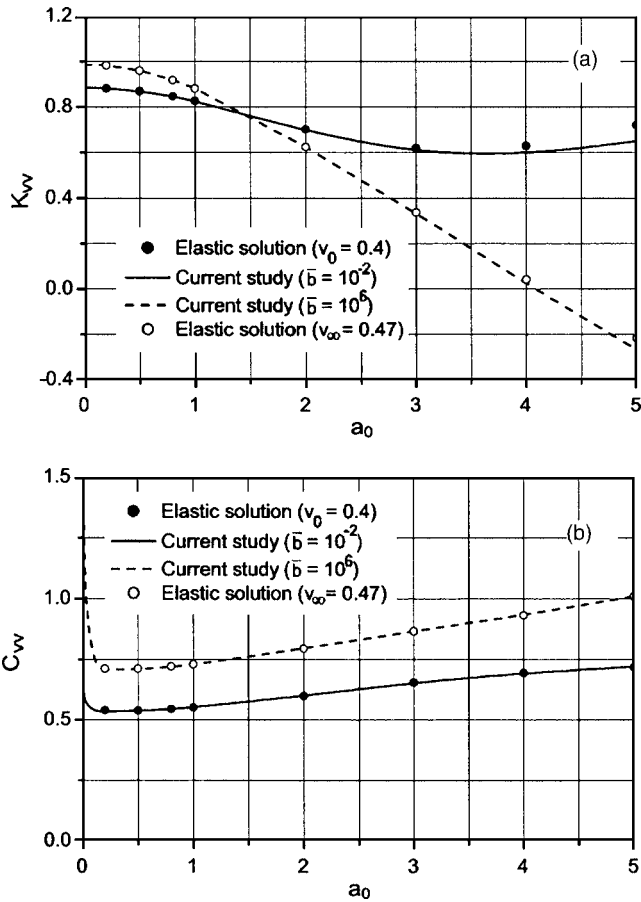
Further, by setting  $a_0=\delta=0$ , the impedance functions for the two limiting cases of  $\bar{b}\rightarrow 0$  and  $\bar{b}\rightarrow\infty$  would ultimately be simplified to the well-known static force-displacement relationships for a rigid foundation bearing on an elastic soil, i.e.,



**Fig. 8.** Influence of plate flexibility on dynamic damping coefficient for  $\bar{b}=10,000$ : (a)  $r_b/r_0=0.25$ ; (b)  $r_b/r_0=0.50$ ; and (c)  $r_b/r_0=0.75$

$4Gr_0\Delta_v/P=1-\nu_0$  and  $1-\nu_\infty$ , respectively. Table 1 shows the calculated compliance functions  $4Gr_0\Delta_v/P$  from the present study for different values of Poisson's ratio of the solid skeleton. As can be seen, the results agree well with the explicit analytical solutions.

Finally, the influences of the plate flexibility on the stress distribution under the circular plate and the plate bending moment are studied. As an example, Figs. 10 and 11 present the numerical results for a flexible circular plate with rigid core ( $\nu_f=0.167$ ,  $\bar{r}_b=0.5$ ) supported on a semi-infinite saturated soil ( $\bar{b}=100$ ,  $\nu=0.4$ ,  $\bar{M}=12.2$ ,  $\bar{m}=1.1$ ,  $\alpha=0.97$ ,  $\bar{\rho}_w=0.53$ ) for four values of  $\delta=1, 10, 100, 1000$ . In Figs. 10 and 11, the contact pressure and bending moment are normalized by



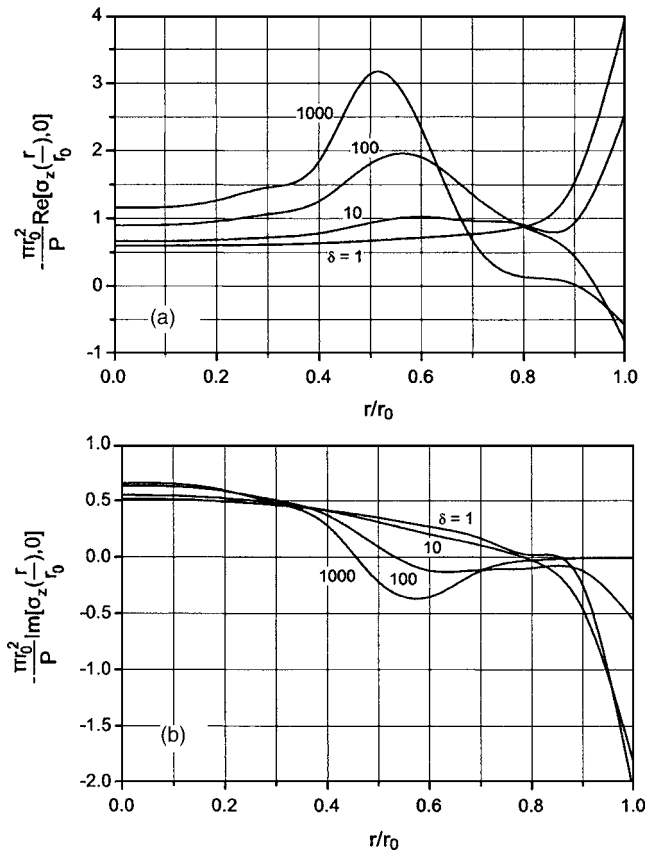
**Fig. 9.** Impedance functions for a foundation on saturated soils with  $\bar{b} \rightarrow 0$  and  $\bar{b} \rightarrow \infty$ : (a) Stiffness coefficient; (b) damping coefficient

$$\frac{\pi r_0^2 \sigma_z(\bar{r}, 0)}{P}$$

and

$$\left| \frac{2\pi \bar{r} M_r(\bar{r})}{P} \right|$$

respectively, and the dimensionless frequency  $a_0=2$  is adopted. As expected, the contact stress distribution depends strongly on the plate flexibility. A relatively stiff plate having  $\delta=1$  or 10 tends to exhibit significant stress concentration near the edge. As the plate flexibility increases, the concentration domain moves toward the boundary between the rigid and flexible portion of the plate ( $\bar{r}_b=0.5$ ). The bending moment distribution of the outer



**Fig. 10.** Contact stress distribution for flexible plates on saturated soils: (a) real part; (b) imaginary part

flexible plate is also greatly influenced by the plate flexibility and its value decreases as  $\delta$  increases.

## Conclusions

This paper presents an analytical solution for the vertical vibration of a flexible plate with a rigid core resting, in smooth contact, on a semi-infinite saturated soil. The soil skeleton and the pore water are both considered to be compressible. The Hankel transform techniques are employed in deriving the fundamental solutions of the skeleton displacements, stresses, and pore pressure, which are subsequently used to formulate a set of dual integral equations associated with the required mixed boundary value problem. It is further shown that these integral equations can be reduced to a Fredholm integral equation of the second kind and

**Table 1.** Static Compliances of a Rigid Plate on Highly Permeable and Nearly Impermeable Saturated Soils

	$4Gr_0\Delta_v/P$			
Poisson's ratio of soil skeleton	Analytical solutions (elastic soil)	Numerical results ( $\bar{b}=10^{-2}$ )	Analytical solutions (elastic soil)	Numerical results ( $\bar{b}=10^6$ )
$\nu=0$ ( $\nu_0=0, \nu_\infty=0.460$ )	1.00	0.9992-0.0044j	0.540	0.5593-0.0098j
$\nu=1/5$ ( $\nu_0=1/5, \nu_\infty=0.462$ )	0.80	0.7995-0.0032j	0.538	0.5515-0.0081j
$\nu=1/4$ ( $\nu_0=1/4, \nu_\infty=0.463$ )	0.75	0.7496-0.0039j	0.537	0.5488-0.0075j
$\nu=1/3$ ( $\nu_0=1/3, \nu_\infty=0.465$ )	0.67	0.6664-0.0036j	0.535	0.5429-0.0062j
$\nu=1/2$ ( $\nu_0=1/2, \nu_\infty=0.500$ )	0.50	0.5009-0.0020j	0.500	0.5008-0.0021j

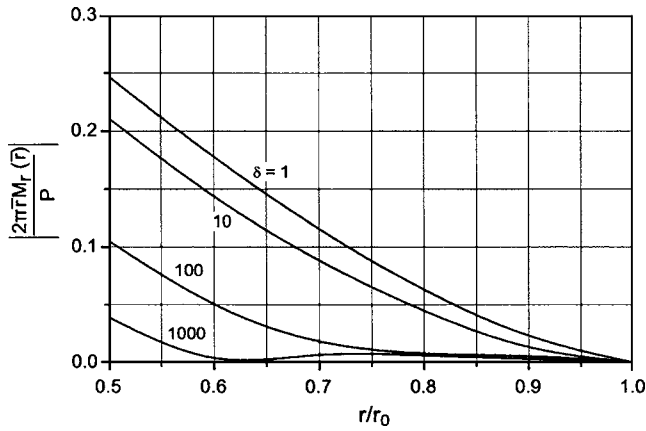


Fig. 11. Bending moment profile for flexible plates on saturated soil

subsequently solved with the standard numerical procedure. The solution of the Fredholm integral equation is then utilized to evaluate the influences of the plate flexibility, the internal friction parameter of the soil, and the exciting frequency on the dynamic stiffness and damping coefficients. For the reduced simple cases, the present solutions are in very good agreement with the existing solutions.

Numerical results presented in this paper show that the plate flexibility leads to a considerable reduction of the dynamic stiffness coefficient at low frequency, while at the high frequency the behavior of the dynamic stiffness coefficient becomes complicated. The dynamic damping coefficient  $C_{vv}$  in general decreases with increasing flexural rigidity and is relatively insensitive to the variation of the internal friction parameter  $b$  as compared to the stiffness coefficient. Our numerical results also indicate that with increasing flexural rigidity of the plate ( $\delta$ ), the dynamic stiffness and damping coefficients become less frequency-dependent, while the influences of the plate flexibility on the stiffness and damping coefficients decrease as the radius of the rigid core in-

creases. Finally, it is found that the distributions of contact stress and bending moment strongly depend on the plate flexibility. A plate with low value of  $\delta$  (stiffer) indicates large stress concentration near the edges and also large bending moment; as the plate flexibility increases, the stress concentration domain moves toward the boundary between the rigid and flexible portion of the plate.

## Appendix

We assume without the loss of generality that  $y > x > \bar{r}_b$ . The function  $\tilde{w}(\bar{r}, p)$  in Eq. (26) can be written as

$$\tilde{w}(\bar{r}, p) = w_1(\bar{r}, p) + w_2(\bar{r}, p) + w_3(\bar{r}, p) + w_4(\bar{r}, p) + w_5(\bar{r}, p) \quad (42)$$

where

$$w_1(\bar{r}, p) = \frac{1}{p^4} [J_0(p\bar{r}) - J_0(p\bar{r}_b)] + \frac{\bar{r}^2 - \bar{r}_b^2 J_1(p\bar{r}_b)}{2\bar{r}_b p^3}$$

$$w_2(\bar{r}, p) = e_0 \left( \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2 - \bar{r}_b^2}{2\bar{r}_b^2} \right) (1 + \nu_f) \left[ \frac{J_1(p\bar{r}_b)}{\bar{r}_b p^3} - \frac{J_1(p)}{p^3} \right]$$

$$w_3(\bar{r}, p) = e_0 \left( \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2 - \bar{r}_b^2}{2\bar{r}_b^2} \right) \left[ \frac{J_2(p)}{p^2} - \frac{J_1(p)}{2p} \right]$$

$$w_4(\bar{r}, p) = e_0 (1 + \nu_f) \ln \bar{r}_b \left( \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2 - \bar{r}_b^2}{2\bar{r}_b^2} \right) \frac{J_1(p)}{2p}$$

$$w_5(\bar{r}, p) = - \left( \bar{r}^2 \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2 - \bar{r}_b^2}{2} \right) \frac{J_1(p)}{4p}$$

By employing the following relations and formulas for the integrals involving Bessel functions

$$\frac{d}{dx} \int_{\bar{r}_b}^x \frac{\bar{r}}{\sqrt{x^2 - \bar{r}^2}} \left\{ \frac{J_0(p\bar{r}) - J_0(p\bar{r}_b)}{p^3} + \frac{\bar{r}^2 - \bar{r}_b^2 J_1(p\bar{r}_b)}{2\bar{r}_b p^2} \right\} d\bar{r} = \frac{d}{dx} \int_{\bar{r}_b}^x \sqrt{x^2 - \bar{r}^2} \left\{ \frac{-J_1(p\bar{r})}{p^2} + \frac{\bar{r} J_1(p\bar{r}_b)}{\bar{r}_b p^2} \right\} d\bar{r} = \frac{x}{p} \int_{\bar{r}_b}^x \left\{ \frac{\sqrt{x^2 - \bar{r}^2} J_2(p\bar{r})}{\bar{r}} \right\} d\bar{r} \quad (43)$$

$$\int_0^\infty \left\{ \frac{J_1(\bar{r}_b p)}{\bar{r}_b p^2} - \frac{J_1(p)}{p^2} \right\} \cos(py) dp = \int_{\bar{r}_b}^1 \frac{1}{\bar{r}_b} \left\{ \int_0^\infty \frac{1}{p} J_2(\bar{r}_b p) \cos(py) dp \right\} d\bar{r}_b$$

$$= \begin{cases} -\frac{1}{2} \left\{ \frac{y^2(1 - \bar{r}_b^2)}{\bar{r}_b^2} + \ln \bar{r}_b \right\} & y < \bar{r}_b \\ -\frac{1}{2} \left\{ \frac{y^2(1 - \bar{r}_b^2)}{\bar{r}_b^2} + \ln \bar{r}_b - \frac{y\sqrt{y^2 - \bar{r}_b^2}}{\bar{r}_b^2} - \ln \frac{\bar{r}_b}{y + \sqrt{y^2 - \bar{r}_b^2}} \right\} & y > \bar{r}_b \end{cases} \quad (44)$$

we obtain

$$\frac{2\delta}{\pi(1 - \nu)} \int_0^\infty p \cos(py) \left\{ \frac{d}{dx} \int_0^x \frac{\bar{r} H(\bar{r} - \bar{r}_b) w_1(\bar{r}, p)}{\sqrt{x^2 - \bar{r}^2}} d\bar{r} \right\} dp$$

$$= \frac{2\delta}{\pi(1 - \nu)} \int_0^\infty \frac{x}{p} \cos(py) \int_{\bar{r}_b}^x \sqrt{x^2 - \bar{r}^2} \frac{J_2(\bar{r} p)}{\bar{r}} d\bar{r} dp = \frac{\delta}{\pi(1 - \nu)} \left\{ \left[ y^2 \ln \frac{x + \sqrt{x^2 - \bar{r}_b^2}}{\bar{r}_b} - \frac{xy^2 \sqrt{x^2 - \bar{r}_b^2}}{\bar{r}_b^2} \right] + \left[ x^2 \ln \frac{x + \sqrt{x^2 - \bar{r}_b^2}}{\bar{r}_b} - x\sqrt{x^2 - \bar{r}_b^2} \right] \right.$$

$$\left. + \left[ \frac{xy}{\bar{r}_b^2} \sqrt{x^2 - \bar{r}_b^2} \sqrt{y^2 - \bar{r}_b^2} + (x^2 + y^2) \ln \frac{\bar{r}_b \sqrt{y^2 - x^2}}{x\sqrt{y^2 - \bar{r}_b^2} + y\sqrt{x^2 - \bar{r}_b^2}} + 2xy \ln \frac{\sqrt{y^2 - \bar{r}_b^2} + \sqrt{x^2 - \bar{r}_b^2}}{\sqrt{y^2 - x^2}} \right] \right\} \quad (45)$$

$$\begin{aligned} & \frac{2\delta}{\pi(1-\nu)} \int_0^\infty p \cos(py) \frac{d}{dx} \int_0^x \frac{\bar{r}H(\bar{r}-\bar{r}_b)w_2(\bar{r},p)}{\sqrt{x^2-\bar{r}^2}} d\bar{r}dp \\ &= -\frac{\delta}{\pi(1-\nu)} e_0(1+\nu_f) \\ & \times \left\{ \frac{y^2(1-\bar{r}_b^2)}{\bar{r}_b^2} + \ln \bar{r}_b - \frac{y\sqrt{y^2-\bar{r}_b^2}}{\bar{r}_b^2} - \ln \frac{\bar{r}_b}{y+\sqrt{y^2-\bar{r}_b^2}} \right\} f(x) \end{aligned} \quad (46)$$

in which

$$\begin{aligned} f(x) &= \frac{d}{dx} \int_{\bar{r}_b}^x \frac{\bar{r}}{\sqrt{x^2-\bar{r}^2}} \left\{ \ln \frac{\bar{r}}{\bar{r}_b} - \frac{\bar{r}^2-\bar{r}_b^2}{2\bar{r}_b^2} \right\} d\bar{r} \\ &= \ln \frac{x+\sqrt{x^2-\bar{r}_b^2}}{\bar{r}_b} - \frac{x\sqrt{x^2-\bar{r}_b^2}}{\bar{r}_b^2} \end{aligned} \quad (47)$$

$$\begin{aligned} & \frac{2\delta}{\pi(1-\nu)} \int_0^\infty p \cos(py) \frac{d}{dx} \int_0^x \frac{\bar{r}H(\bar{r}-\bar{r}_b)w_3(\bar{r},p)}{\sqrt{x^2-\bar{r}^2}} d\bar{r}dp \\ &= -\frac{2\delta}{\pi(1-\nu)} e_0 y^2 f(x) \end{aligned} \quad (48)$$

$$\begin{aligned} & \frac{2\delta}{\pi(1-\nu)} \int_0^\infty p \cos(py) \frac{d}{dx} \int_0^x \frac{\bar{r}H(\bar{r}-\bar{r}_b)w_4(\bar{r},p)}{\sqrt{x^2-\bar{r}^2}} d\bar{r}dp \\ &= \frac{\delta}{\pi(1-\nu)} e_0(1+\nu_f) \ln \bar{r}_b f(x) \end{aligned} \quad (49)$$

$$\begin{aligned} & \frac{2\delta}{\pi(1-\nu)} \int_0^\infty p \cos(py) \frac{d}{dx} \int_0^x \frac{\bar{r}H(\bar{r}-\bar{r}_b)w_5(\bar{r},p)}{\sqrt{x^2-\bar{r}^2}} d\bar{r}dp \\ &= \frac{\delta}{\pi(1-\nu)} \left\{ x\sqrt{x^2-\bar{r}_b^2} + x^2 \ln \frac{x-\sqrt{x^2-\bar{r}_b^2}}{\bar{r}_b} \right\} \end{aligned} \quad (50)$$

Substitution of Eqs. (45)–(50) in Eq. (31) finally yields the closed-form expression for the kernel  $K(x,y)$ , as shown in Eq. (32b) of the main text.

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