Electromagnetic fields induced by a cuboidal inclusion with uniform spontaneous polarization and magnetization

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Exact closed-form solutions are derived for the electromagnetic fields induced by a cuboidal inclusion with uniform spontaneous polarization and magnetization embedded in an infinite uniaxial multiferroic solid. The method of Green's function and direct integration are employed to obtain the solution, with the results for the induced electric and magnetic potentials as well as the electric and magnetic fields being in terms of elementary functions. It is observed that all the electric and magnetic field components exhibit logarithmic singularities near the eight corners of the cuboid. There also exist logarithmic singularities for the electric and magnetic fields near certain edges of the cuboid. Numerical results are presented for a typical multiferroic composite to demonstrate the variation of electromagnetic fields in and near the cuboidal inclusion. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2724728]

I. INTRODUCTION

Magnetoelectric multiferroic materials are compounds that can exhibit ferromagnetism and ferroelectricity simultaneously. In addition magnetic and electric polarizations can be strongly coupled in some magnetoelectric multiferroic materials. Recently Li and Li[1] derived magnetoelectric Green's functions for uniaxial multiferroic materials induced by a point electric or magnetic charge, which was utilized to determine the electromagnetic field in an ellipsoidal inclusion with uniform spontaneous polarization and magnetization. By virtue of the image method, the present authors[2] have recently derived explicit expressions for the magnetoelectric Green's functions in a uniaxial multiferroic half space and bimaterial space, induced by a point electric or magnetic charge.

This paper is focused on a cuboidal inclusion with uniform spontaneous polarization and magnetization, where the cuboid forms the basic building blocks for complex embedded structures found in many practical problems. It shall be mentioned that while closed-form solutions for the elastic problem of a cuboidal inclusion with uniform eigenstrains in an isotropic infinite space and half space have been pursued by many investigators,[3–9] the counterpart problem for the multiferroic material has not been solved yet. In order to find the cuboid-induced magnetoelectric field in a multiferroic material, we employ the exact closed-form Green's function solutions recently derived.[1,2] It is shown that exact closed-form solutions can also be derived for the electric and magnetic potentials as well as the electric and magnetic fields, induced by a cuboidal inclusion with uniform spontaneous polarization and magnetization embedded in an infinite uniaxial multiferroic material. Our solutions clearly demonstrate the continuous, discontinuous, and singular features of the induced fields at the corners and along the edges, which are also verified with numerical examples.

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II. BASIC FORMULATIONS

We consider a cuboidal inclusion V of dimensions $2a_1$, $2a_2$, and $2a_3$ along the x_1, x_2, and x_3 directions, respectively. The cuboid is centered at the origin of an infinite uniaxial multiferroic material having unique axis along the x_3 axis. The inclusion has the same dielectric permittivity constants κ_{11} and κ_{33}, magnetoelectric constants α_{11} and α_{33}, and magnetic permeability constants μ_{11} and μ_{33} as the matrix, but has spontaneous polarization P_i^s and magnetization M_i^s that are absent in the matrix.

As in Li and Li,[1] for the inclusion problem, the electric potential ϕ and magnetic potential ψ can be expressed in terms of the Green's functions via integration,

$$\begin{bmatrix}\phi(\mathbf{x})\\ \psi(\mathbf{x})\end{bmatrix} = -\int_V \begin{bmatrix} G_{\phi P} & G_{\phi M}\\ G_{\psi P} & G_{\psi M}\end{bmatrix}\begin{bmatrix}P_{i,i}^s\\ M_{i,i}^s\end{bmatrix} dV(\mathbf{x}') + \int_{\partial V}\begin{bmatrix} G_{\phi P} & G_{\phi M}\\ G_{\psi P} & G_{\psi M}\end{bmatrix}\begin{bmatrix}P_i^s n_i\\ M_i^s n_i\end{bmatrix} dS(\mathbf{x}'), \tag{1}$$

where the first term is due to the volume charges in V, with the differentiation taken with respect to the x_i'—the source point in the magnetoelectric Green's functions $G_{\phi P}$, $G_{\phi M}$, $G_{\psi P}$, and $G_{\psi M}$ for a uniaxial multiferroic material; the second term is due to the interface charge on ∂V. If we further assume that the spontaneous polarization P_i^s and magnetization M_i^s are uniform within the cuboidal inclusion, then Eq. (1) can be simplified as

$$\begin{bmatrix}\phi(\mathbf{x})\\ \psi(\mathbf{x})\end{bmatrix} = \int_{\partial V}\begin{bmatrix} G_{\phi P} & G_{\phi M}\\ G_{\psi P} & G_{\psi M}\end{bmatrix}\begin{bmatrix}P_i^s n_i\\ M_i^s n_i\end{bmatrix} dS(\mathbf{x}'), \tag{2}$$

which means that only the surface integrals on ∂V need to be carried out in order to determine the induced electric and magnetic potentials.

The involved Green's functions $G_{\alpha\beta}$ have been recently rederived, and they can be expressed in a simple and exact closed form as[2]

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$$4\pi G_{\phi P} = \frac{(\alpha_{11}-\lambda_1\alpha_{33})^2}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+\lambda_1 x_3^2}} + \frac{(\alpha_{11}-\lambda_2\alpha_{33})^2}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+\lambda_2 x_3^2}},$$

$$4\pi G_{\phi M} = 4\pi G_{\psi P} = \frac{(-\alpha_{11}+\lambda_1\alpha_{33})(\kappa_{11}-\lambda_1\kappa_{33})}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+\lambda_1 x_3^2}} + \frac{(-\alpha_{11}+\lambda_2\alpha_{33})(\kappa_{11}-\lambda_2\kappa_{33})}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+\lambda_2 x_3^2}},$$

$$4\pi G_{\psi M} = \frac{(\kappa_{11}-\lambda_1\kappa_{33})^2}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+\lambda_1 x_3^2}} + \frac{(\kappa_{11}-\lambda_2\kappa_{33})^2}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+\lambda_2 x_3^2}}, \tag{3}$$

where

$$\lambda_1 = \frac{\mu_{33}\kappa_{11}+\mu_{11}\kappa_{33}-2\alpha_{11}\alpha_{33}+\sqrt{(\mu_{11}\kappa_{33}-\mu_{33}\kappa_{11})^2+4(\alpha_{11}\mu_{33}-\alpha_{33}\mu_{11})(\alpha_{11}\kappa_{33}-\alpha_{33}\kappa_{11})}}{2(\mu_{33}\kappa_{33}-\alpha_{33}^2)},$$

$$\lambda_2 = \frac{\mu_{33}\kappa_{11}+\mu_{11}\kappa_{33}-2\alpha_{11}\alpha_{33}-\sqrt{(\mu_{11}\kappa_{33}-\mu_{33}\kappa_{11})^2+4(\alpha_{11}\mu_{33}-\alpha_{33}\mu_{11})(\alpha_{11}\kappa_{33}-\alpha_{33}\kappa_{11})}}{2(\mu_{33}\kappa_{33}-\alpha_{33}^2)}, \tag{4}$$

$$\delta_1 = \alpha_{11}^2\kappa_{33}+\kappa_{11}^2\mu_{33}-2\alpha_{11}\alpha_{33}\kappa_{11}+(\mu_{33}\kappa_{33}-\alpha_{33}^2)(\lambda_1^2\kappa_{33}-2\lambda_1\kappa_{11}),$$

$$\delta_2 = \alpha_{11}^2\kappa_{33}+\kappa_{11}^2\mu_{33}-2\alpha_{11}\alpha_{33}\kappa_{11}+(\mu_{33}\kappa_{33}-\alpha_{33}^2)(\lambda_2^2\kappa_{33}-2\lambda_2\kappa_{11}). \tag{5}$$

III. ANALYTICAL RESULTS

A. Electric and magnetic potentials

The surface integrals in Eq. (2) can indeed be explicitly performed. The integrations in Eq. (2) are accomplished using integral forms (A1) and (A2) in the Appendix. Here we ignore the intermediate steps of integrations and lengthy mathematical manipulations, and only list the final results for the induced electric and magnetic potentials,

$$\phi = \sum_{i=1}^{3}[m_{i1}h_i(x_1,x_2,x_3,\lambda_1)+m_{i2}h_i(x_1,x_2,x_3,\lambda_2)],$$

$$\psi = \sum_{i=1}^{3}[q_{i1}h_i(x_1,x_2,x_3,\lambda_1)+q_{i2}h_i(x_1,x_2,x_3,\lambda_2)], \tag{6}$$

where the constants m_{ij} and q_{ij} ($i=1,2,3$, $j=1,2$) are given by

$$\left.\begin{aligned}
m_{i1} &= \frac{M_i^s(\alpha_{11}-\lambda_1\alpha_{33})(\kappa_{11}-\lambda_1\kappa_{33})-P_i^s(\alpha_{11}-\lambda_1\alpha_{33})^2}{4\pi\delta_1\lambda_1}\\
m_{i2} &= \frac{M_i^s(\alpha_{11}-\lambda_2\alpha_{33})(\kappa_{11}-\lambda_2\kappa_{33})-P_i^s(\alpha_{11}-\lambda_2\alpha_{33})^2}{4\pi\delta_2\lambda_2}\\
q_{i1} &= \frac{P_i^s(\alpha_{11}-\lambda_1\alpha_{33})(\kappa_{11}-\lambda_1\kappa_{33})-M_i^s(\kappa_{11}-\lambda_1\kappa_{33})^2}{4\pi\delta_1\lambda_1}\\
q_{i2} &= \frac{P_i^s(\alpha_{11}-\lambda_2\alpha_{33})(\kappa_{11}-\lambda_2\kappa_{33})-M_i^s(\kappa_{11}-\lambda_2\kappa_{33})^2}{4\pi\delta_2\lambda_2}
\end{aligned}\right\} \quad (i=1-2), \tag{7a}$$

$$\left.\begin{aligned}
m_{i1} &= \frac{M_i^s(\alpha_{11}-\lambda_1\alpha_{33})(\kappa_{11}-\lambda_1\kappa_{33})-P_i^s(\alpha_{11}-\lambda_1\alpha_{33})^2}{4\pi\delta_1\sqrt{\lambda_1}}\\
m_{i2} &= \frac{M_i^s(\alpha_{11}-\lambda_2\alpha_{33})(\kappa_{11}-\lambda_2\kappa_{33})-P_i^s(\alpha_{11}-\lambda_2\alpha_{33})^2}{4\pi\delta_2\sqrt{\lambda_2}}\\
q_{i1} &= \frac{P_i^s(\alpha_{11}-\lambda_1\alpha_{33})(\kappa_{11}-\lambda_1\kappa_{33})-M_i^s(\kappa_{11}-\lambda_1\kappa_{33})^2}{4\pi\delta_1\sqrt{\lambda_1}}\\
q_{i2} &= \frac{P_i^s(\alpha_{11}-\lambda_2\alpha_{33})(\kappa_{11}-\lambda_2\kappa_{33})-M_i^s(\kappa_{11}-\lambda_2\kappa_{33})^2}{4\pi\delta_2\sqrt{\lambda_2}}
\end{aligned}\right\} \quad (i=3) \tag{7b}$$

and the functions $h_i(x_1,x_2,x_3,\lambda)$ $(i=1,2,3)$ are defined by

$$h_1(x_1,x_2,x_3,\lambda) = \sum_i \sum_j \sum_k (-1)^{i+j+k}\Bigg\{ \bar{y}_j \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{z}_k] + \bar{z}_k \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{y}_j] + \bar{x}_i \tan^{-1} \frac{\bar{y}_j\bar{z}_k}{\bar{x}_i(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2}} \Bigg\},$$

$$h_2(x_1,x_2,x_3,\lambda) = \sum_i \sum_j \sum_k (-1)^{i+j+k}\Bigg\{ \bar{x}_i \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{z}_k] + \bar{z}_k \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{x}_i] + \bar{y}_j \tan^{-1} \frac{\bar{x}_i\bar{z}_k}{\bar{y}_j(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2}} \Bigg\},$$

$$h_3(x_1,x_2,x_3,\lambda) = \sum_i \sum_j \sum_k (-1)^{i+j+k}\Bigg\{ \bar{x}_i \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{y}_j] + \bar{y}_j \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{x}_i] + \bar{z}_k \tan^{-1} \frac{\bar{x}_i\bar{y}_j}{\bar{z}_k(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2}} \Bigg\}, \quad (8)$$

where the summations for i, j, and k are from 1 to 2, and

$$\bar{x}_1 = x_1 + a_1, \quad \bar{x}_2 = x_1 - a_1,$$

$$\bar{y}_1 = x_2 + a_2, \quad \bar{y}_2 = x_2 - a_2,$$

$$\bar{z}_1 = \sqrt{\lambda}(x_3 + a_3), \quad \bar{z}_2 = \sqrt{\lambda}(x_3 - a_3). \quad (9)$$

It shall be noticed that for the counterpart elastic problem of a cuboidal inclusion with uniform eigenstrains embedded in an infinite isotropic elastic matrix, only explicit expressions for the displacement gradient[3,4] or the stress components[6,7] were obtained previously. It was Glas[8] who derived the analytical solution of the displacement in terms of elementary functions. Here we have obtained the explicit and closed-form expressions of the electric and magnetic potentials induced by a cuboidal inclusion with uniform spontaneous polarization and magnetization. It is of interest to point out that the elementary functions in our expressions are similar to those adopted by Glas.[8] It is observed that even though there exist logarithmic terms in the expressions of electric and magnetic potentials, these potentials are actually *regular* everywhere. It can also be found that the electric and magnetic potentials are further continuous across the inclusion-matrix interface.

B. Electric and magnetic fields

Differentiating Eq. (6) with respect to $\boldsymbol{x}$ yields the electric field $E_j=-\partial\phi/\partial x_j$ and magnetic field $H_j=-\partial\psi/\partial x_j$ as follows:

$$\left.\begin{aligned} E_j &= -\sum_{i=1}^{3}\left[m_{i1}\frac{\partial h_i(x_1,x_2,x_3,\lambda_1)}{\partial x_j} + m_{i2}\frac{\partial h_i(x_1,x_2,x_3,\lambda_2)}{\partial x_j}\right] \\ H_j &= -\sum_{i=1}^{3}\left[q_{i1}\frac{\partial h_i(x_1,x_2,x_3,\lambda_1)}{\partial x_j} + q_{i2}\frac{\partial h_i(x_1,x_2,x_3,\lambda_2)}{\partial x_j}\right] \end{aligned}\right\} \quad (j = 1-3), \quad (10)$$

with

$$\frac{\partial h_1(x_1,x_2,x_3,\lambda)}{\partial x_1} = \sum_i \sum_j \sum_k (-1)^{i+j+k}\left[\frac{\bar{x}_i\bar{y}_j}{\bar{x}_i^2 + \bar{y}_j^2} + \frac{\bar{x}_i\bar{z}_k}{\bar{x}_i^2 + \bar{z}_k^2} + \tan^{-1}\frac{\bar{y}_j\bar{z}_k}{\bar{x}_i(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2}}\right], \quad (11a)$$

$$\frac{\partial h_1(x_1,x_2,x_3,\lambda)}{\partial x_2} = \sum_i \sum_j \sum_k (-1)^{i+j+k}\left\{ \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{z}_k] + \frac{\bar{y}_j^2}{\bar{x}_i^2 + \bar{y}_j^2}\right\}, \quad (11b)$$

$$\frac{\partial h_1(x_1,x_2,x_3,\lambda)}{\partial x_3} = \sum_i \sum_j \sum_k (-1)^{i+j+k}\left\{ \sqrt{\lambda}\ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{y}_j] + \frac{\sqrt{\lambda}\bar{z}_k^2}{\bar{x}_i^2 + \bar{z}_k^2}\right\}, \quad (11c)$$

$$\frac{\partial h_2(x_1,x_2,x_3,\lambda)}{\partial x_1} = \sum_i \sum_j \sum_k (-1)^{i+j+k}\left\{ \ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{z}_k] + \frac{\bar{x}_i^2}{\bar{x}_i^2 + \bar{y}_j^2}\right\}, \quad (12a)$$

$$\frac{\partial h_2(x_1,x_2,x_3,\lambda)}{\partial x_2} = \sum_i \sum_j \sum_k (-1)^{i+j+k}\left[\frac{\bar{x}_i\bar{y}_j}{\bar{x}_i^2 + \bar{y}_j^2} + \frac{\bar{y}_j\bar{z}_k}{\bar{y}_j^2 + \bar{z}_k^2} + \tan^{-1}\frac{\bar{x}_i\bar{z}_k}{\bar{y}_j(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2}}\right], \quad (12b)$$

$$\frac{\partial h_2(x_1,x_2,x_3,\lambda)}{\partial x_3} = \sum_i \sum_j \sum_k (-1)^{i+j+k}\left\{ \sqrt{\lambda}\ln[(\bar{x}_i^2 + \bar{y}_j^2 + \bar{z}_k^2)^{1/2} - \bar{x}_i] + \frac{\sqrt{\lambda}\bar{z}_k^2}{\bar{y}_j^2 + \bar{z}_k^2}\right\}, \quad (12c)$$

$$\frac{\partial h_3(x_1,x_2,x_3,\lambda)}{\partial x_1}=\sum_i\sum_j\sum_k(-1)^{i+j+k}\left\{\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_k^2)^{1/2}-\bar{y}_j]+\frac{\bar{x}_i^2}{\bar{x}_i^2+\bar{z}_k^2}\right\}, \tag{13a}$$

$$\frac{\partial h_3(x_1,x_2,x_3,\lambda)}{\partial x_2}=\sum_i\sum_j\sum_k(-1)^{i+j+k}\left\{\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_k^2)^{1/2}-\bar{x}_i]+\frac{\bar{y}_j^2}{\bar{y}_j^2+\bar{z}_k^2}\right\}, \tag{13b}$$

$$\frac{\partial h_3(x_1,x_2,x_3,\lambda)}{\partial x_3}=\sum_i\sum_j\sum_k(-1)^{i+j+k}\left[\frac{\sqrt{\lambda}\bar{x}_i\bar{z}_k}{\bar{x}_i^2+\bar{z}_k^2}+\frac{\sqrt{\lambda}\bar{y}_j\bar{z}_k}{(\bar{y}_j^2+\bar{z}_k^2)}+\sqrt{\lambda}\tan^{-1}\frac{\bar{x}_i\bar{y}_j}{\bar{z}_k(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_k^2)^{1/2}}\right]. \tag{13c}$$

Therefore, the explicit expressions for the electric and magnetic fields are as follows:

$$\begin{aligned}E_1=&-\sum_i\sum_j\sum_k(-1)^{i+j+k}\Bigg\{m_{21}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{z}_{k1}]+m_{31}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{y}_j]+m_{22}\ln[(\bar{x}_i^2+\bar{y}_j^2+\hat{z}_{k2}^2)^{1/2}-\bar{z}_{k2}]\\&+m_{32}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{y}_j]+\frac{\bar{x}_i[(m_{21}+m_{22})\bar{x}_i+(m_{11}+m_{12})\bar{y}_j]}{\bar{x}_i^2+\bar{y}_j^2}+\frac{\bar{x}_i(m_{31}\bar{x}_i+m_{11}\bar{z}_{k1})}{\bar{x}_i^2+\bar{z}_{k1}^2}+\frac{\bar{x}_i(m_{32}\bar{x}_i+m_{12}\bar{z}_{k2})}{\bar{x}_i^2+\bar{z}_{k2}^2}\\&+m_{11}\tan^{-1}\frac{\bar{y}_j\bar{z}_{k1}}{\bar{x}_i(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}}+m_{12}\tan^{-1}\frac{\bar{y}_j\bar{z}_{k2}}{\bar{x}_i(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}}\Bigg\},\end{aligned} \tag{14a}$$

$$\begin{aligned}E_2=&-\sum_i\sum_j\sum_k(-1)^{i+j+k}\Bigg\{m_{11}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{z}_{k1}]+m_{31}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{x}_i]+m_{12}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{z}_{k2}]\\&+m_{32}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{x}_i]+\frac{\bar{y}_j[(m_{21}+m_{22})\bar{x}_i+(m_{11}+m_{12})\bar{y}_j]}{\bar{x}_i^2+\bar{y}_j^2}+\frac{\bar{y}_j(m_{31}\bar{y}_j+m_{21}\bar{z}_{k1})}{\bar{y}_j^2+\bar{z}_{k1}^2}+\frac{\bar{y}_j(m_{32}\bar{y}_j+m_{22}\bar{z}_{k2})}{\bar{y}_j^2+\bar{z}_{k2}^2}\\&+m_{21}\tan^{-1}\frac{\bar{x}_i\bar{z}_{k1}}{\bar{y}_j(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}}+m_{22}\tan^{-1}\frac{\bar{x}_i\bar{z}_{k2}}{\bar{y}_j(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}}\Bigg\},\end{aligned} \tag{14b}$$

$$\begin{aligned}E_3=&-\sum_i\sum_j\sum_k(-1)^{i+j+k}\Bigg\{m_{11}\sqrt{\lambda_1}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{y}_j]+m_{21}\sqrt{\lambda_1}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{x}_i]+m_{12}\sqrt{\lambda_2}\ln[(\bar{x}_i^2+\bar{y}_j^2\\&+\bar{z}_{k2}^2)^{1/2}-\bar{y}_j]+m_{22}\sqrt{\lambda_2}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{x}_i]+\frac{\sqrt{\lambda_1}\bar{z}_{k1}(m_{31}\bar{x}_i+m_{11}\bar{z}_{k1})}{\bar{x}_i^2+\bar{z}_{k1}^2}+\frac{\sqrt{\lambda_1}\bar{z}_{k1}(m_{31}\bar{y}_j+m_{21}\bar{z}_{k1})}{\bar{y}_j^2+\bar{z}_{k1}^2}\\&+\frac{\sqrt{\lambda_2}\bar{z}_{k2}(m_{32}\bar{x}_i+m_{12}\bar{z}_{k2})}{\bar{x}_i^2+\bar{z}_{k2}^2}+\frac{\sqrt{\lambda_2}\bar{z}_{k2}(m_{32}\bar{y}_j+m_{22}\bar{z}_{k2})}{\bar{y}_j^2+\bar{z}_{k2}^2}+m_{31}\sqrt{\lambda_1}\tan^{-1}\frac{\bar{x}_i\bar{y}_j}{\bar{z}_{k1}(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}}\\&+m_{32}\sqrt{\lambda_2}\tan^{-1}\frac{\bar{x}_i\bar{y}_j}{\bar{z}_{k2}(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}}\Bigg\},\end{aligned} \tag{14c}$$

$$\begin{aligned}H_1=&-\sum_i\sum_j\sum_k(-1)^{i+j+k}\Bigg\{q_{21}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{z}_{k1}]+q_{31}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{y}_j]+q_{22}\ln[(\bar{x}_i^2+\bar{y}_j^2+\hat{z}_{k2}^2)^{1/2}-\bar{z}_{k2}]\\&+q_{32}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{y}_j]+\frac{\bar{x}_i[(q_{21}+q_{22})\bar{x}_i+(q_{11}+q_{12})\bar{y}_j]}{\bar{x}_i^2+\bar{y}_j^2}+\frac{\bar{x}_i(q_{31}\bar{x}_i+q_{11}\bar{z}_{k1})}{\bar{x}_i^2+\bar{z}_{k1}^2}+\frac{\bar{x}_i(q_{32}\bar{x}_i+q_{12}\bar{z}_{k2})}{\bar{x}_i^2+\bar{z}_{k2}^2}\\&+q_{11}\tan^{-1}\frac{\bar{y}_j\bar{z}_{k1}}{\bar{x}_i(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}}+q_{12}\tan^{-1}\frac{\bar{y}_j\bar{z}_{k2}}{\bar{x}_i(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}}\Bigg\},\end{aligned} \tag{15a}$$

$$\begin{aligned}H_2=&-\sum_i\sum_j\sum_k(-1)^{i+j+k}\Bigg\{q_{11}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{z}_{k1}]+q_{31}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{x}_i]+q_{12}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{z}_{k2}]\\&+q_{32}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{x}_i]+\frac{\bar{y}_j[(q_{21}+q_{22})\bar{x}_i+(q_{11}+q_{12})\bar{y}_j]}{\bar{x}_i^2+\bar{y}_j^2}+\frac{\bar{y}_j(q_{31}\bar{y}_j+q_{21}\bar{z}_{k1})}{\bar{y}_j^2+\bar{z}_{k1}^2}+\frac{\bar{y}_j(q_{32}\bar{y}_j+q_{22}\bar{z}_{k2})}{\bar{y}_j^2+\bar{z}_{k2}^2}\\&+q_{21}\tan^{-1}\frac{\bar{x}_i\bar{z}_{k1}}{\bar{y}_j(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}}+q_{22}\tan^{-1}\frac{\bar{x}_i\bar{z}_{k2}}{\bar{y}_j(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}}\Bigg\},\end{aligned} \tag{15b}$$

$$H_3 = -\sum_i \sum_j \sum_k (-1)^{i+j+k} \Bigg\{ q_{11}\sqrt{\lambda_1}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{y}_j] + q_{21}\sqrt{\lambda_1}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}-\bar{x}_i] + q_{12}\sqrt{\lambda_2}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2} - \bar{y}_j] + q_{22}\sqrt{\lambda_2}\ln[(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}-\bar{x}_i] + \frac{\sqrt{\lambda_1}\bar{z}_{k1}(q_{31}\bar{x}_i+q_{11}\bar{z}_{k1})}{\bar{x}_i^2+\bar{z}_{k1}^2} + \frac{\sqrt{\lambda_1}\bar{z}_{k1}(q_{31}\bar{y}_j+q_{21}\bar{z}_{k1})}{\bar{y}_j^2+\bar{z}_{k1}^2} + \frac{\sqrt{\lambda_2}\bar{z}_{k2}(q_{32}\bar{x}_i+q_{12}\bar{z}_{k2})}{\bar{x}_i^2+\bar{z}_{k2}^2} + \frac{\sqrt{\lambda_2}\bar{z}_{k2}(q_{32}\bar{y}_j+q_{22}\bar{z}_{k2})}{\bar{y}_j^2+\bar{z}_{k2}^2} + q_{31}\sqrt{\lambda_1}\tan^{-1}\frac{\bar{x}_i\bar{y}_j}{\bar{z}_{k1}(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k1}^2)^{1/2}} + q_{32}\sqrt{\lambda_2}\tan^{-1}\frac{\bar{x}_i\bar{y}_j}{\bar{z}_{k2}(\bar{x}_i^2+\bar{y}_j^2+\bar{z}_{k2}^2)^{1/2}} \Bigg\}. \tag{15c}$$

In Eqs. (14) and (15),

$$\bar{z}_{11} = \sqrt{\lambda_1}(x_3+a_3), \quad \bar{z}_{21} = \sqrt{\lambda_1}(x_3-a_3),$$

$$\bar{z}_{12} = \sqrt{\lambda_2}(x_3+a_3), \quad \bar{z}_{22} = \sqrt{\lambda_2}(x_3-a_3). \tag{16}$$

It can be observed from the above expressions for the electric and magnetic fields that all the electric and magnetic field components exhibit logarithmic singularities near the eight corners of the cuboidal inclusion. Furthermore, the electric and magnetic fields also exhibit logarithmic singularities near certain edges of the cuboid. The logarithmic singular behaviors of electric and magnetic fields are similar to those of the stresses for a cuboidal inclusion with uniform eigenstrains.[3,6,7] Unlike the case of an ellipsoidal inclusion,[1] the electric and magnetic fields inside a cuboidal inclusion are intrinsically nonuniform. The following can be observed from Eqs. (14) and (15)

(1) On the inclusion-matrix interfaces $x_1 = \pm a_1$, the tangential components E_2 and E_3 and H_2 and H_3 are continuous, while the normal components E_1 and H_1 are discontinuous. More specifically the jumps in E_1 and H_1, which are only caused by polarization P_1^s and magnetization M_1^s, are given by

$$\left.\begin{aligned} E_1^{\text{inclusion}} - E_1^{\text{matrix}} &= 4\pi(m_{11}+m_{12}) \\ H_1^{\text{inclusion}} - H_1^{\text{matrix}} &= 4\pi(q_{11}+q_{12}) \end{aligned}\right\} \quad \text{on } x_1 = \pm a_1. \tag{17}$$

(2) On the interfaces $x_2 = \pm a_2$, the tangential components E_1 and E_3 and H_1 and H_3 are continuous, while the normal components E_2 and H_2 are discontinuous. More specifically the jumps in E_2 and H_2, which are only caused by polarization P_2^s and magnetization M_2^s, are given by

$$\left.\begin{aligned} E_2^{\text{inclusion}} - E_2^{\text{matrix}} &= 4\pi(m_{21}+m_{22}) \\ H_2^{\text{inclusion}} - H_2^{\text{matrix}} &= 4\pi(q_{21}+q_{22}) \end{aligned}\right\} \quad \text{on } x_2 = \pm a_2. \tag{18}$$

(3) On the interfaces $x_3 = \pm a_3$, the tangential components E_1 and E_2 and H_1 and H_2 are continuous, while the normal components E_3 and H_3 are discontinuous. More specifically the jumps in E_3 and H_3, which are only caused by polarization P_3^s and magnetization M_3^s, are given by

$$\left.\begin{aligned} E_3^{\text{inclusion}} - E_3^{\text{matrix}} &= 4\pi(m_{31}\sqrt{\lambda_1}+m_{32}\sqrt{\lambda_2}) \\ H_3^{\text{inclusion}} - H_3^{\text{matrix}} &= 4\pi(q_{31}\sqrt{\lambda_1}+q_{32}\sqrt{\lambda_2}) \end{aligned}\right\} \quad \text{on } x_3 = \pm a_3. \tag{19}$$

It is further observed from Eqs. (17)–(19) that the jumps in normal electric and magnetic fields are independent of the cuboidal geometry; they are only functions of the multiferroic material property and polarization/magnetization.

IV. NUMERICAL RESULTS

We consider a cuboid with sizes $a_1 = a_2 = \alpha a_3$, where α is defined as the shape aspect ratio of the cuboidal inclusion. The constitutive moduli of the multiferroic material are[1] the magnetoelectric coefficient $\alpha_{11}=5$ and $\alpha_{33}=3$ ($\times 10^{-12}$ N s/V C), the dielectric permittivity $\kappa_{11}=8$ and $\kappa_{33}=9.3$ ($\times 10^{-11}$ C^2/N m^2), and the magnetic permeability $\mu_{11}=5.9$ and $\mu_{33}=1.57$ ($\times 10^{-4}$ N s^2/C^2).

Shown in Figs. 1–4 are the distributions of electric potential ϕ, magnetic potential ψ, electric field E_3, and magnetic field H_3 along the x_3 axis induced by a uniform unit spontaneous polarization P_3^s for three different values of the aspect ratio $\alpha = 1/5, 1, 5$ of the cuboid. We also show in Figs. 5 and 6 the distributions of the electric field E_3 and magnetic field H_3 along the x_3 axis by a uniform unit magnetization M_3^s for four different values of the aspect ratio α

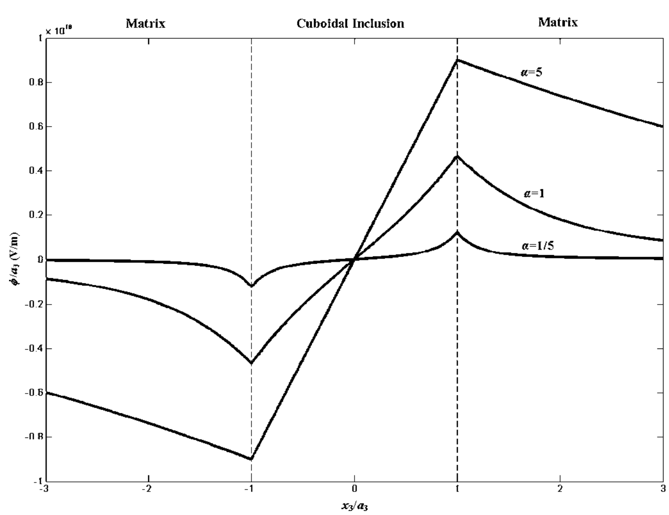

FIG. 1. Distribution of the electric potential along the x_3 axis for three different aspect ratios of the cuboid $\alpha = 1/5, 1, 5$. The cuboid is charged by a uniform unit polarization P_3^s.

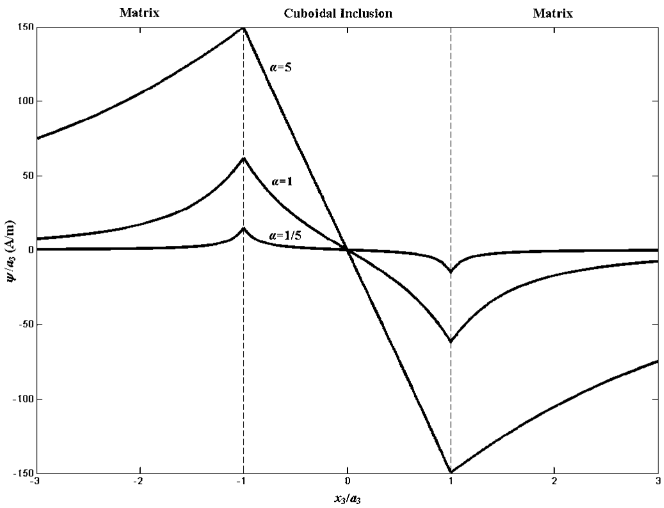

FIG. 2. Distribution of the magnetic potential along the x_3 axis for three different aspect ratios of the cuboid $\alpha=1/5,1,5$. The cuboid is charged by a uniform unit polarization P_3^s.

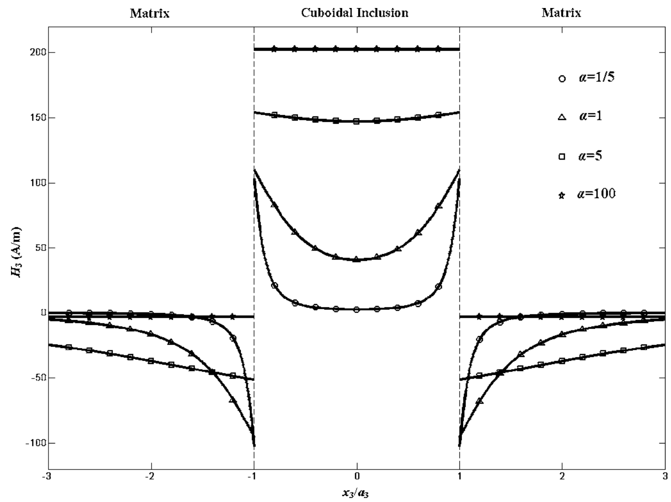

FIG. 4. Distribution of the magnetic field H_3 along the x_3 axis for four different aspect ratios of the cuboid $\alpha=1/5,1,5,100$. The cuboid is charged by a uniform unit polarization P_3^s.

$=1/5,1,5,100$. It is clearly observed from Figs. 1 and 2 that the electric and magnetic potentials are continuous across the inclusion-matrix interfaces $x_3=\pm a_3$. On the other hand, it is seen from Figs. 3–6 that the normal electric and magnetic fields E_3 and H_3 are discontinuous across the interfaces $x_3=\pm a_3$. In addition the jumps in the electric and magnetic fields across the two interfaces are exactly the values predicted by Eq. (19). Also found from Figs. 3–6 is that the electric and magnetic fields are nonuniform inside the cuboidal inclusion. As the value of the aspect ratio α decreases, the nonuniformity in the electric and magnetic fields within the cuboidal inclusion becomes apparent, with the magnitude of the electric and magnetic fields inside the inclusion decreasing. By comparing our results with those of Li and Li,[1] it is found that for the same aspect ratio α, the magnitudes of the electric and magnetic fields inside a cuboidal inclusion are larger than those inside an ellipsoidal inclusion. However, when the aspect ratio is very large, say $\alpha=100$, the electric and magnetic fields along the x_3 axis become uniform inside the cuboidal inclusion and approach zero outside the inclusion, with the nonzero values inside the inclusion very close to those in Li and Li,[1] for the corresponding ellipsoidal case.

Comparison of Fig. 4 with Fig. 6 and Fig. 3 with Fig. 5 reveals that the difference between the magnetic fields induced by a uniform unit magnetization and polarization is much smaller than the difference between the electric fields induced by a uniform unit polarization and magnetization. This result is consistent with that for an ellipsoidal inclusion.[1] Also, our results show that it is easier to manipulate polarization by a magnetic field than to manipulate magnetization by an electric field.

V. CONCLUSIONS

The electromagnetic fields induced by a uniform spontaneous polarization and magnetization within a cuboidal region of an infinite uniaxial multiferroic material are obtained by means of the Green's function and direct integration

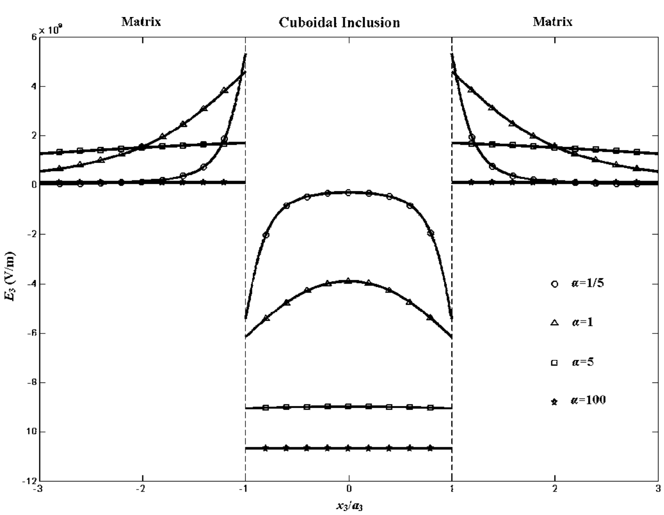

FIG. 3. Distribution of the electric field E_3 along the x_3 axis for four different aspect ratios of the cuboid $\alpha=1/5,1,5,100$. The cuboid is charged by a uniform unit polarization P_3^s.

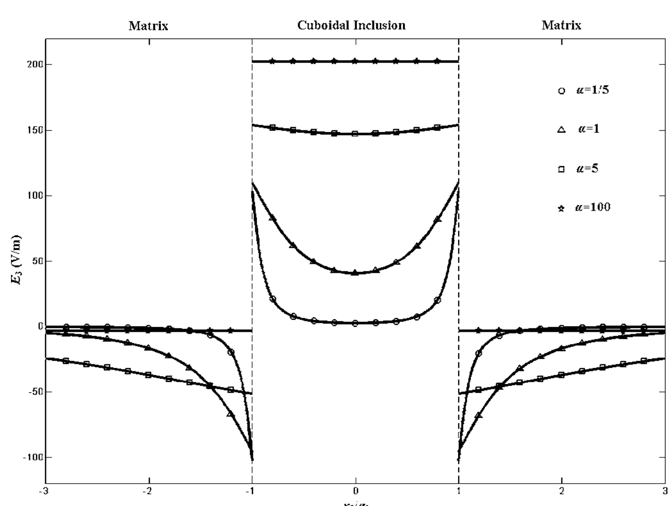

FIG. 5. Distribution of the electric field E_3 along the x_3 axis for four different aspect ratios of the cuboid $\alpha=1/5,1,5,100$. The cuboid is charged by a uniform unit magnetization M_3^s.

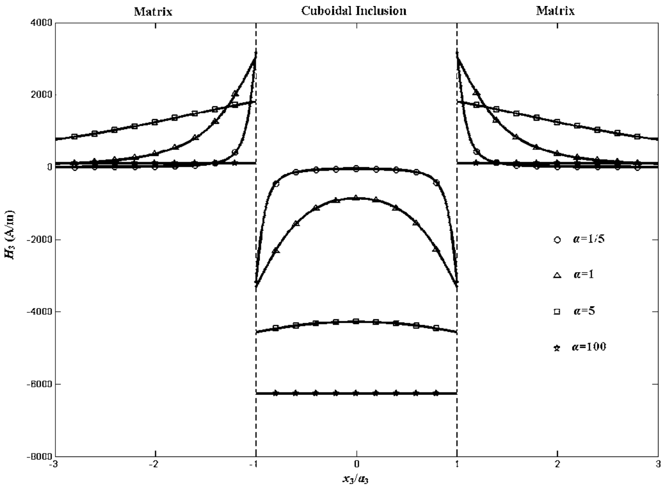

FIG. 6. Distribution of the magnetic field H_3 along the x_3 axis for four different aspect ratios of the cuboid $\alpha=1/5,1,5,100$. The cuboid is charged by a uniform unit magnetization M_3^s.

methods. Exact closed-form expressions are obtained for (1) the electric and magnetic potentials [Eqs. (6)–(9)] and (2) the electric and magnetic fields [Eqs. (14) and (15)]. It is observed from the derived solutions that the electric and magnetic potentials are regular everywhere, while the electric and magnetic fields exhibit logarithmic singularities near the corners and also near certain edges of the cuboid. Numerical results are presented to show the distribution of the electric and magnetic potentials as well as electric and magnetic fields, demonstrating further the discontinuity and singularity features of the field quantity near the edges. Due to the fact that explicit expressions of the Green's functions for a uniaxial multiferroic half space and bimaterial have also been obtained,[2] it becomes possible to investigate a cuboidal inclusion embedded in a uniaxial multiferroic half space or bimaterial.

ACKNOWLEDGMENTS

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APPENDIX

The following are the integrals used in the integration of Eq. (2):

$$\int \frac{1}{\sqrt{\zeta^2+c}} d\zeta = \ln(\zeta+\sqrt{\zeta^2+c}), \tag{A1}$$

$$\int \ln(b+\sqrt{x^2+b^2+\zeta^2}) d\zeta = \zeta \ln(b+\sqrt{x^2+b^2+\zeta^2}) + b \ln(\zeta+\sqrt{x^2+b^2+\zeta^2}) - \zeta + x \tan^{-1}\frac{\zeta}{x} - x \tan^{-1}\frac{b\zeta}{x\sqrt{x^2+b^2+\zeta^2}}. \tag{A2}$$

The integral (A2) is taken from Ref. 10 with the printing error in the last term on the right-hand side of the equation (p. 57, Ref. 10) being corrected.

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