Electromagnetic fields induced by a point source in a uniaxial multiferroic full-space, half-space, and bimaterial space

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Multiferroic magnetoelectrics are materials that are both ferroelectric and ferromagnetic in the same phase. In addition, electric and magnetic polarizations are strongly coupled in some magnetoelectric multiferroic materials. In this work, by virtue of the image method, exact closed-form Green's functions are derived for a uniaxial multiferroic full-space, half-space, and bimaterial space. While for the bimaterial space case the interface is assumed to be perfect, for the half-space case four different sets of surface conditions are considered. The point source can be either an electric or a magnetic charge. Numerical results are presented to demonstrate the differences among the infinite-space, half-space, and bimaterial space Green's functions.

I. INTRODUCTION

Magnetoelectric multiferroic materials are novel compounds that exhibit both ferromagnetism and ferroelectricity simultaneously. In addition, magnetic and electric polarizations are strongly coupled in some magnetoelectric multiferroic materials. These materials may exhibit a spontaneous magnetic polarization that can be switched with an electric field and/or a spontaneous electric polarization that can be switched with a magnetic field. Possible applications of magnetoelectric multiferroics include multiple-state memory elements, giant magnetic resistance devices, electric-field-controlled ferromagnetic-resonance devices, and variable transducers with either magnetically modulated piezoelectricity or electrically modulated piezomagnetism. Recently, various aspects of the fundamental physics of multiferroic materials, including first principles[1–4] and micromechanics,[5–7] were investigated. In terms of fundamental/analytical solutions, while the bimaterial Green's functions in anisotropic and fully coupled magneto-electro-elastic materials were derived by virtue of the Stroh formalism,[8] a three-dimensional (3D) solution of plates made of multiferroic composites,[9] was also available.

In a more recent study, Li and Li[10] derived the explicit Green's functions for a uniaxial multiferroic material full-space induced by a point-electric or magnetic charge, and then used the Green's functions to determine the electromagnetic fields in an ellipsoidal inclusion with spontaneous polarization and magnetization embedded in a multiferroic material.

As the fundamental solutions to various governing systems of equations, Green's functions represent the very basic relation between the response at the field point and the excitation at the source point. Therefore, they can be directly applied to the analysis of material behaviors in certain simple cases or used as kernel functions in a boundary integral equation for analyzing more complicated problems. Green's functions are also fundamental for the prediction of the effective material properties in composites such as the multiferroic material composites.

Because a multiferroic composite would mostly contain interfaces and/or surfaces, a Green's function solution in the corresponding bimaterial space and half-space is important for analyzing the potential effect of the interface and/or surface. Therefore, in this article, we will determine the electromagnetic fields in a uniaxial multiferroic bimaterial or in a uniaxial multiferroic half-space induced by a point-electric or magnetic charge. Our article is arranged as follows: In Sec. II, we first present a different approach to derive the Green's functions for a multiferroic material full-space. In Li and Li,[10] a magnetoelectric potential function, g, which satisfies a dual-harmonic function, is introduced. In our approach, through the introduction of two new functions, the original coupled governing equations for electric and magnetic potentials can be decoupled into two inhomogeneous Laplace equations for the two newly introduced functions, the Green's function solutions of which are well known. As a result, our approach is direct and simple in deriving the Green's functions. In Sec. III, the Green's functions for a uniaxial multiferroic bimaterial

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are derived by means of the method of image.[11] Our results show that one only needs to invert a single 4×4 matrix to find the eight unknown constants appearing in the expressions of magnetoelectric Green's functions for the multiferroic bimaterial. In Sec. IV, the Green's functions for a uniaxial multiferroic half-space with four different sets of boundary conditions on the surface of the half-space are presented also by means of the method of image. The half-space Green's functions are derived explicitly for the four different sets of boundary conditions.

II. GREEN'S FUNCTIONS FOR A MULTIFERROIC FULL-SPACE

The constitutive equations for a uniaxial multiferroic material with its unique axis along the x_3 axis can be written as

$$\begin{bmatrix} D_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix} \begin{bmatrix} E_1 \\ H_1 \end{bmatrix} \quad ,$$
$$\begin{bmatrix} D_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix} \quad ,$$
$$\begin{bmatrix} D_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} \kappa_{33} & \alpha_{33} \\ \alpha_{33} & \mu_{33} \end{bmatrix} \begin{bmatrix} E_3 \\ H_3 \end{bmatrix} \quad , \tag{1}$$

where D_i and B_i ($I = 1, 2, 3$) are, respectively, the electric displacement and magnetic flux components (in the x_1, x_2, and x_3 directions); E_i and H_i are electric-field and magnetic field components, respectively; κ_{11} and κ_{33} are the two dielectric permittivity coefficients in the x_1 and x_3 directions, respectively; α_{11} and α_{33} are the two magnetoelectric coefficients (in the x_1 and x_3 directions, respectively); and μ_{11} and μ_{33} are the two magnetic permeability coefficients (in the x_1 and x_3 directions, respectively). We remark that the multiferroic material behavior exists only if the magnetoelectric coefficient is nonzero, resulting in the coupling between the electric and magnetic fields. In other words, an applied electric field induces a magnetic field, and vice versa. On the other hand, if all the magnetoelectric coefficients are zero, then the electric and magnetic fields are decoupled to each other.

The electric and magnetic fields are related to the electric potential ϕ and magnetic potential ψ through the following 2×1 column matrix relation

$$\begin{bmatrix} E_i \\ H_i \end{bmatrix} = -\begin{bmatrix} \phi_{,i} \\ \psi_{,i} \end{bmatrix} \quad , \tag{2}$$

where the subscript comma "," followed by the index i ($i = 1, 2, 3$) denotes the derivative of the potential with respect to the coordinate x_i.

For the uniaxial multiferroic free-space case, without loss of generality, we assume that there is a point-electric charge P and a point-magnetic charge M, both of which are located at the origin. Then the electric displacement D_i and magnetic flux B_i satisfy the following equations

$$\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3} = P\delta(x_1)\delta(x_2)\delta(x_3) \quad ,$$
$$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = M\delta(x_1)\delta(x_2)\delta(x_3) \quad , \tag{3}$$

where $\delta(\,)$ is the Dirac delta function. We point out that due to the linear relation between electric displacement D_i and spontaneous polarization, and magnetic flux B_i and spontaneous magnetization, the point-electric charge P and the magnetic-charge M can therefore be also physically interpreted as the divergence of a polarization and magnetization, respectively.[10] Therefore, the Green's functions solutions of Eq. (3) are extremely important as they can be directly related to the spontaneous polarization- and magnetization-induced fields, or can be implemented into a boundary-integral formulation for more complicated problem analysis.

Substituting Eq. (2) into Eq. (1), and then substituting the results into Eq. (3), we finally arrive at the following set of inhomogeneous partial differential equations for ϕ and ψ

$$\begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\phi \\ \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi \end{bmatrix} + \begin{bmatrix} \kappa_{33} & \alpha_{33} \\ \alpha_{33} & \mu_{33} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \phi}{\partial x_3^2} \\ \frac{\partial^2 \psi}{\partial x_3^2} \end{bmatrix} = -\begin{bmatrix} P \\ M \end{bmatrix} \delta(x_1)\delta(x_2)\delta(x_3) \quad . \tag{4}$$

In the following, we will present a simple approach to derive the Green's functions for a multiferroic full-space. We first consider the following eigenvalue problem

$$\left(\begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix} - \lambda \begin{bmatrix} \kappa_{33} & \alpha_{33} \\ \alpha_{33} & \mu_{33} \end{bmatrix} \right) \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad . \tag{5}$$

The two eigenvalues λ_1 and λ_2 are given by

$$\lambda_1 = \frac{\mu_{33}\kappa_{11} + \mu_{11}\kappa_{33} - 2\alpha_{11}\alpha_{33} + \sqrt{(\mu_{11}\kappa_{33} - \mu_{33}\kappa_{11})^2 + 4(\alpha_{11}\mu_{33} - \alpha_{33}\mu_{11})(\alpha_{11}\kappa_{33} - \alpha_{33}\kappa_{11})}}{2(\mu_{33}\kappa_{33} - \alpha_{33}^2)} \quad ,$$
$$\lambda_2 = \frac{\mu_{33}\kappa_{11} + \mu_{11}\kappa_{33} - 2\alpha_{11}\alpha_{33} - \sqrt{(\mu_{11}\kappa_{33} - \mu_{33}\kappa_{11})^2 + 4(\alpha_{11}\mu_{33} - \alpha_{33}\mu_{11})(\alpha_{11}\kappa_{33} - \alpha_{33}\kappa_{11})}}{2(\mu_{33}\kappa_{33} - \alpha_{33}^2)} \quad , \tag{6}$$

and the two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated with λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{bmatrix} -\alpha_{11} + \lambda_1\alpha_{33} \\ \kappa_{11} - \lambda_1\kappa_{33} \end{bmatrix} \quad ,$$

$$\mathbf{v}_2 = \begin{bmatrix} -\alpha_{11} + \lambda_2\alpha_{33} \\ \kappa_{11} - \lambda_2\kappa_{33} \end{bmatrix} \quad . \tag{7}$$

Because the two matrices $\begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix}$ and $\begin{bmatrix} \kappa_{33} & \alpha_{33} \\ \alpha_{33} & \mu_{33} \end{bmatrix}$ are real and symmetric, it can be easily verified that the following orthogonal relationships with respect to the two symmetric matrices hold

$$\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \kappa_{33} & \alpha_{33} \\ \alpha_{33} & \mu_{33} \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \quad ,$$

$$\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} \lambda_1\delta_1 & 0 \\ 0 & \lambda_2\delta_2 \end{bmatrix} \quad , \tag{8}$$

where

$$\delta_1 = \alpha_{11}^2\kappa_{33} + \kappa_{11}^2\mu_{33} - 2\alpha_{11}\alpha_{33}\kappa_{11} + (\mu_{33}\kappa_{33} - \alpha_{33}^2)(\lambda_1^2\kappa_{33} - 2\lambda_1\kappa_{11}) \quad ,$$

$$\delta_2 = \alpha_{11}^2\kappa_{33} + \kappa_{11}^2\mu_{33} - 2\alpha_{11}\alpha_{33}\kappa_{11} + (\mu_{33}\kappa_{33} - \alpha_{33}^2)(\lambda_2^2\kappa_{33} - 2\lambda_2\kappa_{11}) \quad . \tag{9}$$

We now introduce two new functions, f and g, which are related to ϕ and ψ through

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{H} \begin{bmatrix} f \\ g \end{bmatrix} \quad , \tag{10}$$

where $\mathbf{H} = [\mathbf{v}_1 \ \mathbf{v}_2]$.

In view of Eqs. (4), (8), and (10), the two new functions f and g satisfy the following two independent inhomogeneous 3D Laplace equations

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{\lambda_1}\frac{\partial^2}{\partial x_3^2}\right) f = -\frac{(-\alpha_{11} + \lambda_1\alpha_{33})P + (\kappa_{11} - \lambda_1\kappa_{33})M}{\delta_1\lambda_1} \delta(x_1)\delta(x_2)\delta(x_3) \quad ,$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{\lambda_2}\frac{\partial^2}{\partial x_3^2}\right) g = -\frac{(-\alpha_{11} + \lambda_2\alpha_{33})P + (\kappa_{11} - \lambda_2\kappa_{33})M}{\delta_2\lambda_2} \delta(x_1)\delta(x_2)\delta(x_3) \quad . \tag{11}$$

Further simplification can be carried for Eq. (11), resulting in the following concise and equivalent expressions

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial\left(\sqrt{\lambda_1}x_3\right)^2}\right) f = -4\pi K_1 \delta(x_1)\delta(x_2)\delta\left(\sqrt{\lambda_1}x_3\right) \quad ,$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial\left(\sqrt{\lambda_2}x_3\right)^2}\right) g = -4\pi K_2 \delta(x_1)\delta(x_2)\delta\left(\sqrt{\lambda_2}x_3\right) \quad , \tag{12}$$

where the two constants K_1 and K_2 are defined as

$$K_1 = \frac{(-\alpha_{11} + \lambda_1\alpha_{33})P + (\kappa_{11} - \lambda_1\kappa_{33})M}{4\pi\delta_1\sqrt{\lambda_1}} \quad ,$$

$$K_2 = \frac{(-\alpha_{11} + \lambda_2\alpha_{33})P + (\kappa_{11} - \lambda_2\kappa_{33})M}{4\pi\delta_2\sqrt{\lambda_2}} \quad . \tag{13}$$

Therefore, solutions to Eq. (12) can be expediently given by

$$f = \frac{K_1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 x_3^2}} \quad ,$$

$$g = \frac{K_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 x_3^2}} \quad . \tag{14}$$

In view of Eqs. (10) and (14), the electric and magnetic potentials can be now obtained as

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{H} \begin{bmatrix} \dfrac{K_1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 x_3^2}} \\ \dfrac{K_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 x_3^2}} \end{bmatrix} \quad . \tag{15}$$

Based on Eq. (15), the full-space multiferroic Green's functions $G_{\alpha\beta}$ are then found to be

$$4\pi G_{\phi P} = \frac{(\alpha_{11} - \lambda_1\alpha_{33})^2}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2 + x_2^2 + \lambda_1 x_3^2}} + \frac{(\alpha_{11} - \lambda_2\alpha_{33})^2}{\delta^2\sqrt{\lambda_2}\sqrt{x_1^2 + x_2^2 + \lambda_2 x_3^2}} \quad ,$$

$$4\pi G_{\phi M} = 4\pi G_{\psi P} = \frac{(-\alpha_{11} + \lambda_1\alpha_{33})(\kappa_{11} - \lambda_1\kappa_{33})}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2 + x_2^2 + \lambda_1 x_3^2}} + \frac{(-\alpha_{11} + \lambda_2\alpha_{33})(\kappa_{11} - \lambda_2\kappa_{33})}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2 + x_2^2 + \lambda_2 x_3^2}} \quad ,$$

$$4\pi G_{\psi M} = \frac{(\kappa_{11} - \lambda_1\kappa_{33})^2}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2 + x_2^2 + \lambda_1 x_3^2}} + \frac{(\kappa_{11} - \lambda_2\kappa_{33})^2}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2 + x_2^2 + \lambda_2 x_3^2}} \quad , \tag{16}$$

where the definitions of the Green's functions are: $G_{\phi P}(x_i)$ is the electric potential ϕ at x_i due to a point-electric charge ($P = 1$) at origin; $G_{\phi M}(x_i)$ is the electric potential ϕ at x_i due to a point-magnetic charge ($M = 1$) at origin; $G_{\psi P}(x_i)$ is the magnetic potential ψ at x_i due to a point-electric charge ($P = 1$) at origin; and $G_{\psi M}(x_i)$ is the magnetic potential ψ at x_i due to a point-magnetic charge ($M = 1$) at origin. It is noted that for the full-space case, the Green's function is symmetric. In other words, the electric potential at x_i due to a point-magnetic charge at the origin equals the magnetic potential at the origin due to a point-electric charge at x_i (also making use of the fact that these Green's functions are even functions of the coordinates). This is an extended result of the well-known Betti–Maxwell reciprocity theorem, which has found numerous applications in various mathematical and physical fields.[12]

It should be mentioned that the above-derived Green's functions are not valid for the decoupling case $\alpha_{11} = \alpha_{33} = 0$ due the fact that in this decoupling case one of the two eigenvectors in Eq. (7) will be zero. For this degenerated case, instead of deriving a new set of solutions for the problem, we can simply assign two very small values for α_{11} and α_{33} to separate the two eigenvalues λ_1 and λ_2 and to make the two eigenvectors in Eq. (7) nonzero and independent of each other. In so doing, the solutions presented in this article can still be applied with neglected errors.[10,13]

We emphasize that we have presented a different approach than that given by Li and Li[10] to obtain the Green's functions for a full-space multiferroic material. In Ref. 10, a magnetoelectric potential function, g, which satisfies a dual-harmonic function, is introduced. In our approach, through the introduction of two new functions the original coupled governing Eq. (4) can be decoupled into two inhomogeneous Laplace equations, Eqs. (11) or (12), the Green's function solutions of which are well known. As a result, our approach is direct and simple in deriving the Green's function. We have verified our solution by comparing our results with those in Ref. 10 and are able to reproduce exactly Figs. 1–3 there.[10] Consequently, our solutions can be utilized with high confidence.

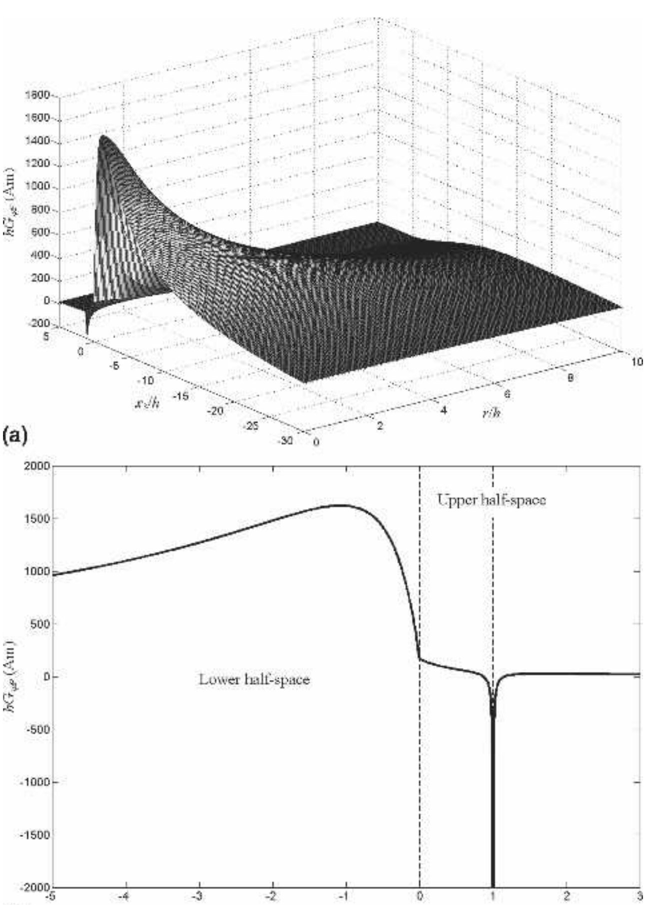

FIG. 1. 3D distribution of the Green's function component $G_{\psi P}$ in a bimaterial space.

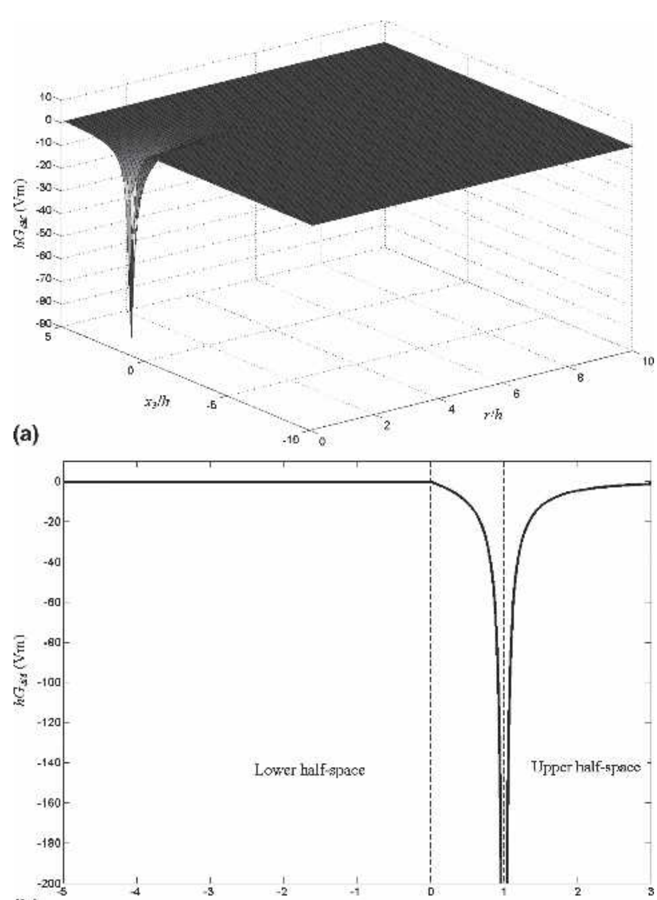

FIG. 2. 3D distribution of the Green's function component $G_{\phi M}$ in a bimaterial space.

III. GREEN'S FUNCTIONS FOR A MULTIFERROIC BIMATERIAL

With the simple full-space Green's function solutions, we can derive the corresponding Green's functions in two bonded multiferroic half-spaces. We assume that the two half-spaces are uniaxial multiferroic materials having a unique axis along the x_3 axis, and that the interface $x_3 = 0$ separating the two half-spaces is perfect. Namely, the electric potential, magnetic potential, normal electric displacement, and normal magnetic flux are all continuous across the interface. Without loss of generality, an electric charge, P, and a magnetic charge, M, are assumed to be located at $x_1 = x_2 = 0$, $x_3 = h$, $(h > 0)$ in the upper half-space of the multiferroic bimaterial.

Following the basic idea of the method of image in static electricity,[11] the electric and magnetic potentials in the two half-spaces induced by the electric and magnetic charges can be assumed as

$$\begin{bmatrix}\phi_1\\ \psi_1\end{bmatrix} = \mathbf{H}_1\begin{bmatrix}\dfrac{K_1}{\sqrt{x_1^2+x_2^2+\lambda_1^{(1)}(x_3-h)^2}}\\ \dfrac{K_2}{\sqrt{x_1^2+x_2^2+\lambda_2^{(1)}(x_3-h)^2}}\end{bmatrix} + \mathbf{H}_1\begin{bmatrix}\dfrac{T_1}{\sqrt{x_1^2+x_2^2+\lambda_1^{(1)}(x_3+h)^2}}\\ \dfrac{T_2}{\sqrt{x_1^2+x_2^2+\lambda_2^{(1)}(x_3+h)^2}}\end{bmatrix} + \mathbf{H}_1\begin{bmatrix}\dfrac{T_3}{\sqrt{x_1^2+x_2^2+\left(\sqrt{\lambda_1^{(1)}}x_3+\sqrt{\lambda_2^{(1)}}h\right)^2}}\\ \dfrac{T_4}{\sqrt{x_1^2+x_2^2+\left(\sqrt{\lambda_2^{(1)}}x_3+\sqrt{\lambda_1^{(1)}}h\right)^2}}\end{bmatrix} ,$$
$$x_3 \geqslant 0 \quad , \tag{17}$$

$$\begin{bmatrix}\phi_2\\ \psi_2\end{bmatrix} = \mathbf{H}_2\begin{bmatrix}\dfrac{L_1}{\sqrt{x_1^2+x_2^2+\left(\sqrt{\lambda_1^{(2)}}x_3-\sqrt{\lambda_1^{(1)}}h\right)^2}}\\ \dfrac{L_2}{\sqrt{x_1^2+x_2^2+\left(\sqrt{\lambda_2^{(2)}}x_3-\sqrt{\lambda_2^{(1)}}\,h\right)^2}}\end{bmatrix} + \mathbf{H}_2\begin{bmatrix}\dfrac{L_3}{\sqrt{x_1^2+x_2^2+\left(\sqrt{\lambda_1^{(2)}}x_3-\sqrt{\lambda_2^{(1)}}h\right)^2}}\\ \dfrac{L_4}{\sqrt{x_1^2+x_2^2+\left(\sqrt{\lambda_2^{(2)}}x_3-\sqrt{\lambda_1^{(1)}}\,h\right)^2}}\end{bmatrix} ,$$
$$x_3 \leqslant 0 \quad , \tag{18}$$

where $\mathbf{H}_1,\lambda_1^{(1)},\lambda_2^{(1)}$ pertain to the upper half-space, whereas $\mathbf{H}_2,\lambda_1^{(2)},\lambda_2^{(2)}$ belong to the lower half-space; K_1 and K_2 have been defined in Eq. (13) with the material properties belonging to the upper half-space; T_I, L_i, $(i = 1–4)$ are eight unknown constants to be determined. Then, enforcing the interface conditions on $x_3 = 0$ results in the following set of linear algebraic equations

$$\begin{aligned}
&H_{11}^{(1)}K_1 + H_{11}^{(1)}T_1 + H_{12}^{(1)}T_4 - H_{11}^{(2)}L_1 - H_{12}^{(2)}L_4 = 0 \quad ,\\
&H_{12}^{(1)}K_2 + H_{12}^{(1)}T_2 + H_{11}^{(1)}T_3 - H_{11}^{(2)}L_3 - H_{12}^{(2)}L_2 = 0 \quad ,\\
&H_{21}^{(1)}K_1 + H_{21}^{(1)}T_1 + H_{22}^{(1)}T_4 - H_{21}^{(2)}L_1 - H_{22}^{(2)}L_4 = 0 \quad ,\\
&H_{22}^{(1)}K_2 + H_{22}^{(1)}T_2 + H_{21}^{(1)}T_3 - H_{21}^{(2)}L_3 - H_{22}^{(2)}L_2 = 0 \quad ,\\
&-J_{11}^{(1)}K_1 + J_{11}^{(1)}T_1 + J_{12}^{(1)}T_4 + J_{11}^{(2)}L_1 + J_{12}^{(2)}L_4 = 0 \quad ,\\
&-J_{12}^{(1)}K_2 + J_{12}^{(1)}T_2 + J_{11}^{(1)}T_3 + J_{11}^{(2)}L_3 + J_{12}^{(2)}L_2 = 0 \quad ,\\
&-J_{21}^{(1)}K_1 + J_{21}^{(1)}T_1 + J_{22}^{(1)}T_4 + J_{21}^{(2)}L_1 + J_{22}^{(2)}L_4 = 0 \quad ,\\
&-J_{22}^{(1)}K_2 + J_{22}^{(1)}T_2 + J_{21}^{(1)}T_3 + J_{21}^{(2)}L_3 + J_{22}^{(2)}L_2 = 0 \quad ,
\end{aligned} \tag{19}$$

where $H_{11}^{(i)}$, $H_{12}^{(i)}$, $H_{21}^{(i)}$, $H_{22}^{(i)}$ are the four components of $\mathbf{H}_i$, and

$$\begin{aligned}
&J_{11}^{(i)} = \sqrt{\lambda_1^{(i)}}(\kappa_{33}^{(i)}H_{11}^{(i)} + \alpha_{33}^{(i)}H_{21}^{(i)}) \quad ,\\
&J_{12}^{(i)} = \sqrt{\lambda_2^{(i)}}(\kappa_{33}^{(i)}H_{12}^{(i)} + \alpha_{33}^{(i)}H_{22}^{(i)}) \quad ,\\
&J_{21}^{(i)} = \sqrt{\lambda_1^{(i)}}(\alpha_{33}^{(i)}H_{11}^{(i)} + \mu_{33}^{(i)}H_{21}^{(i)}) \quad ,\\
&J_{22}^{(i)} = \sqrt{\lambda_2^{(i)}}(\alpha_{33}^{(i)}H_{12}^{(i)} + \mu_{33}^{(i)}H_{22}^{(i)}) \quad ,\\
&(i = 1, 2) \quad .
\end{aligned} \tag{20}$$

As a result, the eight unknown constants T_i, L_i, $(i = 1–4)$ can be uniquely determined by the eight independent algebraic equations in Eq. (18). More specifically

$$\begin{bmatrix} T_1 & T_3\\ T_4 & T_2\\ L_1 & L_3\\ L_4 & L_2\end{bmatrix} = \mathbf{Y}\begin{bmatrix} -H_{11}^{(1)} & -H_{12}^{(1)}\\ -H_{21}^{(1)} & -H_{22}^{(1)}\\ J_{11}^{(1)} & J_{12}^{(1)}\\ J_{21}^{(1)} & J_{22}^{(1)}\end{bmatrix}\begin{bmatrix} K_1 & 0\\ 0 & K_2\end{bmatrix} , \tag{21}$$

where

$$\mathbf{Y} = \begin{bmatrix} H_{11}^{(1)} & H_{12}^{(1)} & -H_{11}^{(2)} & -H_{12}^{(2)}\\ H_{21}^{(1)} & H_{22}^{(1)} & -H_{21}^{(2)} & -H_{22}^{(2)}\\ J_{11}^{(1)} & J_{12}^{(1)} & J_{11}^{(2)} & J_{12}^{(2)}\\ J_{21}^{(1)} & J_{22}^{(1)} & J_{21}^{(2)} & J_{22}^{(2)}\end{bmatrix}^{-1} . \tag{22}$$

The above results demonstrate that we only need to invert a single 4×4 matrix [see Eq. (22)] to find the eight unknown constants appearing in the expressions of the

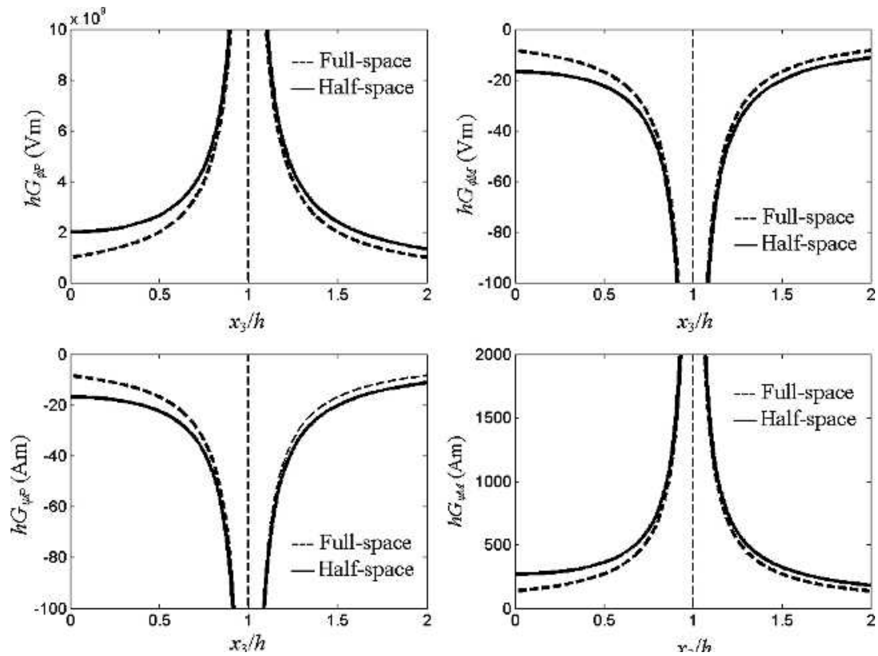

FIG. 3. Variations of the Green's functions $G\phi_P$, $G_{\phi M}$, $G_{\psi P}$, and $G_{\psi M}$ along the positive x_3 axis for the half-space case with $D_3 = B_3 = 0$ on $x_3 = 0$ (solid lines, considering the influence of the surface) and for the full-space (dashed lines, ignoring the influence of the surface).

magnetoelectric Green's functions for the multiferroic bimaterial. Therefore, explicit expressions of the Green's functions $G_{\alpha\beta}$ for the multiferroic bimaterial are finally solved. We also remark that while the bimaterial Green's functions are for the perfect bonded interface case, Green's functions corresponding to other interface conditions, in particular the homogeneous-type and spring-type conditions, could also be derived.[14,15] To demonstrate the derived solutions, we show in Figs. 1 and 2 the Green's functions $G_{\psi P}$ (the magnetic potential induced by a unit point-electric charge) and $G_{\phi M}$ (the electric potential induced by a unit point-magnetic charge). While the material properties of the upper half-space are taken from Ref. 10, the lower half-space is assumed to be $BaTiO_3$. The material coefficients for the two half-spaces are listed in Table I.

Because the magnetoelectric coefficients α_{11} and α_{33} are zero for the lower half-space, which corresponds to the degenerated case we discussed in the previous section, we perturbed α_{11} and α_{33} to be nonzero during calculation so that the Green's function solutions we derived here can still be used. While Figs. 1(a) and 2(a) show the distributions of $G_{\psi P}$ and $G_{\phi M}$ in both the upper and lower half-spaces as functions of the cylindrical coordinates r and x_3, Figs. 1(b) and 2(b) plot the variation of $G_{\psi P}$ and $G_{\phi M}$ along the x_3 axis (i.e., with $r = 0$). It is observed that, due to the existence of the lower half-space the properties of which are quite different from those of the upper multiferroic half-space, the distribution of $G_{\psi P}$ in the lower half-space can be altered significantly (see Fig. 3 in Ref. 10 as a comparison), while $G_{\phi M} \approx 0$ in the lower half-space, which is due to the fact that the dielectric permittivity coefficients for the lower half-space are much higher than those for the upper half-space. More interestingly, there exists a point at $x_1 = x_2 = 0$, $x_3 = -1.08h$ in the lower half-space at which $G_{\psi P}$ attains a maximum value of $1623/h$ A. In addition, our results clearly show that, unlike the full-space multiferroic Green's function, the corresponding bimaterial Green's functions is not symmetric [i.e., $G_{\phi M} \neq G_{\psi P}$ for the bimaterial case (Fig. 1 versus Fig. 2)]. Therefore, the bimaterial Green's functions are required to capture the material-mismatch effect on the field quantities.

TABLE I. The constitutive moduli for the multiferroic bimaterial.[a]

Bimaterial	α_{11}	α_{33}	κ_{11}	κ_{33}	μ_{11}	μ_{33}
Upper half-space	5	3	8	9.3	5.9	1.57
Lower half-space	0	0	1120	1260	0.05	0.1

[a]Unit: magnetoelectric coefficient in 10^{-12} Ns/VC; dielectric permittivity in 10^{-11} C^2/Nm^2; magnetic permeability in 10^{-4} Ns^2/C^2.

IV. GREEN'S FUNCTIONS FOR A MULTIFERROIC HALF-SPACE

In this section, we derive the Green's functions for a multiferroic half-space ($x_3 \geqslant 0$) where a point-electric charge P and a point-magnetic charge M are located at $x_1 = x_2 = 0, x_3 = h, (h > 0)$. Again, we assume that the half-space is a uniaxial multiferroic material having a unique axis along the x_3 axis. Here, we discuss four different types of boundary conditions on the surface

$x_3 = 0$: (1) $\phi = \psi = 0$; (2) $D_3 = B_3 = 0$; (3) $\phi = B_3 = 0$; and (4) $\psi = D_3 = 0$. We point out that although these conditions are special homogeneous conditions, other more complicated boundary conditions, such as the combined homogeneous conditions and even the spring-type conditions can be equally addressed and solved in the exact closed form.[16] On the other hand, for more general cases, one could simply use the present Green's function solution to develop the corresponding boundary integral equation, which can be discretized and solved numerically.[13]

Similar to the discussion in the previous section, the electric potential and magnetic potential in the half-space can be assumed as

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{H} \begin{bmatrix} \dfrac{K_1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} \\ \dfrac{K_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} \end{bmatrix} + \mathbf{H} \begin{bmatrix} \dfrac{T_1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \\ \dfrac{T_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \end{bmatrix} + \mathbf{H} \begin{bmatrix} \dfrac{T_3}{\sqrt{x_1^2 + x_2^2 + \left(\sqrt{\lambda_1} x_3 + \sqrt{\lambda_2} h\right)^2}} \\ \dfrac{T_4}{\sqrt{x_1^2 + x_2^2 + \left(\sqrt{\lambda_2} x_3 + \sqrt{\lambda_1} h\right)^2}} \end{bmatrix} , \quad x_3 \geqslant 0 \quad , \tag{23}$$

where T_i ($i = 1$–4) are four unknown constants to be determined by the prescribed boundary conditions on the surface. In the following, we discuss the four types of boundary conditions one by one.

Type (1). $\phi = \psi = 0$ on $x_3 = 0$

By enforcing the boundary conditions $\phi = \psi = 0$ on $x_3 = 0$, it is found that

$$T_1 = -K_1, \quad T_2 = -K_2, \quad T_3 = T_4 = 0 \quad . \tag{24}$$

Consequently, the electric and magnetic potentials in the half-space take the following simple forms

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{H} \begin{bmatrix} \dfrac{K_1}{\sqrt{x_1^2 + x_2^2 \lambda_1 (x_3 - h)^2}} - \dfrac{K_1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \\ \dfrac{K_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} - \dfrac{K_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \end{bmatrix}, \qquad x_3 \geqslant 0 \quad . \tag{25}$$

Therefore, the explicit expressions of the Green's functions $G_{\alpha\beta}$ are

$$\begin{aligned} 4\pi G_{\phi P} = {} & \frac{(\alpha_{11} - \lambda_1 \alpha_{33})^2}{\delta_1 \sqrt{\lambda_1}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} - \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \right] \\ & + \frac{(\alpha_{11} - \lambda_2 \alpha_{33})^2}{\delta_2 \sqrt{\lambda_2}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} - \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \right] \quad , \end{aligned} \tag{26a}$$

$$\begin{aligned} 4\pi G_{\phi M} = 4\pi G_{\psi P} = {} & \frac{(-\alpha_{11} + \lambda_1 \alpha_{33})(\kappa_{11} - \lambda_1 \kappa_{33})}{\delta_1 \sqrt{\lambda_1}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} - \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \right] \\ & + \frac{(-\alpha_{11} + \lambda_2 \alpha_{33})(\kappa_{11} - \lambda_2 \kappa_{33})}{\delta_2 \sqrt{\lambda_2}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} - \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \right] \quad , \end{aligned} \tag{26b}$$

$$\begin{aligned} 4\pi G_{\phi M} = {} & \frac{(\kappa_{11} - \lambda_1 \kappa_{33})^2}{\delta_1 \sqrt{\lambda_1}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} - \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \right] \\ & + \frac{(\kappa_{11} - \lambda_2 \kappa_{33})^2}{\delta_2 \sqrt{\lambda_2}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} - \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \right] \quad . \end{aligned} \tag{26c}$$

We remark that this set of half-space Green's functions can be reduced from our bimaterial Green's functions [Eqs. (17) and (18)] by assuming the lower half-space with very large κ_{ii} and μ_{ii}.

Type (2). $D_3 = B_3 = 0$ on $x_3 = 0$

By enforcing the boundary conditions $D_3 = B_3 = 0$ on $x_3 = 0$, it is found that

$$T_1 = K_1, \quad T_2 = K_2, \quad T_3 = T_4 = 0 \quad . \tag{27}$$

Thus, the electric and magnetic potentials in the half-space take the following simple forms

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{H} \begin{bmatrix} \dfrac{K_1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} + \dfrac{K_1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \\ \dfrac{K_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} + \dfrac{K_2}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \end{bmatrix}, \qquad x_3 \geqslant 0 \quad . \tag{28}$$

The corresponding Green's functions $G_{\alpha\beta}$ can be finally derived to be

$$\begin{aligned} 4\pi G_{\phi P} = {} & \frac{(\alpha_{11} - \lambda_1 \alpha_{33})^2}{\delta_1 \sqrt{\lambda_1}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} + \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \right] \\ & + \frac{(\alpha_{11} - \lambda_2 \alpha_{33})^2}{\delta_2 \sqrt{\lambda_2}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} + \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \right] \quad , \end{aligned} \tag{29a}$$

$$\begin{aligned} 4\pi G_{\phi M} = 4\pi G_{\psi P} = {} & \frac{(-\alpha_{11} + \lambda_1 \alpha_{33})(\kappa_{11} - \lambda_1 \kappa_{33})}{\delta_1 \sqrt{\lambda_1}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} + \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \right] \\ & + \frac{(-\alpha_{11} + \lambda_2 \alpha_{33})(\kappa_{11} - \lambda_2 \kappa_{33})}{\delta_2 \sqrt{\lambda_2}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} + \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \right] \quad , \end{aligned} \tag{29b}$$

$$\begin{aligned} 4\pi G_{\psi M} = {} & \frac{(\kappa_{11} - \lambda_1 \kappa_{33})^2}{\delta_1 \sqrt{\lambda_1}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 - h)^2}} + \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_1 (x_3 + h)^2}} \right] \\ & + \frac{(\kappa_{11} - \lambda_2 \kappa_{33})^2}{\delta_2 \sqrt{\lambda_2} \sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} \left[\frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 - h)^2}} + \frac{1}{\sqrt{x_1^2 + x_2^2 + \lambda_2 (x_3 + h)^2}} \right] \quad . \end{aligned} \tag{29c}$$

Similar to the first set of half-space Green's functions [Eq. (26)], this set of half-space Green's functions again can be reduced from our bimaterial Green's functions [Eqs. (17) and (18)] by assuming the lower half-space with very small κ_{ii}, α_{ii}, and μ_{ii}.

Type (3). $\phi = B_3 = 0$ on $x_3 = 0$

By enforcing the boundary conditions $\phi = B_3 = 0$ on $x_3 = 0$, then T_i ($i = 1$–4) can be uniquely determined to be

$$\begin{aligned} T_1 &= \frac{K_1 (H_{11} J_{22} + H_{12} J_{21})}{H_{12} J_{21} - H_{11} J_{22}}, \qquad & T_2 &= \frac{K_2 (H_{11} J_{22} + H_{12} J_{21})}{H_{11} J_{22} - H_{12} J_{21}}, \\ T_3 &= \frac{2 K_2 H_{12} J_{22}}{H_{12} J_{21} - H_{11} J_{22}}, & T_4 &= \frac{2 K_1 H_{11} J_{21}}{H_{11} J_{22} - H_{12} J_{21}} \quad , \end{aligned} \tag{30}$$

with $J_{\alpha\beta}$ having been defined in Eq. (20).

The explicit expressions of the Green's functions $G_{\alpha\beta}$ for this case can be finally derived to be

$$\begin{aligned}4\pi G_{\phi P}=&\frac{H_{11}^2}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{12}J_{21}-H_{11}J_{22})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]\\&+\frac{H_{12}^2}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{11}J_{22}-H_{12}J_{21})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]\\&+\frac{2H_{11}H_{12}}{H_{12}J_{21}-H_{11}J_{22}}\left[\frac{H_{12}J_{22}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{11}J_{21}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\ ,\end{aligned} \tag{31a}$$

$$\begin{aligned}4\pi G_{\phi M}=&\frac{H_{11}H_{21}}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{12}J_{21}-H_{11}J_{22})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]\\&+\frac{H_{12}H_{22}}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{11}J_{22}-H_{12}J_{21})\sqrt{x_1^2+x_2^2+\lambda_2(x_3+h)^2}}\right]\\&+\frac{2H_{11}H_{12}}{H_{12}J_{21}-H_{11}J_{22}}\left[\frac{H_{22}J_{22}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{21}J_{21}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\ ,\end{aligned} \tag{31b}$$

$$\begin{aligned}4\pi G_{\psi P}=&\frac{H_{11}H_{21}}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{12}J_{21}-H_{11}J_{22})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]\\&+\frac{H_{12}H_{22}}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{11}J_{22}-H_{12}J_{21})\sqrt{x_1^2+x_2^2+\lambda_2(x_3+h)^2}}\right]\\&+\frac{2}{H_{12}J_{21}-H_{11}J_{22}}\left[\frac{H_{12}^2H_{21}J_{22}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{11}^2H_{22}J_{21}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\ ,\end{aligned} \tag{31c}$$

$$\begin{aligned}4\pi G_{\psi M}=&\frac{H_{21}^2}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{12}J_{21}-H_{11}J_{22})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]\\&+\frac{H_{22}^2}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{11}J_{22}+H_{12}J_{21}}{(H_{11}J_{22}-H_{12}J_{21})\sqrt{x_1^2+x_2^2+\lambda_2(x_3+h)^2}}\right]\\&+\frac{2H_{21}H_{22}}{H_{12}J_{21}-H_{11}J_{22}}\left[\frac{H_{12}J_{22}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{11}J_{21}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\ .\end{aligned} \tag{31d}$$

Type (4). $\psi = D_3 = 0$ on $x_3 = 0$

By enforcing the boundary conditions $\psi = \mathrm{D}_3 = 0$ on $x_3 = 0$, then T_i (i = 1–4) can be uniquely determined to be

$$T_1=\frac{K_1(H_{21}J_{12}+H_{22}J_{11})}{H_{22}J_{11}-H_{21}J_{12}},\qquad T_2=\frac{K_2(H_{22}J_{11}+H_{21}J_{12})}{H_{21}J_{12}-H_{22}J_{11}},$$
$$T_3=\frac{2K_2H_{22}J_{12}}{H_{22}J_{11}-H_{21}J_{12}},\qquad T_4=\frac{2K_1H_{21}J_{11}}{H_{21}J_{12}-H_{22}J_{11}}\quad . \tag{32}$$

The corresponding Green's functions $G_{\alpha\beta}$ are found to be

$$4\pi G_{\phi P}=\frac{H_{11}^2}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{21}J_{12}+H_{22}J_{11}}{(H_{22}J_{11}-H_{21}J_{12})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]$$
$$+\frac{H_{12}^2}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{22}J_{11}+H_{21}J_{12}}{(H_{21}J_{12}-H_{22}J_{11})\sqrt{x_1^2+x_2^2+\lambda_2(x_3+h)^2}}\right]$$
$$+\frac{2H_{11}H_{12}}{H_{22}J_{11}-H_{21}J_{12}}\left[\frac{H_{22}J_{12}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{21}J_{11}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\quad, \tag{33a}$$

$$4\pi G_{\phi M}=\frac{H_{11}H_{21}}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{21}J_{12}+H_{22}J_{11}}{(H_{22}J_{11}-H_{21}J_{12})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]$$
$$+\frac{H_{12}H_{22}}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{22}J_{11}+H_{21}J_{12}}{(H_{21}J_{12}-H_{22}J_{11})\sqrt{x_1^2+x_2^2+\lambda_2(x_3+h)^2}}\right]$$
$$+\frac{2}{H_{22}J_{11}-H_{21}J_{12}}\left[\frac{H_{11}H_{22}^2J_{12}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{12}H_{21}^2J_{11}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\quad, \tag{33b}$$

$$4\pi G_{\psi P}=\frac{H_{11}H_{21}}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{21}J_{12}+H_{22}J_{11}}{(H_{22}J_{11}-H_{21}J_{12})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]$$
$$+\frac{H_{12}H_{22}}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{22}J_{11}+H_{21}J_{12}}{(H_{21}J_{12}-H_{22}J_{11})\sqrt{x_1^2+x_2^2+\lambda_2(x_3+h)^2}}\right]$$
$$+\frac{2H_{21}H_{22}}{H_{22}J_{11}-H_{21}J_{12}}\left[\frac{H_{12}J_{12}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{11}J_{11}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\quad, \tag{33c}$$

$$4\pi G_{\psi M}=\frac{H_{21}^2}{\delta_1\sqrt{\lambda_1}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_1(x_3-h)^2}}+\frac{H_{21}J_{12}+H_{22}J_{11}}{(H_{22}J_{11}-H_{21}J_{12})\sqrt{x_1^2+x_2^2+\lambda_1(x_3+h)^2}}\right]$$
$$+\frac{H_{22}^2}{\delta_2\sqrt{\lambda_2}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+\lambda_2(x_3-h)^2}}+\frac{H_{22}J_{11}+H_{21}J_{12}}{(H_{21}J_{12}-H_{22}J_{11})\sqrt{x_1^2+x_2^2+\lambda_2(x_3+h)^2}}\right]$$
$$+\frac{2H_{21}H_{22}}{H_{22}J_{11}-H_{21}J_{12}}\left[\frac{H_{22}J_{12}}{\delta_2\sqrt{\lambda_2}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_1}x_3+\sqrt{\lambda_2}h)^2}}-\frac{H_{21}J_{11}}{\delta_1\sqrt{\lambda_1}\sqrt{x_1^2+x_2^2+(\sqrt{\lambda_2}x_3+\sqrt{\lambda_1}h)^2}}\right]\quad. \tag{33d}$$

Remark 1:

The four unknowns appearing in Eq. (23) have been determined explicitly for the four types of boundary conditions on the surface of the uniaxial multiferroic half-space. As a result, the explicit expressions of the Green's function $G_{\alpha\beta}$ can be obtained. For types (1) and (2), the Green's function are symmetric (i.e., $G_{\phi M} = G_{\psi P}$); while for types (3) and (4), the Green's functions are not symmetric (i.e., $G_{\phi M} \neq G_{\psi P}$) due to the effect of the boundary conditions on the surface. It can be easily checked from the above expressions that $G_{\phi P} = G_{\phi M} = G_{\psi P} = G_{\psi M} = 0$ on $x_3 = 0$ for type (1), $G_{\phi P} = G_{\phi M} = 0$ on $x_3 = 0$ for type (3), and $G_{\psi P} = G_{\psi M} = 0$ on $x_3 = 0$ for type (4), which correspondingly satisfy the boundary conditions on the surface of the half-space.

Remark 2:

We emphasize that while exact closed-form Green's functions have been derived for the four types of boundary conditions on the surface [i.e., (1) $\phi = \psi = 0$; (2) $D_3 = B_3 = 0$; (3) $\phi = B_3 = 0$; (4) $\psi = D_3 = 0$], the more general problems with complicated boundary conditions would need a numerical approach to solve them. One example is the inhomogeneous spring-type boundary condition $D_3 = k\phi + b$ on the surface, where k is the "spring coefficient" and b is an arbitrary constant. The boundary-integral-equation method would be an excellent candidate to attack this type of problem by making use of the present exact closed-form Green's functions as the kernels.[13]

To demonstrate the influence of the surface of the half-space on the distribution of the Green's function, we present in Fig. 3 the distributions of $G_{\phi P}$, $G_{\phi M}$, $G_{\psi P}$, and $G_{\psi M}$ along the positive x_3 axis in solid lines for type (2) (e.g., $D_3 = B_3 = 0$ on $x_3 = 0$). The material properties of the half-space are those in Ref. 10 (or see Table I). As a comparison, we also present the corresponding full-space results in dashed lines where the influence of the surface is ignored. It is obvious that the surface condition has a significant effect on the Green's functions, especially for field points that are very close to the surface. It is further observed from Fig. 3 that, due to the effect of the surface condition, the magnitudes of the half-space $G_{\alpha\beta}$ near the surface are larger than those for the full-space. While we only present the results for type (2) of the boundary conditions, the half-space Green's functions corresponding to other types of boundary conditions will also affect the near-surface response compared to the full-space Green's functions. In other words, if one is interested in the near-field response, the suitable half-space Green's functions, instead of the full-space Green's functions, need to be used. On the other hand, the simple full-space Green's functions can be used if the interested domain is far from the surface because in this case the half-space Green's functions are insensitive to the boundary condition on the surface (when the source and field points are both far from the surface).

The half-space Green's functions presented in this section should be extremely useful when connecting to experimental studies. Because the Green's function is a fundamental solution representing the field response due to a unit source at another location, it can be directly utilized to calculate accurately and efficiently the response due to any possible defect in the multiferroic material, and therefore a program can be designed to backcalculate the defect size and location. Another application, which is closely related to our half-space Green's functions, is to combine our solutions with experimental tests (e.g., an indentation test) to understand multiferroic material behaviors and even to invert the effective composite material properties. To simulate an indentation test on the multiferroic half-space, one only needs to move the source from an inner location to the surface (i.e., let $h = 0$ in our half-space Green's functions). The corresponding piezoelastic half-space Green's functions have been applied very successfully in the continuum mechanics community.[17–19]

V. CONCLUSIONS

In this research, we have derived magnetoelectric Green's functions for a uniaxial multiferroic full-space, bimaterial space, and half-space by means of the method of image. The expressions of these Green's functions are rather simple [e.g., Eqs. (17) and (18) for the bimaterial case] and explicit [see, e.g., Eqs. (26), (29), (31), and (33) for the half-space case]. Numerical results are also presented to show the influence of the surface conditions on the Green's functions, indicating clearly that for points close to the surface, the half-space Green's functions, instead of the full-space Green's functions, have to be used.

Due to their simple and explicit form of expression, these Green's functions could find numerous applications. They can be conveniently applied to investigate the electromagnetic fields induced by the inclusion of various shapes with spontaneous polarization and magnetization embedded in a multiferroic bimaterial space or half-space. Combined with the basic micromechanics theory, the corresponding inhomogeneity problem can also be investigated so that the effective material behavior in multiferroic composites can be accurately analyzed. These results could be connected to the future experimental measurement for verification. Furthermore, using these Green's functions as the kernels in the boundary integral equation, the corresponding formulation can be applied to the analysis of multiferroic material-based novel devices.

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