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Three-dimensional Green's functions for a steady point heat source in a functionally graded half-space and some related problems

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Abstract

Three-dimensional Green's functions are derived for a steady point heat source in a functionally graded half-space where the thermal conductivity varies exponentially along an arbitrary direction. We first introduce an auxiliary function which satisfies an inhomogeneous Helmholtz equation. Then by virtue of the image method which was first proposed by Sommerfeld for the homogeneous half-space Green's function of a steady point heat source, we arrive at an explicit expression for this function. Finally with this auxiliary function, we derive the three-dimensional Green's functions due to a steady point heat source in a functionally graded half-space. Also investigated in this paper are the temperature field induced by a point heat source moving at a constant speed in a functionally graded full-space; the electric potential due to a static point electric charge in a dielectric full-space with electric field gradient effects; and the two-dimensional time-harmonic dynamic Green's function for homogeneous and functionally graded materials with strain gradient effects. © 2007 Elsevier Ltd. All rights reserved.

Keywords: Green's function; Heat conduction; Functionally graded material; Electric field gradient effects; Strain gradient effect; Half-space; Helmholtz equation

1. Introduction

At the beginning of last century, Sommerfeld [1] derived the half-space Green's function for a steady point heat source using the image method. As pointed out by Ochmann [2] that, using the method of superposition, Sommerfeld [3] solved the half-space problem by summing the contribution from the original and mirror heat sources and that from a line integration over the single thermal source placed along the z -axis below the mirror source. In Sommerfeld's original problem, the half-space is thermally homogeneous, i.e., the thermal conduc-

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tivity is constant everywhere in the half-space. Recently, however, Sommerfeld's solution and approach have been extended to the sound field study caused by a monopole source above an impedance plane [2,4,5] and to the investigation of a steady point heat source interacting with either a weakly or highly conducting interface between two half-spaces [6].

Functionally graded materials (FGMs), in which the material properties vary smoothly (usually in a fixed direction), have been utilized as thermal barrier coatings in high-temperature environment. Recently, some Green's functions in FGMs have been derived for the development of the corresponding FGM boundary element method (BEM) (see, i.e., [7–9]). In the work by Gray et al. [7], the *full-space* Green's functions of a steady heat source was obtained for the graded material where the thermal conductivity varies exponentially in a fixed direction. So far, however, the corresponding *half-space* Green's function is still unavailable. Such a half-space Green's function is important itself in the study of the FGM effect on the Green's function response, and further it can be implemented into a BEM formulation, avoiding the discretization of the planar surface of the half space.

Therefore, the main focus of this investigation is to derive the three-dimensional (3D) Green's functions for a steady point heat source in a FGM half-space with thermal conductivity varying exponentially along an arbitrary direction. After introducing an auxiliary function, we find that this new function satisfies an inhomogeneous Helmholtz equation, with the boundary value problem being very similar to that for a point source above an impedance plane [2,5]. Subsequently the Sommerfeld's method is employed to derive an explicit expression for the temperature field induced by the steady point heat source in the FGM half-space. Also in this paper, we have studied a couple of associated problems: The first problem is for a point heat source moving at a constant speed in a FGM full-space. Again, we introduce a new function which satisfies an inhomogeneous Helmholtz equation in a moving coordinate system which moves together with the point heat source. Based on a similar approach, we have found the temperature field induced by the moving heat source. The second problem considered is a static point electric charge in a dielectric full-space with electric field gradient effects. We employ the theories first proposed by Mindlin [10] and recently developed by Yang et al. [11], which account for the size effects resulting from the microstructure, similar to the strain gradient elasticity theory [12–14]. It is found that the electric potential satisfies an inhomogeneous 3D Helmholtz–Laplace equation with its solution being subsequently derived. As a third associated problem, we consider the out-of-plane displacement induced by an anti-plane time-harmonic line force in a homogeneous and FGM plane with strain gradient effect. For this problem, we adopt the formulation similar to Paulino et al. [14] and find that the displacement field for a homogeneous plane and the newly introduced function for a FGM plane both satisfy an inhomogeneous two-dimensional (2D) Helmholtz–Helmholtz equation, with its solution being easily found. It is remarked that when adopting the electric field gradient or strain gradient theories, the solution is non-singular at the location of the point charge or the line force, in contrast to the solution for the corresponding homogeneous space.

2. A steady point heat source in a FGM half-space

In a fixed Cartesian coordinate system (x, y, z) , we consider the upper half-space $z \geq 0$ with its boundary at $z = 0$. A steady point heat source of strength H is located at point $(0, 0, h)$, ($h > 0$) in the half-space. Let T be the temperature field, then the heat fluxes q_x , q_y , q_z are given by

$$q_x = -k \frac{\partial T}{\partial x}, \quad q_y = -k \frac{\partial T}{\partial y}, \quad q_z = -k \frac{\partial T}{\partial z}, \quad (1)$$

where $k(x, y, z)$ is the thermal conductivity of the half-space. In this investigation $k(x, y, z)$ is assumed to vary exponentially along an arbitrary direction as

$$k = k_0 \exp(\beta_1 x + \beta_2 y + \beta_3 z), \quad (2)$$

where k_0 , β_1 , β_2 , β_3 are material constants. It's apparent that the gradient direction cosines are given by $\left[\frac{\beta_1}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}, \frac{\beta_2}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}, \frac{\beta_3}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} \right]$. On the other hand, the heat fluxes in the half-space must satisfy the following relationship:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = H\delta(x)\delta(y)\delta(z-h), \quad (z \geq 0), \quad (3)$$

where $\delta()$ is the Dirac delta function.

It follows from Eqs. (1)–(3) that the temperature field obeys the following inhomogeneous perturbed Laplace equation:

$$\nabla^2 T + \beta_1 \frac{\partial T}{\partial x} + \beta_2 \frac{\partial T}{\partial y} + \beta_3 \frac{\partial T}{\partial z} = -\frac{H}{k_0 \exp(\beta_3 h)} \delta(x)\delta(y)\delta(z-h), \quad (z \geq 0), \quad (4)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the 3D Laplace operator.

The boundary condition on the surface $z = 0$ is given by

$$\frac{\partial T}{\partial z} - \lambda T = 0, \quad (z = 0), \quad (5)$$

where the relative heat transfer coefficient λ is a non-negative constant. When $\lambda = 0$, the surface $z = 0$ is insulated, whilst when $\lambda \rightarrow \infty$, it becomes a conducting surface.

Our task now is to find the temperature field satisfying Eqs. (4) and (5). If we first introduce a new function G which is related to the temperature T through

$$T = \exp\left(-\frac{\beta_1 x + \beta_2 y + \beta_3 z}{2}\right) G, \quad (6)$$

then Eqs. (4) and (5) can be expressed in terms of G as follows:

$$\nabla^2 G - \alpha^2 G = -\frac{H}{k_0} \exp\left(-\frac{\beta_3 h}{2}\right) \delta(x)\delta(y)\delta(z-h), \quad (z \geq 0), \quad (7)$$

$$\frac{\partial G}{\partial z} - \gamma G = 0, \quad \text{on } z = 0, \quad (8)$$

where $\alpha = \frac{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}{2}$ and $\gamma = \lambda + \frac{\beta_3}{2}$. It is found that Eq. (7) is in fact an inhomogeneous Helmholtz equation, and Eqs. (7) and (8) are very similar to the equations for a point source above an impedance plane [2,5].

Following the approach by Sommerfeld [3] and Ochmann [2], we can easily solve Eqs. (7) and (8) with the function G being explicitly as

$$G = \frac{H}{4\pi k_0} \exp\left(-\frac{\beta_3 h}{2}\right) \left\{ \frac{\exp\left[-\alpha\sqrt{x^2 + y^2 + (z-h)^2}\right]}{\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{\exp\left[-\alpha\sqrt{x^2 + y^2 + (z+h)^2}\right]}{\sqrt{x^2 + y^2 + (z+h)^2}} - 2\gamma \int_0^{+\infty} \frac{\exp\left[-\alpha\sqrt{x^2 + y^2 + (z+h+\eta)^2} - \gamma\eta\right]}{\sqrt{x^2 + y^2 + (z+h+\eta)^2}} d\eta \right\}, \quad (z \geq 0). \quad (9)$$

It is mentioned that the first term in the parenthesis $\{ \}$ on the right-hand side of Eq. (9) represents the monopole source located at $(0, 0, h)$; the second term a mirror source at $(0, 0, -h)$; and the third term a line integral over the single source placed along the z -axis below the mirror point. We further point out that the line integral in Eq. (9) is always convergent due to the fact that the term $\alpha\sqrt{x^2 + y^2 + (z+h+\eta)^2} + \gamma\eta$ is always non-negative.

It follows from Eqs. (6) and (9) that the temperature field T in the half-space ($z \geq 0$) is given by

$$T = \frac{H}{4\pi k_0} \exp \left[-\frac{\beta_1 x + \beta_2 y + \beta_3(z+h)}{2} \right] \left\{ \frac{\exp \left[-\alpha \sqrt{x^2 + y^2 + (z-h)^2} \right]}{\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{\exp \left[-\alpha \sqrt{x^2 + y^2 + (z+h)^2} \right]}{\sqrt{x^2 + y^2 + (z+h)^2}} - 2\gamma \int_0^{+\infty} \frac{\exp \left[-\alpha \sqrt{x^2 + y^2 + (z+h+\eta)^2} - \gamma\eta \right]}{\sqrt{x^2 + y^2 + (z+h+\eta)^2}} d\eta \right\}, \quad (z \geq 0). \quad (10)$$

Particularly the temperature along the positive z -axis ($x = y = 0$) can be concisely given by

$$\tilde{T} = \exp \left[-\frac{\tilde{\beta}_3(\tilde{z}+1)}{2} \right] \left\{ \frac{\exp[-\tilde{\alpha}|\tilde{z}-1|]}{|\tilde{z}-1|} + \frac{\exp[-\tilde{\alpha}(\tilde{z}+1)]}{\tilde{z}+1} - 2\tilde{\gamma} \exp[\tilde{\gamma}(\tilde{z}+1)] E_1[(\tilde{\alpha} + \tilde{\gamma})(\tilde{z}+1)] \right\}, \quad (\tilde{z} \geq 0), \quad (11)$$

where $\tilde{T} = \frac{4\pi h k_0}{H} T$, $\tilde{z} = \frac{z}{h}$, $\tilde{\beta}_3 = \beta_3 h$, $\tilde{\alpha} = \alpha h$, $\tilde{\gamma} = \gamma h$ are all dimensionless and $E_1(\lambda)$ is the exponential integral function [15]

$$E_1(\lambda) = \int_{\lambda}^{\infty} \frac{\exp(-t)}{t} dt. \quad (12)$$

Fig. 1 illustrates the temperature distribution along the positive z -axis for four different values of the gradient parameter $\tilde{\beta}_3 = 0, 0.5, 1, 3$ with fixed $\beta_1 = \beta_2 = 0$ and $\lambda = 0$ (an insulating surface). Apparently the boundary condition $\frac{\partial T}{\partial z} = 0$ on $z = 0$ is satisfied. Fig. 1 also shows clearly that the gradient parameter $\tilde{\beta}_3$ has a significant

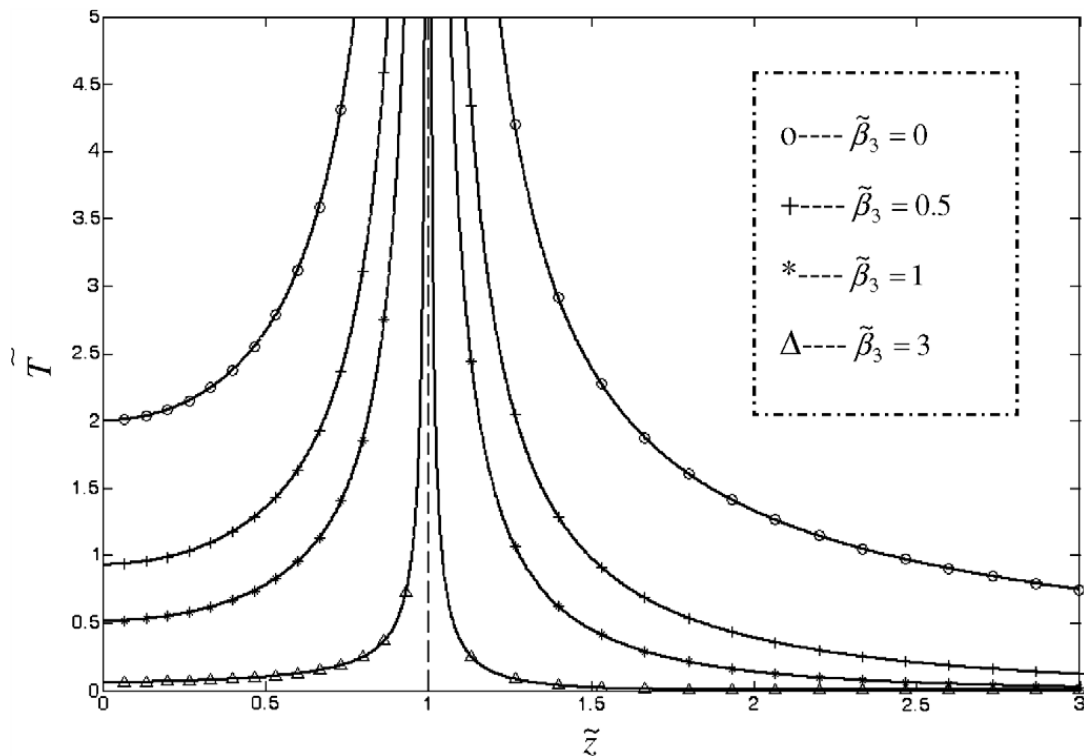


Fig. 1. Distribution of the temperature field along the positive z -axis for four values of the gradient parameter $\tilde{\beta}_3 = 0, 0.5, 1, 3$ with $\beta_1 = \beta_2 = 0$ and $\lambda = 0$.

influence on the temperature distribution in the half-space. When $\tilde{\beta}_3 \geq 3$, the temperature quickly approaches zero as the field point moves far away from the source ($\tilde{z} = 1$) into the half space ($\tilde{z} \gg 1$).

Several special cases from our general solution are discussed below:

- (1) For the boundary condition $T = 0$ on the surface $z = 0$ (i.e., $\lambda \rightarrow \infty$), the temperature field in the half-space is simplified to

$$T = \frac{H}{4\pi k_0} \exp\left[-\frac{\beta_1 x + \beta_2 y + \beta_3(z+h)}{2}\right] \times \left\{ \frac{\exp\left[-\alpha\sqrt{x^2 + y^2 + (z-h)^2}\right]}{\sqrt{x^2 + y^2 + (z-h)^2}} - \frac{\exp\left[-\alpha\sqrt{x^2 + y^2 + (z+h)^2}\right]}{\sqrt{x^2 + y^2 + (z+h)^2}} \right\}, \quad (z \geq 0), \quad (13)$$

- (2) If $\beta_1 = \beta_2 = \beta_3 = 0$, then it follows from Eq. (10) that

$$T = \frac{H}{4\pi k_0} \left\{ \frac{1}{\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{1}{\sqrt{x^2 + y^2 + (z+h)^2}} - 2\lambda \int_0^{+\infty} \frac{\exp(-\lambda\eta)}{\sqrt{x^2 + y^2 + (z+h+\eta)^2}} d\eta \right\}, \quad (z \geq 0), \quad (14)$$

which is just the classical result of Sommerfeld for a homogeneous half-space.

- (3) It follows from Eq. (10) that the temperature field induced by a steady point heat source at the origin of the coordinate system in a FGM full-space is expediently given by

$$T = \frac{H}{4\pi k_0} \exp\left(-\frac{\beta_1 x + \beta_2 y + \beta_3 z}{2}\right) \frac{\exp\left(-\alpha\sqrt{x^2 + y^2 + z^2}\right)}{\sqrt{x^2 + y^2 + z^2}}, \quad (15)$$

which is consistent with Gray et al. [7]. However, it is noted that by introducing the new function G , the derivation of the full-space Green's function becomes much simple due to the fact that the new function satisfies a Helmholtz equation with its Green's function being well known.

3. A moving point heat source in a FGM full-space

Through introduction of a new function, the temperature field induced by a point heat source moving at a constant speed in a FGM full-space can also be easily obtained. We assume that a point heat source of strength H in a FGM full-space is located at the origin of a fixed Cartesian coordinate system (x, y, z) at time $t = 0$, and that the heat source moves with a constant speed V along the positive x -direction. We further assume that the thermal conductivity of the FGM full-space varies exponentially as

$$k = k_0 \exp(\beta_2 y + \beta_3 z), \quad (16)$$

where k_0, β_2, β_3 are material constants. Eq. (16) implies that the thermal conductivity is *constant* in the x -direction. In addition the thermal diffusivity k_d is assumed to be constant everywhere. Consequently the temperature field satisfies the following equation:

$$\nabla^2 T + \beta_2 \frac{\partial T}{\partial y} + \beta_3 \frac{\partial T}{\partial z} - \frac{1}{k_d} \frac{\partial T}{\partial t} = -\frac{H}{k_0} \delta(x - Vt) \delta(y) \delta(z). \quad (17)$$

In order to solve the above equation, we introduce a new moving coordinate system (ξ, η, ζ) which is related to the fixed coordinate system (x, y, z) through

$$\xi = x - Vt, \quad \eta = y, \quad \zeta = z. \quad (18)$$

In view of the fact that the moving coordinate system moves at the same speed as the heat source, the temperature field in the new moving coordinate system does not explicitly depends on the time t . Therefore, in the new coordinate system, Eq. (17) is changed into the following inhomogeneous perturbed Laplace equation

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial \zeta^2} + \frac{V}{k_d} \frac{\partial T}{\partial \xi} + \beta_2 \frac{\partial T}{\partial \eta} + \beta_3 \frac{\partial T}{\partial \zeta} = -\frac{H}{k_0} \delta(\xi) \delta(\eta) \delta(\zeta). \quad (19)$$

Following the similar approach utilized in the previous section, we now introduce a new function G which is related to T through

$$T = \exp \left[-\frac{(V/k_d)\xi + \beta_2\eta + \beta_3\zeta}{2} \right] G. \quad (20)$$

As a result, Eq. (19) is changed into an inhomogeneous Helmholtz equation

$$\frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2} + \frac{\partial^2 G}{\partial \zeta^2} - \alpha^2 G = -\frac{H}{k_0} \delta(\xi) \delta(\eta) \delta(\zeta), \quad (21)$$

where $\alpha = \frac{\sqrt{(V/k_d)^2 + \beta_2^2 + \beta_3^2}}{2}$. The solution to Eq. (20) is readily given by

$$G = \frac{H}{4\pi k_0} \frac{\exp \left(-\alpha \sqrt{\xi^2 + \eta^2 + \zeta^2} \right)}{\sqrt{\xi^2 + \eta^2 + \zeta^2}}. \quad (22)$$

In view of Eqs. (18) and (20), the temperature field in the fixed coordinate system is obtained as

$$T = \frac{H}{4\pi k_0} \frac{1}{\sqrt{(x - Vt)^2 + y^2 + z^2}} \exp \left[-\frac{(V/k_d)(x - Vt) + \beta_2 y + \beta_3 z}{2} - \frac{\sqrt{(V/k_d)^2 + \beta_2^2 + \beta_3^2}}{2} \sqrt{(x - Vt)^2 + y^2 + z^2} \right]. \quad (23)$$

When $\beta_2 = \beta_3 = 0$, Eq. (23) reduces to

$$T = \frac{H}{4\pi k_0} \frac{1}{\sqrt{(x - Vt)^2 + y^2 + z^2}} \exp \left\{ -\frac{V}{2k_d} \left[\sqrt{(x - Vt)^2 + y^2 + z^2} + (x - Vt) \right] \right\}, \quad (24)$$

which is exactly the solution for a homogeneous full-space [16].

4. A static point charge in a dielectric full-space with electric field gradient effect

In this section, we investigate the electric potential induced by a static point electric charge Q at the origin of a Cartesian coordinate system (x, y, z) in a dielectric full-space with electric gradient effects. Here we employ the theories first proposed by Mindlin [10] and recently enhanced by Yang et al. [11], which account for the size effects resulting from the underlying microstructure. In order to address the problem in a simplified setting, we assume that the piezoelectric effect is ignored and the material is dielectrically isotropic. Therefore, the electric potential ϕ satisfies the following inhomogeneous 3D Helmholtz–Laplace equation (see, i.e., Yang et al. [11] for comparison)

$$\nabla^2 \phi - l^2 \nabla^2 \nabla^2 \phi = -\frac{Q}{\epsilon} \delta(x) \delta(y) \delta(z), \quad (25)$$

where ∇^2 is again the 3D Laplace operator; $l \geq 0$ the material characteristic length; and ϵ the dielectric constant in the dielectric space. The electric displacements D_x , D_y , D_z are related to the electric potential ϕ through

$$D_x = -\epsilon (1 - l^2 \nabla^2) \frac{\partial \phi}{\partial x}, \quad D_y = -\epsilon (1 - l^2 \nabla^2) \frac{\partial \phi}{\partial y}, \quad D_z = -\epsilon (1 - l^2 \nabla^2) \frac{\partial \phi}{\partial z}. \quad (26)$$

To find the solution to Eq. (25), we express it equivalently as

$$\nabla^2 P_1 = -\frac{Q}{\epsilon} \delta(x)\delta(y)\delta(z), \tag{27}$$

$$(\nabla^2 - l^2)P_2 = \frac{Q}{\epsilon l^2} \delta(x)\delta(y)\delta(z), \tag{28}$$

where $P_1 = (1 - l^2 \nabla^2)\phi$, $P_2 = \nabla^2 \phi$.

Apparently, the solution to the Laplace Eq. (27) is

$$P_1 = (1 - l^2 \nabla^2)\phi = \frac{Q}{4\pi \epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \tag{29}$$

and the solution to the Helmholtz Eq. (28) is

$$P_2 = \nabla^2 \phi = -\frac{Q}{4\pi \epsilon l^2} \frac{\exp\left(-l^{-1} \sqrt{x^2 + y^2 + z^2}\right)}{\sqrt{x^2 + y^2 + z^2}}. \tag{30}$$

It follows from Eqs. (29) and (30) that the electric potential induced by the electric charge is

$$\phi = \frac{Q}{4\pi \epsilon} \frac{1 - \exp\left(-l^{-1} \sqrt{x^2 + y^2 + z^2}\right)}{\sqrt{x^2 + y^2 + z^2}}. \tag{31}$$

As $l \rightarrow 0$, which corresponds to the dielectric ceramics without electric field gradient effects, Eq. (31) reduces to the classical solution

$$\phi = \frac{Q}{4\pi \epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}}. \tag{32}$$

There are a couple of features associated with Eqs. (31) and (32), and we briefly discuss them below:

- (1) It can be easily shown that when $r \rightarrow \infty$, Eq. (31) approaches the classical solution Eq. (32).

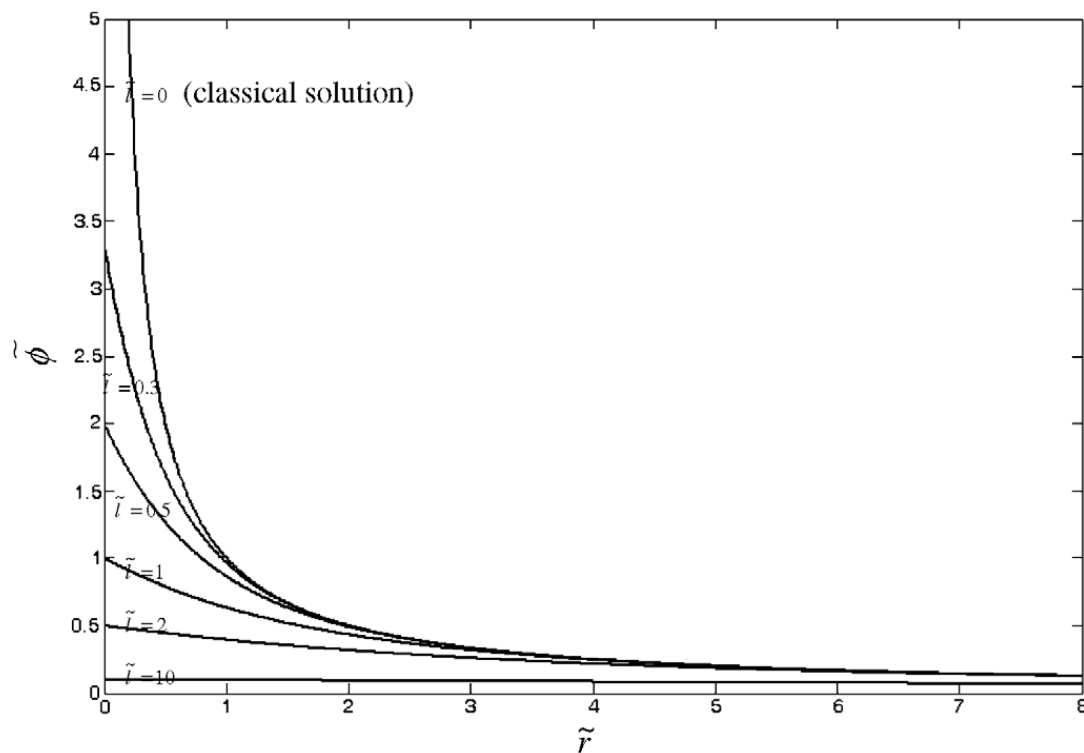


Fig. 2. Distribution of $\tilde{\phi}$ as a function of \tilde{r} for different values of the characteristic length \tilde{l} .

- (2) It is of interest to observe from Eq. (31) that the electric potential due to a point charge in a dielectric ceramic with electric field gradient effects is still *finite* (or non-singular) when $r = \sqrt{x^2 + y^2 + z^2} \rightarrow 0$ (i.e., when approaching the location of the electric charge). More specifically

$$\lim_{r \rightarrow 0} \phi = \frac{Q}{4\pi \epsilon l}, \tag{33}$$

which is inversely proportional to l . Furthermore it can be proved that

$$\lim_{r \rightarrow 0} \frac{d^n \phi}{dr^n} = (-1)^n \frac{Q}{4\pi \epsilon (n+1)l^{n+1}}, \quad (n = 0, 1, 2, 3, \dots), \tag{34}$$

where $\frac{d^0 \phi}{dr^0} = \phi$. A detailed proof of Eq. (34) is presented in Appendix A.

- (3) On the other hand, we clearly observe from Eq. (32) that, in a dielectric ceramic without electric field gradient, the electric potential as well as its derivatives with respect to r is *singular* when $r \rightarrow 0$.

The above limiting behaviors for the electric potential due to a point charge are in agreement with those observed by Yang et al. [11] for electric potential due to a line charge. Fig. 2 demonstrates the distribution of $\tilde{\phi} = \frac{1 - \exp(-\tilde{l}^{-1}\tilde{r})}{\tilde{r}}$, where $\tilde{\phi} = \frac{4\pi\epsilon r_0}{Q} \phi$, $\tilde{r} = \frac{r}{r_0}$, $\tilde{l} = \frac{l}{r_0}$ with r_0 being the nominal length, as a function of \tilde{r} for various values of the characteristic length \tilde{l} . The limiting behaviors at the origin and the far field can be clearly observed from this figure.

5. Two-dimensional time-harmonic Green's functions for homogeneous materials with strain gradient effect

In this section, we consider the 2D time-harmonic Green's functions for homogeneous elastic materials with strain gradient effects. To simplify the analysis, we only consider the anti-plane deformation case. A time-harmonic line force $pe^{-i\omega t}$ in the z -direction with circular frequency ω is distributed on the z -axis. In what follows, the time dependence of all the field components will be suppressed. Adopting the related expressions presented in Paulino et al. [14] and assuming the material to be homogeneous, it is found that the out-of-plane displacement w satisfies the following equation:

$$\nabla^2 w - l^2 \nabla^2 \nabla^2 w + k^2 w = -\frac{p}{\mu} \delta(x) \delta(y), \tag{35}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2D Laplace operator, μ shear modulus of the elastic material, l the material characteristic length, and $k = \omega / \sqrt{\mu/\rho}$ with ρ being the mass density of the elastic material.

Eq. (35) can also be equivalently expressed via the following 2D inhomogeneous Helmholtz–Helmholtz equation:

$$(\nabla^2 + k_1^2)(\nabla^2 - k_2^2)w = \frac{p}{l^2 \mu} \delta(x) \delta(y), \tag{36}$$

where the two positive constants k_1 and k_2 are given by

$$k_1 = \frac{\sqrt{2}k}{\sqrt{1 + \sqrt{1 + 4k^2 l^2}}}, \quad k_2 = \frac{\sqrt{1 + \sqrt{1 + 4k^2 l^2}}}{\sqrt{2}l}. \tag{37}$$

Taking $\nabla^2 w - k_2^2 w$ as a new function in Eq. (36) and $\nabla^2 w + k_1^2 w$ as the other, it can be easily shown that

$$\nabla^2 w - k_2^2 w = -\frac{ip}{4l^2 \mu} H_0^{(1)}(k_1 r), \tag{38}$$

$$\nabla^2 w + k_1^2 w = -\frac{p}{2\pi l^2 \mu} K_0(k_2 r), \tag{39}$$

where $H_n^{(1)}$ is the n th-order Hankel function of the first kind, K_n is the n th-order modified Bessel function of the second kind, and $r = \sqrt{x^2 + y^2}$.

Consequently, the out-of-plane displacement w can be obtained from Eqs. (38) and (39) as

$$w = \frac{P}{2\pi\sqrt{1+4k^2l^2}\mu} \left[\frac{i\pi}{2} H_0^{(1)}(k_1r) - K_0(k_2r) \right]. \quad (40)$$

For the classical elastic material without strain gradient (i.e., $l \rightarrow 0$), the above solution reduces to the well known Green's solution for 2D Helmholtz equation

$$w = \frac{ip}{4\mu} H_0^{(1)}(kr). \quad (41)$$

On the other hand if we let $\omega \rightarrow 0$ (or $k \rightarrow 0$) for a static line force, Eq. (40) reduces to

$$w = -\frac{P}{2\pi\mu} [\ln r + K_0(l^{-1}r)], \quad (42)$$

which is in agreement with the result in Yang et al. [11].

In view of the following asymptotic behaviors of the Hankel and Bessel functions

$$H_0^{(1)}(k_1r) \rightarrow 1 + \frac{2i}{\pi} \ln\left(\frac{\gamma k_1r}{2}\right), \quad K_0(k_2r) \rightarrow -\ln(k_2r/2), \quad \text{when } r \rightarrow 0 \quad (43)$$

one can easily verify that Eq. (40) is finite (non-singular) at origin, while Eq. (41) exhibits logarithmic singularity at origin.

On the other hand, at the far field $r \gg 1$, we have the following asymptotic behaviors

$$H_0^{(1)}(k_1r) \rightarrow \sqrt{\frac{2}{\pi k_1r}} e^{i(k_1r - \frac{\pi}{4})}, \quad K_0(k_2r) \rightarrow \sqrt{\frac{\pi}{2k_2r}} e^{-k_2r}, \quad \text{when } r \rightarrow \infty. \quad (44)$$

Then the asymptotic expression of Eq. (40) for w at infinity is

$$w = \frac{P}{\mu\sqrt{8\pi k_1(1+4k^2l^2)}r} e^{i(k_1r - \frac{\pi}{4})}, \quad \text{when } r \rightarrow \infty \quad (45)$$

whilst the asymptotic expression of the classical solution Eq. (41) at infinity is

$$w = \frac{P}{\mu\sqrt{8\pi kr}} e^{i(kr - \frac{\pi}{4})} \quad \text{when } r \rightarrow \infty. \quad (46)$$

Comparing Eq. (45) with Eq. (46), we find that the behavior of w at the far field in the strain gradient space is different to the classical solution. This phenomenon is different to the corresponding static case in which w at far field approaches the classical solution.

6. Two-dimensional time-harmonic Green's functions for FGMs with strain gradient effect

In Section 5, it is assumed that the elastic plane is homogeneous. In this section, we consider the more difficult problem: a time-harmonic line force in a FGM plane with strain gradient effect. The problem considered here is very similar to the one considered in the previous section except that the shear modulus and mass density of the material vary exponentially according to

$$\mu = \mu_0 \exp(\beta y), \quad \rho = \rho_0 \exp(\beta y), \quad (47)$$

where μ_0 , ρ_0 and β are material constants. Also adopting the basic equations in Paulino et al. [14], it is found that the displacement w satisfies the following equation:

$$\left(1 - \beta l^2 \frac{\partial}{\partial y} - l^2 \nabla^2\right) \left(\nabla^2 + \beta \frac{\partial}{\partial y}\right) w + k^2 w = -\frac{P}{\mu_0} \delta(x) \delta(y), \quad (48)$$

where $k = \omega / \sqrt{\mu_0 / \rho_0}$.

First we introduce a new function Θ which is related to w through

$$w = \exp\left(-\frac{\beta}{2}y\right)\Theta. \tag{49}$$

As a result, Eq. (48) can now be expressed as

$$\left(1 + \frac{\beta^2 l^2}{4} - l^2 \nabla^2\right)\left(\nabla^2 - \frac{\beta^2}{4}\right)\Theta + k^2\Theta = -\frac{P}{\mu_0}\delta(x)\delta(y), \tag{50}$$

or equivalently

$$\nabla^4\Theta - \tilde{l}^{-2}\nabla^2\Theta - \left[k^2 l^{-2} - \frac{\beta^2}{4}\left(\tilde{l}^{-2} - \frac{\beta^2}{4}\right)\right]\Theta = \frac{P}{l^2\mu_0}\delta(x)\delta(y), \tag{51}$$

where $\tilde{l} = \frac{1}{\sqrt{l^{-2} + \frac{\beta^2}{2}}}$.

We now discuss the solution to the above equation for the following two cases.

6.1. Case one: $k^2 \geq \frac{\beta^2 l^2}{4}\left(\tilde{l}^{-2} - \frac{\beta^2}{4}\right)$

For this case, Eq. (51) can be expressed as the following inhomogeneous Helmholtz–Helmholtz equation

$$(\nabla^2 + k_1^2)(\nabla^2 - k_2^2)\Theta = \frac{P}{l^2\mu_0}\delta(x)\delta(y), \tag{52}$$

where the two positive constants k_1 and k_2 are given by

$$k_1 = \frac{\sqrt{2}\tilde{k}}{\sqrt{1 + \sqrt{1 + 4\tilde{k}^2\tilde{l}^2}}}, \quad k_2 = \frac{\sqrt{1 + \sqrt{1 + 4\tilde{k}^2\tilde{l}^2}}}{\sqrt{2}\tilde{l}}, \quad \tilde{k} = \frac{\tilde{l}}{l}\sqrt{k^2 - \frac{\beta^2 l^2}{4}\left(\tilde{l}^{-2} - \frac{\beta^2}{4}\right)}. \tag{53}$$

Eq. (52) is identical to Eq. (36), and its solution is expediently given by

$$\Theta = \frac{P\tilde{l}^2}{2\pi l^2\sqrt{1 + 4\tilde{k}^2\tilde{l}^2}\mu_0}\left[\frac{i\pi}{2}H_0^{(1)}(k_1 r) - K_0(k_2 r)\right]. \tag{54}$$

Inserting the above expression into Eq. (49) results in the expression of the displacement w as

$$w = \frac{P\tilde{l}^2}{2\pi l^2\sqrt{1 + 4\tilde{k}^2\tilde{l}^2}\mu_0}\exp\left(-\frac{\beta}{2}y\right)\left[\frac{i\pi}{2}H_0^{(1)}(k_1 r) - K_0(k_2 r)\right]. \tag{55}$$

Apparently, the above expression is non-singular as $r \rightarrow 0$.

6.2. Case two: $k^2 < \frac{\beta^2 l^2}{4}\left(\tilde{l}^{-2} - \frac{\beta^2}{4}\right)$

For this case, Eq. (51) can be expressed as the following inhomogeneous Helmholtz–Helmholtz equation

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\Theta = \frac{P}{l^2\mu_0}\delta(x)\delta(y), \tag{56}$$

where the two positive constants k_1 and k_2 are given by

$$k_1 = \frac{\sqrt{2}\tilde{k}}{\sqrt{1 + \sqrt{1 - 4\tilde{k}^2\tilde{l}^2}}}, \quad k_2 = \frac{\sqrt{1 + \sqrt{1 - 4\tilde{k}^2\tilde{l}^2}}}{\sqrt{2}\tilde{l}}, \quad \tilde{k} = \frac{\tilde{l}}{l}\sqrt{\frac{\beta^2 l^2}{4}\left(\tilde{l}^{-2} - \frac{\beta^2}{4}\right) - k^2}. \tag{57}$$

It is noted that $1 - 4\tilde{k}^2\tilde{l}^2 = \left(1 - \frac{\beta^2\tilde{l}^2}{2}\right)^2 + 4k^2\tilde{l}^4 l^{-2} \geq 0$.

It follows from Eq. (56) that

$$\nabla^2 \Theta - k_2^2 \Theta = -\frac{P}{2\pi l^2 \mu_0} K_0(k_1 r), \quad (58)$$

$$\nabla^2 \Theta - k_1^2 \Theta = -\frac{P}{2\pi l^2 \mu_0} K_0(k_2 r). \quad (59)$$

As a result we have

$$\Theta = \frac{p \tilde{l}^2}{2\pi l^2 \sqrt{1 - 4\tilde{k}^2 \tilde{l}^2} \mu_0} [K_0(k_1 r) - K_0(k_2 r)]. \quad (60)$$

Inserting the above solution into Eq. (49) results in the expression of the displacement w as

$$w = \frac{p \tilde{l}^2}{2\pi l^2 \sqrt{1 - 4\tilde{k}^2 \tilde{l}^2} \mu_0} \exp\left(-\frac{\beta}{2} y\right) [K_0(k_1 r) - K_0(k_2 r)], \quad (61)$$

which is regular as $r \rightarrow 0$.

7. Conclusions

The Green's function for a steady point heat source in a FGM half-space is derived in Section 2 by virtue of a new function and the image method of Sommerfeld. Since the Green's function and its derivatives are very simple, it can be conveniently implemented into a BEM formulation for practical applications in the future.

Some associated problems are also studied in this paper: a constantly moving point heat source in a FGM full-space in Section 3; a static point charge in a dielectric full-space with electric field gradient effects in Section 4; and a time-harmonic line force in a homogeneous and FGM planes with strain gradient effects in Sections 5 and 6. For the point charge case (Section 4), we find that the electric potential and its derivatives due to the point charge in a dielectric full-space with electric field gradient effects are still finite at the source point. This is in contrast to the classical solution where the source point is singular. For the case of a time-harmonic line force (Section 5), we find that the out-of-plane displacement is non-singular at the source location and that at the far field it does not approach the classical solution. This is in contrast to both the classical solution and the corresponding static solution. We expect that the new fundamental solution Eq. (31) can be incorporated into a BEM formulation suitable for the dielectric ceramics accounting for electric field gradient effects, and that Eqs. (40), (55) and (61) can also be applied to formulate a BEM for time-harmonic problems in homogeneous materials and FGMs with strain gradient effects.

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Appendix A. Proof of Eq. (34)

Apparently Eq. (34) is valid for $n = 0$, i.e.,

$$\lim_{r \rightarrow 0} \phi = \frac{Q}{4\pi \epsilon l}. \quad (A.1)$$

If we assume that Eq. (34) is valid for an integer $n = N$, then

$$\lim_{r \rightarrow 0} \frac{d^N \phi}{dr^N} = (-1)^N \frac{Q}{4\pi \epsilon (N+1) l^{N+1}}. \quad (A.2)$$

It can be easily verified from Eq. (31) that

$$r \frac{d\phi}{dr} + \phi = \frac{Q}{4\pi \in l} \exp(-l^{-1}r). \quad (\text{A.3})$$

Then it follows from the above expression that

$$r \frac{d^{N+1}\phi}{dr^{N+1}} + (N+1) \frac{d^N\phi}{dr^N} = (-1)^N \frac{Q}{4\pi \in l^{N+1}} \exp(-l^{-1}r). \quad (\text{A.4})$$

The above can also be equivalently expressed as

$$\frac{d^{N+1}\phi}{dr^{N+1}} = \frac{-(N+1) \frac{d^N\phi}{dr^N} + (-1)^N \frac{Q}{4\pi \in l^{N+1}} \exp(-l^{-1}r)}{r}. \quad (\text{A.5})$$

Taking the limit $r \rightarrow 0$ to both sides of the above expression, we have

$$\lim_{r \rightarrow 0} \frac{d^{N+1}\phi}{dr^{N+1}} = \lim_{r \rightarrow 0} \frac{-(N+1) \frac{d^N\phi}{dr^N} + (-1)^N \frac{Q}{4\pi \in l^{N+1}} \exp(-l^{-1}r)}{r}. \quad (\text{A.6})$$

In view of (A.2), the numerator of the right hand side approaches zero when $r \rightarrow 0$. Then by applying the L'Hospital's Rule to the right hand side of (A.6), we finally arrive at

$$\lim_{r \rightarrow 0} \frac{d^{N+1}\phi}{dr^{N+1}} = (-1)^{N+1} \frac{Q}{4\pi \in (N+2)l^{N+2}}, \quad (\text{A.7})$$

which states that Eq. (34) is also valid for $n = N + 1$.

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