A Functionally Graded Plane With a Circular Inclusion Under Uniform Antiplane Eigenstrain

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The problem of a functionally graded plane with a circular inclusion under a uniform antiplane eigenstrain is investigated, where the shear modulus varies exponentially along the x direction. By introducing a new function which satisfies the Helmholtz equation, the general solution to the original problem is derived in terms of series expansion. Numerical results are then presented which demonstrate clearly that for a functionally graded plane, the strain and stress fields inside the circular inclusion under uniform antiplane eigenstrains are intrinsically nonuniform. This phenomenon differs from the corresponding homogeneous material case where both the strain and stress fields are uniform inside the circular inclusion. [DOI: 10.1115/1.2745391]

Keywords: functionally graded material, Eshelby eigenstrain, circular inclusion, antiplane deformation

1 Introduction

The well-known result of Eshelby [1] for an elastic space shows that the strain and stress fields inside an ellipsoidal (and elliptical) inclusion under uniform eigenstrains are uniform. Eshelby's result is based on the assumption that the infinite elastic space is isotropic and homogeneous. Recently, this classic Eshelby problem has been extended to material anisotropy and even piezoelectric coupling [2–4], with applications in novel strained semiconductor quantum structures (see, e.g., [5,6]).

As a new type of composites, functionally graded materials (FGMs) were initially designed as thermal barrier materials for aerospace structures (Koizumi [7]), in which the volume fractions of different constituent materials vary continuously from one side to the other, resulting in smooth variation of material properties. If a FGM space contains an ellipsoidal or elliptical inclusion with uniform eigenstrains, are the strain and stress fields inside the inclusion still uniform? To the best of the authors' knowledge, the Eshelby problem in FGMs has not been addressed, although the fracture problem (see, e.g., [8]) and some Green's function prob-

lems (see, e.g., [9-13]) in FGMs were investigated before. Since the general Eshelby problem in FGMs is very difficult, we consider here only the simple situation in which a FGM plane contains a circular cylindrical inclusion under uniform antiplane eigenstrains. Furthermore we assume that the shear modulus of the FGM varies exponentially along a fixed direction, say the *x* direction, as adopted by Erdogan et al. [8]. In doing so, it is possible for us to derive a general solution to this problem by introducing a new function φ which satisfies the Helmholtz equation. The final series solution is expressed in terms of the modified Bessel functions.

2 General Solution

We consider an infinite FGM in the *x*-*y* plane as shown in Fig. 1, and assume that the shear modulus μ of the FGM varies exponentially in the *x* direction as (e.g., [8])

$$\mu = e^{2\beta x} \mu_0 \tag{1}$$

where μ_0 is the homogeneous shear modulus and β is the gradient factor of the FGM.

We point out that while various processing techniques have been proposed for FGMs (e.g. [14-17]), including the isotropic FGM as a special case [15], the exponential variation described by Eq. (1) could be difficult to achieve experimentally. Therefore, Eq. (1) should be regarded as a simplified FGM model to the more complicated FGMs fabricated from laboratories. We further mention that an isotropic FGM, as the one assumed here, could be realized only for certain spatial variations of composition [14] since random distributed microstructures (i.e., two distinct phases distributed in a disordered fashion) would be locally *anisotropic* [18–20], with the latter requires more involved analysis.

We also assume that, within the FGM, there is a circular inclusion $r = \sqrt{x^2 + y^2} \leq R$ which undergoes uniform antiplane eigenstrains ε_{zx} and ε_{zy} . The boundary condition along the inclusionmatrix interface r=R is assumed to be fully bonded, and can be expressed in terms of the out-of-plane elastic displacements $w^{(1)}$ inside the inclusion and $w^{(2)}$ outside, as

$$w^{(1)} + w^* = w^{(2)}$$
$$\frac{\partial w^{(1)}}{\partial r} = \frac{\partial w^{(2)}}{\partial r} \qquad (r = R)$$
(2)

where $w^* = r(\varepsilon_{zx}^* - i\varepsilon_{zy}^*)e^{i\theta} + r(\varepsilon_{zx}^* + i\varepsilon_{zy}^*)e^{i\theta}$ is the additional displacement corresponding to the uniform eigenstrains $\varepsilon_{zx}^*, \varepsilon_{zy}^*$. The first condition in Eq. (2) states that the displacement is continuous across the interface; while the second one in Eq. (2) implies that the traction σ_{zr} is continuous across the interface. Furthermore, it is easy to show that $w^{(1)}$ and $w^{(2)}$ satisfy the following partial differential equations:

$$\frac{\partial^2 w^{(1)}}{\partial x^2} + \frac{\partial^2 w^{(1)}}{\partial y^2} + 2\beta \frac{\partial w^{(1)}}{\partial x} = 0 \quad \sqrt{x^2 + y^2} \le R$$
$$\frac{\partial^2 w^{(2)}}{\partial x^2} + \frac{\partial^2 w^{(2)}}{\partial y^2} + 2\beta \frac{\partial w^{(2)}}{\partial x} = 0 \quad \sqrt{x^2 + y^2} \ge R \tag{3}$$

We now introduce a new function φ which is related to w through the following relation:

$$w = e^{-\beta x}\varphi \tag{4}$$

It is easy to show that, in terms of the new function φ , the boundary condition (2) can be equivalently expressed as

$$\varphi^{(2)} - \varphi^{(1)} = e^{\beta x} w^*$$

$$\frac{\partial \varphi^{(2)}}{\partial r} - \frac{\partial \varphi^{(1)}}{\partial r} = \beta e^{\beta x} \cos \theta w^* \quad (r = R)$$
(5)

where $\varphi^{(1)}$ and $\varphi^{(2)}$ are within and outside the inclusion, respectively. They satisfy the following Helmholtz equations:

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Fig. 1 An infinite FGM plane containing a circular inclusion with uniform antiplane eigenstrains

$$\frac{\partial^2 \varphi^{(1)}}{\partial x^2} + \frac{\partial^2 \varphi^{(1)}}{\partial y^2} - \beta^2 \varphi^{(1)} = 0 \quad \sqrt{x^2 + y^2} \le R$$
$$\frac{\partial^2 \varphi^{(2)}}{\partial x^2} + \frac{\partial^2 \varphi^{(2)}}{\partial y^2} - \beta^2 \varphi^{(2)} = 0 \quad \sqrt{x^2 + y^2} \ge R \tag{6}$$

In view of Eq. (6), $\varphi^{(1)}$ and $\varphi^{(2)}$ can be expressed in terms of series expansion as

$$\varphi^{(1)} = \sum_{n=-\infty}^{+\infty} A_n^{(1)} I_n(|\beta|r) e^{in\theta} \quad 0 \le r \le R$$
(7)

$$\varphi^{(2)} = \sum_{n=-\infty}^{+\infty} A_n^{(2)} K_n(|\beta|r) e^{in\theta} \quad r \ge R$$
(8)

where I_n and K_n are the modified nth-order Bessel functions of the first and second kinds, respectively; $A_n^{(1)}$ and $A_n^{(2)}$ are unknown coefficients to be determined. In addition, the exponential function $e^{\beta x}$ can be expanded as follows:

$$e^{\beta x} = \sum_{n=-\infty}^{+\infty} I_n(\beta r) e^{in\theta}$$
(9)

Therefore, the two terms on the right-hand side of Eq. (5) can be expanded as

$$e^{\beta x}w^* = R\sum_{n=-\infty}^{+\infty} \left[I_{n-1}(\beta R)(\varepsilon_{zx}^* - i\varepsilon_{zy}^*) + I_{n+1}(\beta R)(\varepsilon_{zx}^* + i\varepsilon_{zy}^*) \right] e^{in\theta}$$
(10)

$$\beta \cos \theta e^{\beta x} w^* = \frac{\beta R}{2} \sum_{n=-\infty}^{+\infty} \left\{ \left[I_{n-2}(\beta R) + I_n(\beta R) \right] (\varepsilon_{zx}^* - i\varepsilon_{zy}^*) + \left[I_n(\beta R) + I_{n+2}(\beta R) \right] (\varepsilon_{zx}^* + i\varepsilon_{zy}^*) \right\} e^{in\theta}$$
(11)

By enforcing the boundary condition (5), we determine the unknown expansion coefficients in Eqs. (7) and (8) as

$$\begin{bmatrix} A_{n}^{(1)} \\ A_{n}^{(2)} \end{bmatrix} = \frac{R}{K_{n}(|\beta|R)I_{n}'(|\beta|R) - I_{n}(|\beta|R)K_{n}'(|\beta|R)} \begin{bmatrix} K_{n}'(|\beta|R) - K_{n}(|\beta|R) \\ I_{n}'(|\beta|R) - I_{n}(|\beta|R) \end{bmatrix}$$

$$\times \begin{bmatrix} I_{n-1}(\beta R)(\varepsilon_{zx}^{*} - i\varepsilon_{zy}^{*}) + I_{n+1}(\beta R)(\varepsilon_{zx}^{*} + i\varepsilon_{zy}^{*}) \\ \frac{\beta}{2|\beta|} \{ [I_{n-2}(\beta R) + I_{n}(\beta R)](\varepsilon_{zx}^{*} - i\varepsilon_{zy}^{*}) + [I_{n}(\beta R) + I_{n+2}(\beta R)](\varepsilon_{zx}^{*} + i\varepsilon_{zy}^{*}) \} \end{bmatrix}$$
(12)

where the prime (') denotes the derivative with respect to the variable in the parentheses.

We mention that the following identities are useful in the calculation of the coefficients:

$$I'_{n}(x) = \frac{I_{n-1}(x) + I_{n+1}(x)}{2}$$
$$K'_{n}(x) = -\frac{K_{n-1}(x) + K_{n+1}(x)}{2}$$
(13)

which can be easily derived using the definitions that

$$I_n(x) = \mathrm{i}^{-n} J_n(\mathrm{i}x)$$

$$K_n(x) = \frac{\pi}{2} \mathbf{i}^{n+1} H_n^{(1)}(\mathbf{i}x) \tag{14}$$

where J_n and $H_n^{(1)}$ are the *n*th order Bessel and Hankel functions of the first kind. The other useful identities are

$$J'_{n}(x) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2}$$

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 $H_n^{(1)\prime}(x) = \frac{H_{n-1}^{(1)}(x) - H_{n+1}^{(1)}(x)}{2}$ (15)

We add that Eq. (9) can be easily derived from the following Jacobi-Anger expansion [21]:

$$e^{ikx} = \sum_{n=-\infty}^{+\infty} i^n J_n(kr) e^{in\theta}$$
(16)

by taking $\beta = ik$ and using the definition of I_n in Eq. (14).

3 Numerical Results

As a numerical example, we consider a circular inclusion with uniform eigenstrains $\varepsilon_{zx}^* \neq 0$ and $\varepsilon_{zy}^* = 0$. We truncate the series in Eqs. (7) and (8) at $n = \pm 10$ in order to obtain a result with a relative truncation error less than 0.1%.

Figure 2 shows the distribution of the normalized stress component

$$\sigma = -\frac{\sigma_{zx}}{\mu_0 \varepsilon_{zx}}$$

where

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Fig. 2 Distribution of the normalized stress component $\sigma = -(\sigma_{zx}/\mu_0 \hat{\epsilon}_{zx})$ along the *x* axis for different gradient parameters $\beta^* = \beta R = 0, 0.2, 0.5, 1$

$$\sigma_{zx} = \mu \frac{\partial w}{\partial x} = \mu_0 e^{\beta x} \left(-\beta \varphi + \frac{\partial \varphi}{\partial x} \right)$$

along the x axis for four different gradient parameters $\beta^* = \beta R$ =0,0.2,0.5,1. It is noted that, for $\beta^* = 0$, which corresponds to a homogeneous plane, variation of σ obeys the following exact expression, which can also be derived from the result of Ru and Schiavone [22]:

$$\sigma = \begin{cases} 1, & |x| \le R \\ \left(\frac{R}{x}\right)^2, & |x| \ge R \end{cases}$$
(17)

It is well known that the stress field inside the circular inclusion is uniform when the plane is homogeneous, which is the classic Eshelby result. When $\beta^* \neq 0$ for an FGM plane, however, the stress field inside the circular inclusion is no longer uniform with its maximum value being always reached at x=R (Fig. 2).

Figure 3 shows the variation of the maximum stress σ_{max} as a function of β^* . It is observed that σ_{max} is a monotonic increasing function of β^* . The influence of the gradient parameter β^* on σ_{max} is significant. For example, when $\beta^*=3$, $\sigma_{\text{max}}=98.3133$, a value nearly 100 times of the one corresponding to the homogeneous material case ($\sigma_{\text{max}}=1$ for $\beta^*=0$).



Fig. 4 Distribution of the normalized total strain component γ along the *x* axis for different gradient parameters $\beta^{*} = 0, 0.2, 0.5, 1$

Besides the stress distribution, we also show in Fig. 4 the distribution of the normalized total strain component (being an element of the Eshelby tensor [1])

$$\gamma = \begin{cases} (\varepsilon_{zx}^{(1)} + \varepsilon_{zx}^*)/\varepsilon_{zx}^*, & |x| \le R\\ \varepsilon_{zx}^{(2)}/\varepsilon_{zx}^*, & |x| \ge R \end{cases}$$
(18)

along the x axis for four different gradient parameters $\beta^* = 0, 0.2, 0.5, 1$. In Eq. (18),

$$\varepsilon_{zx}^{(1)} = \frac{1}{2} \frac{\partial w^{(1)}}{\partial x}$$

and

$$\varepsilon_{zx}^{(2)} = \frac{1}{2} \frac{\partial w^{(2)}}{\partial x}$$

are the elastic strains inside and outside the inclusion, respectively. Similarly, Fig. 4 demonstrates that the Eshelby tensor within the circular inclusion is no longer uniform for an FGM plane ($\beta^* \neq 0$).

Figure 5 shows the distribution of the normalized total displacement



Fig. 3 Variation of the maximum stress σ_{\max} as a function of β^*

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100

90 80 70

60

30 20

10

0

60 Sec



Fig. 5 Distribution of the normalized total displacement δ along the *x* axis for different gradient parameters $\beta^* = 0, 0.2, 0.5, 1$

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$$\delta = \begin{cases} (w^{(1)} + w^{*})/R\varepsilon_{zx}^{*}, & |x| \le R \\ w^{(2)}/R\varepsilon_{zx}^{*}, & |x| \ge R \end{cases}$$
(19)

along the *x* axis for four different gradient parameters $\beta^* = 0, 0.2, 0.5, 1$. It is observed that the displacement field inside the circular inclusion in an FGM plane (especially when $\beta^*=1$) is no longer a linear function of the coordinate *x*. Furthermore, the magnitude of δ for $x \ge R$ is very small when the gradient parameter β^* is large. For example, for a large β^* , say $\beta^* \ge 1$, the magnitude of δ at x=R is $\delta \approx 0$. It is further interesting that, for a large β^* , the corresponding stress σ at x=R is also large ($\sigma > 3.5$ at x=R). On the other hand, for $\beta^* \ge 1$, the magnitude of δ at x=-R is very large ($\delta < -2.5$ at x=-R) while that of σ at x=-R is small ($\sigma \approx 0$ at x=-R). Finally, when $\beta^*=0$, i.e., for the corresponding homogeneous material case, the displacement δ along the *x* axis obeys the following exact expression [22]:

$$\delta = \begin{cases} x/R, & |x| \le R\\ R/x, & |x| \ge R \end{cases}$$
(20)

which implies that δ within the inclusion is proportional to the coordinate *x*, while outside the inclusion δ is inversely proportional to the coordinate *x*.

We remark that the dimensionless gradient parameter β^* cannot be arbitrarily large as this would result in a FGM with a very large shear modulus. For example, the modulus corresponding to β^* =3 at *x*=*R* would be more than 400 times larger than the one corresponding to the homogeneous material case (i.e., μ_0 when $\beta^*=0$).

4 Conclusions

We have analyzed the displacement, strain, and stress fields for an infinite FGM plane containing a circular inclusion under uniform antiplane eigenstrains. The solution is expressed in terms of series expansion by virtue of a new function. Numerical results show that, inside the circular inclusion, the stress and strain fields are nonuniform and the displacement field is no longer a linear function of the coordinates x and y when the elastic plane is functionally graded (or inhomogeneous in shear modulus). A similar problem that could be addressed in the future is for the corresponding transversely isotropic and piezoelectric FGM plane with a circular inclusion under uniform antiplane eigenstrains and inplane eigenelectric fields. Finally we indicate again that the FGM plane studied in this research is assumed to be isotropic and exponentially graded to simplify the analysis. Introduction of local anisotropy [18–20] to our model will require more involved investigation and thus form the subject of future research.

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