Available online at www.sciencedirect.com
ScienceDirect
European Journal of Mechanics A/Solids 27 (2008) 478–486

Anti-plane Green's functions and cracks for piezoelectric material with couple stress and electric field gradient effects

X. Wang [a,*], E. Pan [a], W.J. Feng [b]

[a] *Department of Civil Engineering and Department of Applied Mathematics, University of Akron, Akron, OH 44325-3905, USA*
[b] *Department of Engineering Mechanics, Shijiazhuang Railway Institute, Shijiazhuang 050043, PR China*

Received 15 May 2007; accepted 25 September 2007

Available online 29 September 2007

Abstract

This research is concerned with the anti-plane strain problems of polarized ceramics with both the couple stress and electric field gradient effects. This theory possesses two characteristic lengths l_1 and l_2 which are determined explicitly. In addition the two characteristic lengths can be either positive real or complex conjugate with positive real part. We first investigate the electroelastic field induced by a static line force and a line charge. It is found that the displacement and the electric potential are regular at the point where the line force and line charge are located. We then consider the near-tip asymptotic electroelastic field for a mode III crack. The analysis demonstrates that the near-tip asymptotic electroelastic field is governed by two parameters B and D. The total stresses and in-plane electric displacements exhibit the stronger $r^{-3/2}$ singularity near the crack tip; while the couple stresses, the out-of-plane electric displacement, and those associated with electric quadrupole densities exhibit the weaker $r^{-1/2}$ singularity near the crack tip.

Keywords: Couple stress elasticity; Electric field gradient; Anti-plane problem; Piezoelectric material

1. Introduction

Up to now various continuum mechanics theories have been proposed to account for the size effect experimentally observed in problems with a geometric length scale comparable to materials' microstructural length. To name a few, (i) *micropolar and couple stress elasticity* (see for example Mindlin and Tiersten, 1962; Muki and Sternberg, 1965; Hartranft and Sih, 1965; Reddy and Venkatasubramanian, 1978; Jasiuk and Ostoja-Starzewski, 1995; Cheng and He, 1995, 1997; Lubarda, 2003; Shodja and Ghazisaeidi, 2007); (ii) *nonlocal elasticity* (see for example Eringen, 1983, 1992, 2002; Lazar et al., 2006); (iii) *strain gradient elasticity* (see for example Ru and Aifantis, 1993; Zhang et al., 1998; Gutkin and Aifantis, 1999; Paulino et al., 2003; Lazar and Maugin, 2004, 2005; Zhang and Sharma, 2005) and *strain gradient plasticity* (see for example Fleck and Hutchinson, 1993, 1997; Fleck et al., 1994; Nix and Gao, 1998a); (iv) *electric field gradient theory* (see for example Mindlin, 1968; Kalpakides and Agiasofitou, 2002; Yang et al.,

* Corresponding author.
E-mail address: xuwang@uakron.edu (X. Wang).

0997-7538/$ – see front matter
doi:10.1016/j.euromechsol.2007.09.005

2004, 2006); (v) *surface elasticity* (see for example Nix and Gao, 1998b; Gurtin et al., 1998; Sharma and Ganti, 2004; Yang, F.Q., 2004; Wang and Wang, 2006; Chen et al., 2007).

Most recently the electric field gradient theory (Yang, J.S., 2004; Yang et al., 2004, 2006) and the couple stress elasticity (Shodja and Ghazisaeidi, 2007) have been individually applied to anti-plane strain problems in piezoelectric materials, which exhibit intrinsic electromechanical coupling phenomenon, to account for the size effect of small scale electronic devices. This research endeavors to combine the electric field gradient theory and the coupled stress elasticity into a unified framework to study anti-plane problems in piezoelectric ceramics. We keep the notations used for electric field gradient theory (Yang, J.S., 2004; Yang et al., 2004, 2006) and those for the couple stress elasticity (Shodja and Ghazisaeidi, 2007) in order to make a connection between our results and the previous ones (Yang et al., 2004, 2006; Shodja and Ghazisaeidi, 2007). In Section 2 we investigate the electroelastic field induced by a static line force and a line charge in an unbounded piezoelectric plane. Then we further address in Section 3 the near-tip asymptotic electroelastic field for a mode III semi-infinite crack.

2. Green's functions

We consider a hexagonal piezoelectric material of 6 mm symmetry with poling in the x_3 direction. Furthermore we confine our attention to the anti-plane strain problems described by

$$u_1 = u_2 = 0, \quad u_3 = w(x_1, x_2), \quad \phi = \phi(x_1, x_2). \tag{1}$$

Within the context of couple stress elasticity (Shodja and Ghazisaeidi, 2007) and electric field gradient theory (Yang et al., 2006), the non-trivial total stress components t_{13}, t_{31}, t_{23}, t_{32} and the non-vanishing electric displacements D_1, D_2, D_3 can be expressed in terms of the displacement w and the electric potential ϕ as

$$\begin{aligned}
t_{13} &= c_{44}\frac{\partial w}{\partial x_1} + e_{15}\frac{\partial \phi}{\partial x_1} - \frac{k_{66}}{4}\frac{\partial(\nabla^2 w)}{\partial x_1}, & t_{31} &= c_{44}\frac{\partial w}{\partial x_1} + e_{15}\frac{\partial \phi}{\partial x_1} + \frac{k_{66}}{4}\frac{\partial(\nabla^2 w)}{\partial x_1},\\
t_{23} &= c_{44}\frac{\partial w}{\partial x_2} + e_{15}\frac{\partial \phi}{\partial x_2} - \frac{k_{66}}{4}\frac{\partial(\nabla^2 w)}{\partial x_2}, & t_{32} &= c_{44}\frac{\partial w}{\partial x_2} + e_{15}\frac{\partial \phi}{\partial x_2} + \frac{k_{66}}{4}\frac{\partial(\nabla^2 w)}{\partial x_2},\\
D_1 &= e_{15}\frac{\partial w}{\partial x_1} - \varepsilon_{11}\frac{\partial \phi}{\partial x_1} + \varepsilon_0\alpha\frac{\partial(\nabla^2 \phi)}{\partial x_1}, & D_2 &= e_{15}\frac{\partial w}{\partial x_2} - \varepsilon_{11}\frac{\partial \phi}{\partial x_2} + \varepsilon_0\alpha\frac{\partial(\nabla^2 \phi)}{\partial x_2},\\
D_3 &= -\varepsilon_0(\gamma_{31} - \gamma_{15})\nabla^2\phi - \frac{f_{36}}{2}\nabla^2 w,
\end{aligned} \tag{2}$$

where ∇^2 is the two-dimensional Laplacian; ε_0 is the dielectric permittivity constant of free space; c_{44}, e_{15} and ε_{11} are respectively the elastic, piezoelectric and dielectric coefficients of the piezoelectric material; k_{66} (> 0) and f_{36} are new material constants due to the introduction of the couple stress, while $\alpha = \alpha_{11}$ (> 0), γ_{31} and γ_{15} are also new material constants due to the introduction of the electric field gradient.

In addition the couple stress components m_{11}, m_{22}, m_{12} and m_{21} are related to w through

$$\begin{aligned}
m_{11} &= -m_{22} = \frac{k_{11} - k_{12}}{2}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
m_{12} &= -\frac{k_{66}}{2}\frac{\partial^2 w}{\partial x_1^2} + \frac{k_{69}}{2}\frac{\partial^2 w}{\partial x_2^2},\\
m_{21} &= -\frac{k_{69}}{2}\frac{\partial^2 w}{\partial x_1^2} + \frac{k_{66}}{2}\frac{\partial^2 w}{\partial x_2^2},
\end{aligned} \tag{3}$$

and Π_{11}, Π_{22}, Π_{12}, which are associated with electric quadrupole densities (Yang et al., 2006), can be expressed as

$$\begin{aligned}
\Pi_{11} &= -\varepsilon_0\left(\alpha_{11}\frac{\partial^2 \phi}{\partial x_1^2} + \alpha_{12}\frac{\partial^2 \phi}{\partial x_2^2}\right),\\
\Pi_{22} &= -\varepsilon_0\left(\alpha_{12}\frac{\partial^2 \phi}{\partial x_1^2} + \alpha_{11}\frac{\partial^2 \phi}{\partial x_2^2}\right),\\
\Pi_{12} &= \Pi_{21} = -2\varepsilon_0\alpha_{66}\frac{\partial^2 \phi}{\partial x_1 \partial x_2},
\end{aligned} \tag{4}$$

where k_{11}, k_{12}, k_{69} are new material constants due to the introduction of the couple stress, while α_{12} and α_{66} are also new material constants due to the introduction of the electric field gradient.

In polar coordinates (r, θ), the non-vanishing total stress components $t_{r3}, t_{3r}, t_{\theta 3}$ and $t_{3\theta}$, the electric displacements D_r, D_θ, the couple stresses $m_{rr}, m_{\theta\theta}, m_{r\theta}, m_{\theta r}$ and $\Pi_{rr}, \Pi_{\theta\theta}, \Pi_{r\theta}$ associated with electric quadrupole densities are expressed as

$$\begin{aligned}
t_{r3} &= c_{44}\frac{\partial w}{\partial r} + e_{15}\frac{\partial \phi}{\partial r} - \frac{k_{66}}{4}\frac{\partial(\nabla^2 w)}{\partial r}, & t_{3r} &= c_{44}\frac{\partial w}{\partial r} + e_{15}\frac{\partial \phi}{\partial r} + \frac{k_{66}}{4}\frac{\partial(\nabla^2 w)}{\partial r},\\
t_{\theta 3} &= \frac{c_{44}}{r}\frac{\partial w}{\partial \theta} + \frac{e_{15}}{r}\frac{\partial \phi}{\partial \theta} - \frac{k_{66}}{4r}\frac{\partial(\nabla^2 w)}{\partial \theta}, & t_{3\theta} &= \frac{c_{44}}{r}\frac{\partial w}{\partial \theta} + \frac{e_{15}}{r}\frac{\partial \phi}{\partial \theta} + \frac{k_{66}}{4r}\frac{\partial(\nabla^2 w)}{\partial \theta},\\
D_r &= e_{15}\frac{\partial w}{\partial r} - \varepsilon_{11}\frac{\partial \phi}{\partial r} + \varepsilon_0\alpha\frac{\partial(\nabla^2 \phi)}{\partial r}, & D_\theta &= \frac{e_{15}}{r}\frac{\partial w}{\partial \theta} - \frac{\varepsilon_{11}}{r}\frac{\partial \phi}{\partial \theta} + \frac{\varepsilon_0\alpha}{r}\frac{\partial(\nabla^2 \phi)}{\partial \theta},
\end{aligned} \tag{5}$$

$$\begin{aligned}
m_{rr} &= -m_{\theta\theta} = \frac{k_{11}-k_{12}}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right),\\
m_{r\theta} &= -\frac{k_{66}+k_{69}}{2}\frac{\partial^2 w}{\partial r^2} + \frac{k_{69}}{2}\nabla^2 w,\\
m_{\theta r} &= -\frac{k_{66}+k_{69}}{2}\frac{\partial^2 w}{\partial r^2} + \frac{k_{66}}{2}\nabla^2 w,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\Pi_{rr} &= -\varepsilon_0\left[(\alpha_{11}-\alpha_{12})\frac{\partial^2 \phi}{\partial r^2} + \alpha_{12}\nabla^2\phi\right],\\
\Pi_{\theta\theta} &= -\varepsilon_0\left[(\alpha_{12}-\alpha_{11})\frac{\partial^2 \phi}{\partial r^2} + \alpha_{11}\nabla^2\phi\right],\\
\Pi_{r\theta} &= \Pi_{\theta r} = -2\varepsilon_0\alpha_{66}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right).
\end{aligned} \tag{7}$$

In the presence of a static antiplane line force p and line charge q both located at the origin, the static equilibrium equations for the piezoelectric body are given by

$$\begin{aligned}
t_{13,1} + t_{23,2} &= -p\delta(x_1)\delta(x_2),\\
D_{1,1} + D_{2,2} &= q\delta(x_1)\delta(x_2),
\end{aligned} \tag{8}$$

where $\delta(\cdot)$ is the Dirac delta function.

Substitution of Eq. (2) into the above expression will lead to the following set of coupled inhomogeneous partial differential equations for w and ϕ

$$\begin{aligned}
c_{44}\nabla^2 w + e_{15}\nabla^2\phi - \frac{k_{66}}{4}\nabla^4 w &= -p\delta(x_1)\delta(x_2),\\
e_{15}\nabla^2 w - \varepsilon_{11}\nabla^2\phi + \varepsilon_0\alpha\nabla^4\phi &= q\delta(x_1)\delta(x_2).
\end{aligned} \tag{9}$$

In order to solve the above set of coupled differential equations, we consider the following eigenvalue problem:

$$\left(\begin{bmatrix} \frac{k_{66}}{4} & 0 \\ 0 & -\varepsilon_0\alpha \end{bmatrix} - l^2\begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix}\right)\mathbf{v} = \mathbf{0}, \tag{10}$$

where l and $\mathbf{v}$ are respectively the eigenvalue and the associated eigenvector.

When k_{66} and α satisfy the following inequality

$$\sqrt{\frac{k_{66}}{4\varepsilon_0\alpha}} \leqslant \frac{\sqrt{c_{44}\varepsilon_{11} + e_{15}^2} - |e_{15}|}{\varepsilon_{11}} \quad \text{or} \quad \sqrt{\frac{k_{66}}{4\varepsilon_0\alpha}} \geqslant \frac{\sqrt{c_{44}\varepsilon_{11} + e_{15}^2} + |e_{15}|}{\varepsilon_{11}}, \tag{11}$$

then Eq. (10) possesses two positive real eigenvalues given by

$$\begin{aligned}
l_1 &= \sqrt{\xi}\left[\frac{1}{2}(\rho+1)\right]^{1/2} + \sqrt{\xi}\left[\frac{1}{2}(\rho-1)\right]^{1/2},\\
l_2 &= \sqrt{\xi}\left[\frac{1}{2}(\rho+1)\right]^{1/2} - \sqrt{\xi}\left[\frac{1}{2}(\rho-1)\right]^{1/2},
\end{aligned} \qquad (\rho \geqslant 1) \tag{12}$$

where $l_1 \geqslant l_2 > 0$, and

$$\xi = \frac{1}{2}\sqrt{\frac{k_{66}\varepsilon_0\alpha}{c_{44}\varepsilon_{11} + e_{15}^2}}, \qquad \rho = \frac{4c_{44}\varepsilon_0\alpha + k_{66}\varepsilon_{11}}{4\sqrt{k_{66}\varepsilon_0\alpha(c_{44}\varepsilon_{11} + e_{15}^2)}}. \tag{13}$$

When k_{66} and α satisfy the following inequality

$$\frac{\sqrt{c_{44}\varepsilon_{11} + e_{15}^2} - |e_{15}|}{\varepsilon_{11}} < \sqrt{\frac{k_{66}}{4\varepsilon_0\alpha}} < \frac{\sqrt{c_{44}\varepsilon_{11} + e_{15}^2} + |e_{15}|}{\varepsilon_{11}}, \tag{14}$$

then Eq. (10) possesses two complex conjugate eigenvalues given by

$$\begin{aligned} l_1 &= \sqrt{\xi}\left[\frac{1}{2}(1+\rho)\right]^{1/2} + \mathrm{i}\sqrt{\xi}\left[\frac{1}{2}(1-\rho)\right]^{1/2}, \\ l_2 &= \sqrt{\xi}\left[\frac{1}{2}(1+\rho)\right]^{1/2} - \mathrm{i}\sqrt{\xi}\left[\frac{1}{2}(1-\rho)\right]^{1/2}, \end{aligned} \quad (0 < \rho < 1) \tag{15}$$

where $\mathrm{Re}\{l_1\} = \mathrm{Re}\{l_2\} > 0$ with ξ and ρ being defined by Eq. (13).

It is mentioned that in deriving the explicit expressions (12) and (15) for l_1 and l_2, we adopt an approach similar to that used by Suo (1990) and Yang et al. (1991) in deriving the Stroh eigenvalues for elastically orthotropic materials.

The two eigenvectors associated with the two eigenvalues are

$$\mathbf{v}_1 = \begin{bmatrix} e_{15}l_1^2 \\ \frac{k_{66}}{4} - c_{44}l_1^2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} e_{15}l_2^2 \\ \frac{k_{66}}{4} - c_{44}l_2^2 \end{bmatrix}. \tag{16}$$

Since the fact that the two matrices $\left[\begin{smallmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \frac{k_{66}}{4} & 0 \\ 0 & -\varepsilon_0\alpha \end{smallmatrix}\right]$ are real and symmetric, then it can be easily verified that when $l_1 \neq l_2$ the following orthogonal relationships with respect to the two symmetric matrices hold

$$\begin{aligned} &\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix} [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \\ &\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \dfrac{k_{66}}{4} & 0 \\ 0 & -\varepsilon_0\alpha \end{bmatrix} [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] = \begin{bmatrix} l_1^2\delta_1 & 0 \\ 0 & l_2^2\delta_2 \end{bmatrix}, \end{aligned} \tag{17}$$

where the two constants δ_1 and δ_2 are given by

$$\begin{aligned} \delta_1 &= \frac{k_{66}e_{15}^2 l_1^2}{4} - \varepsilon_0\alpha l_1^{-2}\left(c_{44}l_1^2 - \frac{k_{66}}{4}\right)^2, \\ \delta_2 &= \frac{k_{66}e_{15}^2 l_2^2}{4} - \varepsilon_0\alpha l_2^{-2}\left(c_{44}l_2^2 - \frac{k_{66}}{4}\right)^2. \end{aligned} \tag{18}$$

Now we introduce two new functions f and g, which are related to w and ϕ through

$$\begin{bmatrix} w \\ \phi \end{bmatrix} = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} f \\ g \end{bmatrix}. \tag{19}$$

In view of Eqs. (9), (17) and (19), the two new functions f and g satisfy the following two independent inhomogeneous partial differential equations

$$\nabla^2 f - l_1^2\nabla^4 f = \frac{q(k_{66}/4 - c_{44}l_1^2) - e_{15}l_1^2 p}{\delta_1}\delta(x_1)\delta(x_2), \tag{20}$$

$$\nabla^2 g - l_2^2\nabla^4 g = \frac{q(k_{66}/4 - c_{44}l_2^2) - e_{15}l_2^2 p}{\delta_2}\delta(x_1)\delta(x_2). \tag{21}$$

It is observed that the two parameters l_1 and l_2, which depend on the electromechanical properties of the piezoelectric material (or more specifically depend on ξ with the dimension of (length)^2 and the dimensionless parameter ρ),

have the dimension of length and are called the characteristic lengths. Thus for piezoelectric materials within the framework of couple stress elasticity and electric field gradient theory, two intrinsic length scales l_1 and l_2 are needed to describe size effect. The introduction of two length scales is different from the situation of a single length scale for the piezoelectric media in the context of couple stress elasticity (Shodja and Ghazisaeidi, 2007) or in the context of electric field gradient theory (Yang et al., 2006). In addition the two characteristic lengths l_1 and l_2 can be positive real (see Eq. (12)) or complex conjugate with positive real part (see Eq. (15)).

When ignoring the electric field gradient effect, i.e., $\alpha = 0$, it is observed from Eq. (13) that $\xi \to 0$ while $\rho \cong k_{66}\varepsilon_{11}/(4\sqrt{k_{66}\varepsilon_0\alpha(c_{44}\varepsilon_{11} + e_{15}^2)}) \gg 1$. Consequently it follows from Eq. (12) that

$$l_1 \cong 2\sqrt{\xi}\left[\frac{1}{2}\rho\right]^{1/2} = \sqrt{\frac{k_{66}\varepsilon_{11}}{4(c_{44}\varepsilon_{11} + e_{15}^2)}}, \qquad l_2 = 0, \tag{22}$$

which is just the result obtained by Shodja and Ghazisaeidi (2007).

On the other hand when ignoring the couple stress effect, i.e., $k_{66} = 0$, it is observed from Eq. (13) that $\xi \to 0$ while $\rho \cong c_{44}\varepsilon_0\alpha/(\sqrt{k_{66}\varepsilon_0\alpha(c_{44}\varepsilon_{11} + e_{15}^2)}) \gg 1$. Consequently it follows from Eq. (12) that

$$l_1 \cong 2\sqrt{\xi}\left[\frac{1}{2}\rho\right]^{1/2} = \sqrt{\frac{c_{44}\varepsilon_0\alpha}{c_{44}\varepsilon_{11} + e_{15}^2}}, \qquad l_2 = 0, \tag{23}$$

which is just the result obtained by Yang et al. (2004, 2006).

In the following we solve the two independent partial differential equations (20) and (21). It follows from Eq. (20) that

$$f - l_1^2\nabla^2 f = \frac{q(k_{66}/4 - c_{44}l_1^2) - e_{15}l_1^2 p}{2\pi\delta_1}\ln r, \tag{24}$$

and

$$l_1^2\nabla^2 f = \frac{q(k_{66}/4 - c_{44}l_1^2) - e_{15}l_1^2 p}{2\pi\delta_1}K_0(l_1^{-1}r), \tag{25}$$

where K_n is the nth order modified Bessel function of the second kind.

Adding Eqs. (24) and (25) leads to the following expression for f

$$f = \frac{q(k_{66}/4 - c_{44}l_1^2) - e_{15}l_1^2 p}{2\pi\delta_1}\left[\ln r + K_0(l_1^{-1}r)\right]. \tag{26}$$

Similarly it follows from Eq. (21) that

$$g - l_2^2\nabla^2 g = \frac{q(k_{66}/4 - c_{44}l_2^2) - e_{15}l_2^2 p}{2\pi\delta_2}\ln r, \tag{27}$$

and

$$l_2^2\nabla^2 g = \frac{q(k_{66}/4 - c_{44}l_2^2) - e_{15}l_2^2 p}{2\pi\delta_2}K_0(l_2^{-1}r). \tag{28}$$

Adding Eqs. (27) and (28) leads to the following expression of g

$$g = \frac{q(k_{66}/4 - c_{44}l_2^2) - e_{15}l_2^2 p}{2\pi\delta_2}\left[\ln r + K_0(l_2^{-1}r)\right]. \tag{29}$$

Now that we have obtained the expressions of f and g, then it is not difficult to arrive at the expressions of the original w and ϕ as follows in view of Eq. (19)

$$\begin{aligned} w = {} & \frac{e_{15}l_1^2[q(k_{66} - 4c_{44}l_1^2) - 4e_{15}l_1^2 p]}{8\pi\delta_1}\left[\ln r + K_0(l_1^{-1}r)\right] \\ & + \frac{e_{15}l_2^2[q(k_{66} - 4c_{44}l_2^2) - 4e_{15}l_2^2 p]}{8\pi\delta_2}\left[\ln r + K_0(l_2^{-1}r)\right], \end{aligned} \tag{30}$$

$$\phi = \frac{(k_{66} - 4c_{44}l_1^2)[q(k_{66} - 4c_{44}l_1^2) - 4e_{15}l_1^2 p]}{32\pi\delta_1}\left[\ln r + K_0(l_1^{-1}r)\right] + \frac{(k_{66} - 4c_{44}l_2^2)[q(k_{66} - 4c_{44}l_2^2) - 4e_{15}l_2^2 p]}{32\pi\delta_2}\left[\ln r + K_0(l_2^{-1}r)\right].$$

Based on the above, the Green's functions $G_{\alpha\beta}$ are found to be

$$\begin{aligned} G_{wp} &= -\frac{e_{15}^2 l_1^4}{2\pi\delta_1}\left[\ln r + K_0(l_1^{-1}r)\right] - \frac{e_{15}^2 l_2^4}{2\pi\delta_2}\left[\ln r + K_0(l_2^{-1}r)\right], \\ G_{wq} &= -G_{\phi p} = \frac{e_{15}l_1^2(k_{66} - 4c_{44}l_1^2)}{8\pi\delta_1}\left[\ln r + K_0(l_1^{-1}r)\right] + \frac{e_{15}l_2^2(k_{66} - 4c_{44}l_2^2)}{8\pi\delta_2}\left[\ln r + K_0(l_2^{-1}r)\right], \\ G_{\phi q} &= \frac{(k_{66} - 4c_{44}l_1^2)^2}{32\pi\delta_1}\left[\ln r + K_0(l_1^{-1}r)\right] + \frac{(k_{66} - 4c_{44}l_2^2)^2}{32\pi\delta_2}\left[\ln r + K_0(l_2^{-1}r)\right], \end{aligned} \tag{31}$$

where the definitions of the Green's functions are: $G_{wp}(x_i)$ is the displacement w at x_i due to a line force ($p = 1$) at $x_i = 0$; $G_{wq}(x_i)$ is the displacement w at x_i due to a line charge ($q = 1$) at $x_i = 0$; $G_{\phi p}(x_i)$ is the electric potential ϕ at x_i due to a line force ($p = 1$) at $x_i = 0$; and $G_{\phi q}(x_i)$ is the electric potential ϕ at x_i due to a line charge ($q = 1$) at $x_i = 0$.

The following can be observed from the derived Green's functions $G_{\alpha\beta}$

(i) At origin $r = 0$ where the line force and line charge are located, the Green's functions $G_{\alpha\beta}$ remain *finite values* in view of the following asymptotic behavior for K_0 at origin

$$K_0(l^{-1}r) \to -\ln(l^{-1}r/2), \quad \text{when } r \to 0; \tag{32}$$

(ii) The Green's functions exhibit the following asymptotic behaviors at infinity

$$\begin{aligned} G_{wp} &= -\frac{e_{15}^2}{2\pi}\left(\frac{l_1^4}{\delta_1} + \frac{l_2^4}{\delta_2}\right)\ln r, \\ G_{wq} &= -G_{\phi p} = \frac{e_{15}}{8\pi}\left[\frac{l_1^2(k_{66} - 4c_{44}l_1^2)}{\delta_1} + \frac{l_2^2(k_{66} - 4c_{44}l_2^2)}{\delta_2}\right]\ln r, \quad \text{when } r \to \infty, \\ G_{\phi q} &= \frac{1}{32\pi}\left[\frac{(k_{66} - 4c_{44}l_1^2)^2}{\delta_1} + \frac{(k_{66} - 4c_{44}l_2^2)^2}{\delta_2}\right]\ln r. \end{aligned} \tag{33}$$

The far-field asymptotic behaviors imply that the couple stress and the electric field gradient effects will influence the values of the Green's functions at infinity.

(iii) The Green's functions exhibit the following anti-symmetric property in index: $G_{wq} = -G_{\phi p}$.

3. Mode III crack problem

In this section we consider the simplest problem of an anti-plane, semi-infinite crack at $\theta = \pi$ in an unbounded and source free ($p = q = 0$) polarized ceramics with both the couple stress and the electric field gradient effects.

Based on previous results for reduced boundary conditions in anti-plane shear (Zhang et al., 1998; Lubarda, 2003; Yang et al., 2006; Shodja and Ghazisaeidi, 2007), the boundary conditions on a traction-free and charge-free crack face are

$$\begin{aligned} t_{\theta 3} + \frac{1}{2}\frac{\partial m_{\theta\theta}}{\partial r} = 0, \quad m_{\theta r} = 0, \\ D_\theta + \frac{\partial \Pi_{\theta r}}{\partial r} = 0, \quad \Pi_{\theta\theta} = 0, \end{aligned} \quad \text{for } \theta = \pi. \tag{34}$$

In addition the anti-symmetry of anti-plane shear deformation requires that

$$w = \phi = 0, \quad \text{for } \theta = 0. \tag{35}$$

Following Williams' expansion, the displacement w and the electric potential ϕ near the crack tip can be expanded as

$$w = r^s w_s(\theta), \qquad \phi = r^s \phi_s(\theta), \tag{36}$$

where the power s and the angular functions $w_s(\theta)$ and $\phi_s(\theta)$ are to be determined. Keeping the dominant singular terms, we find the governing equations (9) with $p = q = 0$ and the boundary condition equations (34) on the crack surface become

$$\begin{aligned} &\nabla^4 w = 0, \\ &\frac{\partial}{\partial\theta}\left[\frac{\partial^2 w}{\partial r^2} - \frac{2}{r}\frac{\partial w}{\partial r} + \frac{2w}{r^2} + \frac{k_{66}}{k_{11}-k_{12}}\nabla^2 w\right] = 0, \quad \text{for } \theta = \pi, \\ &\frac{k_{69}}{k_{66}}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} = 0, \end{aligned} \tag{37}$$

and

$$\begin{aligned} &\nabla^4 \phi = 0, \\ &\frac{\partial}{\partial\theta}\left[\frac{\partial^2 \phi}{\partial r^2} - \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{2\phi}{r^2} - \frac{\alpha_{11}}{2\alpha_{66}}\nabla^2 \phi\right] = 0, \quad \text{for } \theta = \pi, \\ &\frac{\alpha_{12}}{\alpha_{11}}\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial\theta^2} = 0. \end{aligned} \tag{38}$$

The general solutions of w and ϕ satisfying Eqs. $(37)_1$ and $(38)_1$ and the anti-symmetry condition (35) at $\theta = 0$ are given by (Zhang et al., 1998)

$$\begin{aligned} w &= r^s\{A\sin(s\theta) + B\sin[(s-2)\theta]\}, \\ \phi &= r^s\{C\sin(s\theta) + D\sin[(s-2)\theta]\}, \end{aligned} \tag{39}$$

where A, B, C and D are undetermined constants.

Substituting Eq. (39) into Eqs. $(37)_{2,3}$ and $(38)_{2,3}$, we obtain the following

$$\begin{aligned} &(s-1)(s-2)\cos(s\pi)\left[sA + \left(s - 2 + \frac{4k_{66}}{k_{11}-k_{12}}\right)B\right] = 0, \\ &(s-1)\sin(s\pi)\left[\frac{k_{69}}{k_{66}}s(A+B) + sA + (s-4)B\right] = 0, \end{aligned} \tag{40}$$

and

$$\begin{aligned} &(s-1)(s-2)\cos(s\pi)\left[sC + \left(s - 2 - \frac{2\alpha_{11}}{\alpha_{66}}\right)D\right] = 0, \\ &(s-1)\sin(s\pi)\left[\frac{\alpha_{12}}{\alpha_{11}}s(C+D) - sC - (s-4)D\right] = 0. \end{aligned} \tag{41}$$

Similar to the argument by Zhang et al. (1998), we can determine

$$s = \frac{3}{2}, \tag{42}$$

and

$$A = \frac{5k_{66} - 3k_{69}}{3(k_{66} + k_{69})}B, \tag{43}$$

$$C = \frac{5\alpha_{11} + 3\alpha_{12}}{3(\alpha_{11} - \alpha_{12})}D. \tag{44}$$

Thus the distributions of displacement, electric potential, total stresses, couple stresses, electric displacements near the crack tip are given by

$$
\begin{aligned}
w &= Br^{3/2}\left[\frac{5k_{66}-3k_{69}}{3(k_{66}+k_{69})}\sin\left(\frac{3}{2}\theta\right)-\sin\left(\frac{\theta}{2}\right)\right],\\
\phi &= Dr^{3/2}\left[\frac{5\alpha_{11}+3\alpha_{12}}{3(\alpha_{11}-\alpha_{12})}\sin\left(\frac{3}{2}\theta\right)-\sin\left(\frac{\theta}{2}\right)\right],
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&t_{r3}=-t_{3r}=-\frac{k_{66}}{4}Br^{-3/2}\sin\left(\frac{\theta}{2}\right), \qquad t_{\theta 3}=-t_{3\theta}=\frac{k_{66}}{4}Br^{-3/2}\cos\left(\frac{\theta}{2}\right),\\
&D_r=\varepsilon_0\alpha Dr^{-3/2}\sin\left(\frac{\theta}{2}\right), \qquad D_\theta=-\varepsilon_0\alpha Dr^{-3/2}\cos\left(\frac{\theta}{2}\right),\\
&D_3=\left[2\varepsilon_0(\gamma_{31}-\gamma_{15})D+f_{36}B\right]r^{-1/2}\sin\left(\frac{\theta}{2}\right),
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
m_{rr} &= -m_{\theta\theta}=\frac{k_{11}-k_{12}}{8}Br^{-1/2}\left[\frac{5k_{66}-3k_{69}}{k_{66}+k_{69}}\cos\left(\frac{3}{2}\theta\right)-\cos\left(\frac{\theta}{2}\right)\right],\\
m_{r\theta} &= -\frac{5k_{66}-3k_{69}}{8}Br^{-1/2}\left[\sin\left(\frac{3}{2}\theta\right)-\frac{3k_{66}-5k_{69}}{5k_{66}-3k_{69}}\sin\left(\frac{\theta}{2}\right)\right],\\
m_{\theta r} &= -\frac{5k_{66}-3k_{69}}{8}Br^{-1/2}\left[\sin\left(\frac{3}{2}\theta\right)+\sin\left(\frac{\theta}{2}\right)\right],
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\Pi_{rr} &= -\frac{\varepsilon_0(5\alpha_{11}+3\alpha_{12})}{4}Dr^{-1/2}\left[\sin\left(\frac{3}{2}\theta\right)-\frac{3\alpha_{11}+5\alpha_{12}}{5\alpha_{11}+3\alpha_{12}}\sin\left(\frac{\theta}{2}\right)\right],\\
\Pi_{\theta\theta} &= \frac{\varepsilon_0(5\alpha_{11}+3\alpha_{12})}{4}Dr^{-1/2}\left[\sin\left(\frac{3}{2}\theta\right)+\sin\left(\frac{\theta}{2}\right)\right],\\
\Pi_{r\theta} &= \Pi_{\theta r}=-\frac{\varepsilon_0\alpha_{66}}{2}Dr^{-1/2}\left[\frac{5\alpha_{11}+3\alpha_{12}}{\alpha_{11}-\alpha_{12}}\cos\left(\frac{3}{2}\theta\right)-\cos\left(\frac{\theta}{2}\right)\right].
\end{aligned}
\tag{48}
$$

It is observed from the above near-tip asymptotic expressions for the field variables that:

(i) Similar to the classical mode III crack tip electroelastic field (Pak, 1990), the near-tip asymptotic electroelastic field in a piezoelectric material with both the couple stress and the electric field gradient effects is governed by two parameters B and D in Eqs. (45)–(48);
(ii) The total stresses t_{r3}, t_{3r}, $t_{\theta 3}$ and $t_{3\theta}$ and the in-plane electric displacements D_r, D_θ all exhibit the $r^{-3/2}$ singularity near the crack tip. This singularity is much stronger than the classical square root singularity (Pak, 1990) and is in agreement with the results of Zhang et al. (1998), Paulino et al. (2003) and Yang, J.S. (2004) when the strain gradient or the electric field gradient effect is considered;
(iii) The couple stresses m_{rr}, $m_{\theta\theta}$, $m_{r\theta}$, $m_{\theta r}$, the electric displacement D_3, and Π_{rr}, $\Pi_{\theta\theta}$, $\Pi_{r\theta}$ all exhibit $r^{-1/2}$ singularity near the crack tip.

4. Conclusions

We have incorporated the effects of both the electric field gradient and the couple stress into anti-plane problems of piezoelectric materials to account for the size effect of small-scale problems. It is found that the out-of-plane displacement and the electric potential are regular at the point where a static line force and line charge are located. The asymptotic electroelastic field near a mode III crack in a piezoelectric material with both the couple stress and the electric field gradient effects is governed by two parameters B and D. Couple stresses, the out-of-plane electric displacement, and those associated with electric quadruple densities have inverse square-root singularity near the crack tip, while the total stresses and the in-plane electric displacements exhibit the stronger $r^{-3/2}$ singularity. Another interesting application of the obtained results is to investigate a circular piezoelectric inclusion or inhomogeneity embedded in a piezoelectric matrix, which was previously investigated within the framework of electric field gradient theory (Yang et al., 2006) and within the framework of couple stress elasticity (Shodja and Ghazisaeidi, 2007).

Acknowledgements

This work was supported in part by AFOSR FA9550-06-1-0317.

References

Chen, T., Dvorak, G.J., Yu, C.C., 2007. Size-dependent elastic properties of unidirectional nano-composites with interface stresses. Acta Mech. 188, 39–54.

Cheng, Z.Q., He, L.H., 1995. Micropolar elastic fields due to a spherical inclusion. Int. J. Engrg. Sci. 33, 389–397.

Cheng, Z.Q., He, L.H., 1997. Micropolar elastic fields due to a circular cylindrical inclusion. Int. J. Engrg. Sci. 35, 659–668.

Eringen, A.C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J. Appl. Phys. 54, 4703–4710.

Eringen, A.C., 1992. Vistas of nonlocal continuum physics. Int. J. Engrg. Sci. 30, 1551–1565.

Eringen, A.C., 2002. Nonlocal Continuum Field Theories. Springer, New York.

Fleck, N.A., Hutchinson, J.W., 1993. A phenomenological theory for strain gradient effects in plasticity. J. Mech. Phys. Solids 41, 1825–1857.

Fleck, N.A., Hutchinson, J.W., 1997. Strain gradient plasticity. Adv. Appl. Mech. 33, 195–361.

Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W., 1994. Strain gradient plasticity: theory and experiment. Acta Metall. Mater. 42, 475–487.

Gurtin, M.E., Weissmuller, J., Larche, F., 1998. The general theory of curved deformable interfaces in solids at equilibrium. Philos. Mag. A 78, 1093.

Gutkin, M.Yu., Aifantis, E.C., 1999. Dislocations in the theory of gradient elasticity. Scripta Mater. 40, 559–566.

Hartranft, R.J., Sih, G.C., 1965. The effect of couple stresses on the stress concentration of a circular inclusion. J. Appl. Mech. 32, 429–431.

Jasiuk, I., Ostoja-Starzewski, M., 1995. Planar Cosserat elasticity of materials with holes and inclusions. Appl. Mech. Rev. 48, S11–S18.

Kalpakides, V.K., Agiasofitou, E.K., 2002. On material equations in second order gradient electroelasticity. J. Elast. 67, 205–227.

Lazar, M., Maugin, G.A., 2004. Defects in gradient micropolar elasticity–I: screw dislocation. J. Mech. Phys. Solids 52, 2263–2284.

Lazar, M., Maugin, G.A., 2005. Nonsingular stress and strain fields of dislocations and declinations in first strain gradient elasticity. Int. J. Engrg. Sci. 43, 1157–1184.

Lazar, M., Maugin, G.A., Aifantis, E.C., 2006. On a theory of nonlocal elasticity of bi-Helmholtz type and some applications. Int. J. Solids Struct. 43, 1404–1421.

Lubarda, V.A., 2003. Circular inclusions in anti-plane strain couple stress elasticity. Int. J. Solids Struct. 40, 3827–3851.

Mindlin, R.D., Tiersten, H.F., 1962. Effects of couple-stresses in linear elasticity. Arch. Ration. Mech. Anal. 11, 415–448.

Mindlin, R.D., 1968. Polarization gradient in elastic dielectric. Int. J. Solids Struct. 4, 637–642.

Muki, R., Sternberg, E., 1965. The influence of couple-stresses on singular stress concentrations in elastic solids. Z. Angew. Math. Phys. 16, 611–648.

Nix, W.D., Gao, H.J., 1998a. Indentation size effects in crystalline materials: A law for strain gradient plasticity. J. Mech. Phys. Solids 46, 411–425.

Nix, W.D., Gao, H.J., 1998b. An atomic interpretation of interface stress. Scripta Mater. 39, 1653–1661.

Pak, Y.E., 1990. Crack extension force in a piezoelectric material. J. Appl. Mech. 57, 647–653.

Paulino, G.H., Fannjiang, A.C., Chan, Y.S., 2003. Gradient elasticity theory for mode III fracture in functionally graded materials – Part I: crack perpendicular to the material gradation. J. Appl. Mech. 70, 531–542.

Reddy, K., Venkatasubramanian, N.K., 1978. On the flexural rigidity of a micropolar elastic circular cylinder. J. Appl. Mech. 45, 429–431.

Ru, C.Q., Aifantis, E.C., 1993. A simple approach to solve boundary-value problems in gradient elasticity. Acta Mech. 101, 59–68.

Sharma, P., Ganti, S., 2004. Size-dependent Eshelby's tensor for embedded nano-inclusions incorporating surface/interface energies. J. Appl. Mech. 71, 663–671.

Shodja, H.M., Ghazisaeidi, M., 2007. Effects of couple stress on anti-plane problems of piezoelectric media with inhomogeneities. Eur. J. Mech. A Solids 26, 647–658.

Suo, Z.G., 1990. Singularities, interfaces and cracks in dissimilar anisotropic materials. Proc. R. Soc. Lond. Ser. A 427, 331–358.

Wang, G.F., Wang, T.J., 2006. Surface effects on the diffraction of plane compressional waves by a nanosized circular hole. Appl. Phys. Lett. 89, 231923.

Yang, F.Q., 2004. Size-dependent effective modulus of elastic composite materials: spherical nanocavities at dilute concentrations. J. Appl. Phys. 95, 3516–3520.

Yang, J.S., 2004. Effects of electric field gradient on an anti-plane crack in piezoelectric ceramics. Int. J. Fract. 127, L111–L116.

Yang, J.S., Zhou, H.G., Li, J.Y., 2006. Electric field gradient effects in an anti-plane circular inclusion in polarized ceramics. Proc. R. Soc. Lond. Ser. A 462, 3511–3522.

Yang, W., Suo, Z., Shih, C.F., 1991. Mechanics of dynamic debonding. Proc. R. Soc. Lond. Ser. A 433, 679–697.

Yang, X.M., Hu, Y.T., Yang, J.S., 2004. Electric field gradient effects in anti-plane problems of polarized ceramics. Int. J. Solids Struct. 41, 6801–6811.

Zhang, L., Huang, Y., Chen, J.Y., Hwang, K.C., 1998. The mode III full-field solution in elastic materials with strain gradient effects. Int. J. Fract. 92, 325–348.

Zhang, X., Sharma, P., 2005. Inclusions and inhomogeneities in strain gradient elasticity with couple stresses and related problems. Int. J. Solids Struct. 42, 3833–3851.