Exact closed-form electromagnetic Green's functions for graded uniaxial multiferroic materials

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Functionally graded multiferroic composites are being investingated in order to tailor the electromagnetic properties of synthetic surfaces and material interfaces. Theoretical representations of multiferroic materials simultaneously account for magnetization and ferroelectric polarization in a strongly coupled system. In this paper, exact closed-form electromagnetic Green's functions due to electric and magnetic point sources are derived for the general class of uniaxial multiferroics with functionally graded compositions where the material property varies exponentially in an arbitrary direction. After the introduction of two new functions, the set of coupled governing partial differential equations is separated into two independent inhomogeneous partial differential equations. These resultant equations are further cast in Helmholtz form so that solutions can be derived expediently. We observed that the derived Green's functions in the graded case possess symmetric properties in their indices, but not in their spatial variables. Numerical results show that the amplitudes of the field response along the direction of the compositional gradient can be either matched to a uniform material or systematically reduced, as necessary. Finally, Green's functions are applied to study the electric dipole-induced electric and magnetic potentials in the functionally graded multiferroics. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2939263]

I. INTRODUCTION

Multiferroic materials that combine a spontaneous magnetization with a ferroelectric polarization are of great theoretical and practical interest.[1–3] Strong coupling between the polarization and magnetization in multiferroic materials would allow ferroelectric data storage combined with a magnetic read. The ability to tune or switch the magnetic properties with an electric field and vice versa could lead to unexpected developments in conventional devices such as transducers. Various models in the framework of micromechanics have been proposed for multiferroic materials.[4–7]

Green's function–based methods are mathematically elegant and computationally powerful.[8] Green's functions in multiferroic materials can be utilized to predict the magnetoelectric effect and to investigate inclusion problems,[4–7] to name a few. It is observed that so far the derived Green's functions for multiferroic materials are restricted to homogeneous or piecewise homogeneous materials.[5–7] The graded spatial compositions associated with functionally graded materials (FGMs) provide additional freedom to optimize the design and fabrication of novel structures for interface modulation and also for exploiting other unique features of these systems.[9,10] Very recently, multiferroic FGMs based on piezoelectric $BaTiO_3$ and magnetostrictive $CoFe_2O_4$, on $PbTiO_3$–$CoFe_2O_4$, and on $BiMnO_3$ and Mn_3O_4 have been successfully fabricated by using different approaches for better dielectric and magnetic properties.[11] Furthermore, an interesting theoretical analysis on the behavior of piezoelectric FGMs was also carried out, which showed the flexibility of FGMs for modulating and adjusting the structure response.[12] To the best of the authors' knowledge, however, Green's functions in FGMs are currently limited to heat conduction,[13] elastostatics,[14,15] and piezoelectricity.[16]

Therefore, in this paper, we derive three-dimensional (3D) electromagnetic Green's functions for a uniaxial multiferroic FGM which is exponentially graded in an arbitrary direction. Upon introduction of two new functions,[7] the set of coupled governing partial differential equations can be decoupled into two independent inhomogeneous partial differential equations. It is observed that the original problem can be reduced to the determination of Green's functions for two independent 3D Helmholtz equations. Explicit expressions of Green's functions and their derivatives are thus derived, along with the corresponding boundary integral equations. Numerical examples (of Green's functions and electric-dipole solutions) are also presented for different exponential factors in the FGMs, and it is found that, along the compositional gradient direction, the amplitudes of the field response in a FGM multiferroic space can be either reduced or kept in the same values as those in the corresponding homogeneous material (for Green's function solutions).

II. GREEN'S FUNCTIONS FOR A FUNCTIONALLY GRADED MULTIFERROIC MATERIAL

The constitutive equations for a uniaxial multiferroic material with its unique axis along the x_3-direction can be

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written as[6]

$$\begin{bmatrix} D_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix} \begin{bmatrix} E_1 \\ H_1 \end{bmatrix}, \quad \begin{bmatrix} D_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \kappa_{11} & \alpha_{11} \\ \alpha_{11} & \mu_{11} \end{bmatrix} \times \begin{bmatrix} E_2 \\ H_2 \end{bmatrix}, \quad \begin{bmatrix} D_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} \kappa_{33} & \alpha_{33} \\ \alpha_{33} & \mu_{33} \end{bmatrix} \begin{bmatrix} E_3 \\ H_3 \end{bmatrix}, \tag{1}$$

where D_i and B_i $(i=1,2,3)$ are the electric displacement and magnetic flux components (in the x_1-, x_2-, and x_3-directions), E_i and H_i are the electric field and magnetic field components, κ_{11} and κ_{33} are the two dielectric permittivity coefficients in the x_1- and x_3-directions, respectively, α_{11} and α_{33} are the two magnetoelectric coefficients (in the x_1- and x_3-directions), and μ_{11} and μ_{33} are the two magnetic permeability coefficients (in the x_1- and x_3-directions).

For a FGM uniaxial multiferroic material with exponential variation in an arbitrary direction, the material coefficients in Eq. (1) can be described by (assuming a uniform variation in space for all the material coefficients[13–16])

$$[\kappa_{ii}(\mathbf{x})\ \alpha_{ii}(\mathbf{x})\ \mu_{ii}(\mathbf{x})] = [\kappa_{ii}^0\ \alpha_{ii}^0\ \mu_{ii}^0]\exp(2\beta_1 x_1 + 2\beta_2 x_2 + 2\beta_3 x_3) \quad (i=1,3), \tag{2}$$

where β_1, β_2, and β_3 are three exponential factors characterizing the degree of material gradient in the x_1-, x_2-, and x_3-directions, respectively. The superscript "0" indicates the coordinate-independent factor in the material coefficient, and the factor of 2 in the exponential is introduced to simplify the derivation of Green's function expressions. It is obvious that $\beta_1=\beta_2=\beta_3=0$ corresponds to the homogeneous multiferroic material case investigated previously.[6–8] We point out that the special exponential variation is assumed so that the exact closed-form Green's functions can be derived, and that the assumed exponential function can be used to piecewise approximate any smooth variation of the FGM in a small space domain determined experimentally from the graded multiferroic composition profile. We further remark that for either positive or negative β_i, some of Green's function components may become unbounded at infinity, and therefore use of them in the boundary integral equation for boundary value problems associated with infinite domains should be very cautious.

The electric and magnetic fields are related to the electric potential ϕ and magnetic potential ψ through the following 2×1 column matrix relation:

$$\begin{bmatrix} E_i \\ H_i \end{bmatrix} = -\begin{bmatrix} \phi_{,i} \\ \psi_{,i} \end{bmatrix}, \tag{3}$$

where the subscript comma "," followed by the index i $(i=1,2,3)$ denotes the derivative of the potential with respect to the coordinate x_i.

For the FGM uniaxial multiferroic free space, we assume, without loss of generality, that there is a point electric charge P and a point magnetic charge M both located at the origin (see Appendix A for the case where the source is not at the origin). Thus, the electric displacement D_i and magnetic flux B_i satisfy the following equations:[7]

$$\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3} = P\delta(x_1)\delta(x_2)\delta(x_3),$$

$$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = M\delta(x_1)\delta(x_2)\delta(x_3), \tag{4}$$

where δ is the Dirac delta function.

Substituting Eqs. (2) and (3) Eq. (1), and then the results into Eq. (4), we arrive at the following set of inhomogeneous partial differential equations for ϕ and ψ:

$$\begin{bmatrix} \kappa_{11}^0 & \alpha_{11}^0 \\ \alpha_{11}^0 & \mu_{11}^0 \end{bmatrix} \begin{bmatrix} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\phi \\ \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi \end{bmatrix} + \begin{bmatrix} \kappa_{33}^0 & \alpha_{33}^0 \\ \alpha_{33}^0 & \mu_{33}^0 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \phi}{\partial x_3^2} \\ \frac{\partial^2 \psi}{\partial x_3^2} \end{bmatrix} + 2\begin{bmatrix} \kappa_{11}^0 & \alpha_{11}^0 \\ \alpha_{11}^0 & \mu_{11}^0 \end{bmatrix} \begin{bmatrix} \beta_1 \frac{\partial \phi}{\partial x_1} + \beta_2 \frac{\partial \phi}{\partial x_2} \\ \beta_1 \frac{\partial \psi}{\partial x_1} + \beta_2 \frac{\partial \psi}{\partial x_2} \end{bmatrix} + 2\beta_3 \begin{bmatrix} \kappa_{33}^0 & \alpha_{33}^0 \\ \alpha_{33}^0 & \mu_{33}^0 \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x_3} \\ \frac{\partial \psi}{\partial x_3} \end{bmatrix} = -\begin{bmatrix} P \\ M \end{bmatrix} \delta(x_1)\delta(x_2)\delta(x_3), \tag{5}$$

where the well-known properties of the delta function have been employed.

In order to solve the above set of coupled differential equations, we first consider the following eigenvalue problem:[7]

$$\left(\begin{bmatrix} \kappa_{11}^0 & \alpha_{11}^0 \\ \alpha_{11}^0 & \mu_{11}^0 \end{bmatrix} - \lambda \begin{bmatrix} \kappa_{33}^0 & \alpha_{33}^0 \\ \alpha_{33}^0 & \mu_{33}^0 \end{bmatrix} \right) \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

The two eigenvalues λ_1 and λ_2 are given by

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$$\lambda_1 = \frac{\mu_{33}^0\kappa_{11}^0 + \mu_{11}^0\kappa_{33}^0 - 2\alpha_{11}^0\alpha_{33}^0 + \sqrt{(\mu_{11}^0\kappa_{33}^0 - \mu_{33}^0\kappa_{11}^0)^2 + 4(\alpha_{11}^0\mu_{33}^0 - \alpha_{33}^0\mu_{11}^0)(\alpha_{11}^0\kappa_{33}^0 - \alpha_{33}^0\kappa_{11}^0)}}{2(\mu_{33}^0\kappa_{33}^0 - \alpha_{33}^{02})},$$

$$\lambda_2 = \frac{\mu_{33}^0\kappa_{11}^0 + \mu_{11}^0\kappa_{33}^0 - 2\alpha_{11}^0\alpha_{33}^0 - \sqrt{(\mu_{11}^0\kappa_{33}^0 - \mu_{33}^0\kappa_{11}^0)^2 + 4(\alpha_{11}^0\mu_{33}^0 - \alpha_{33}^0\mu_{11}^0)(\alpha_{11}^0\kappa_{33}^0 - \alpha_{33}^0\kappa_{11}^0)}}{2(\mu_{33}^0\kappa_{33}^0 - \alpha_{33}^{02})}, \tag{7}$$

and the corresponding eigenvectors associated with λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{bmatrix} -\alpha_{11}^0 + \lambda_1\alpha_{33}^0 \\ \kappa_{11}^0 - \lambda_1\kappa_{33}^0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -\alpha_{11}^0 + \lambda_2\alpha_{33}^0 \\ \kappa_{11}^0 - \lambda_2\kappa_{33}^0 \end{bmatrix}. \tag{8}$$

Since the two matrices $\begin{bmatrix} \kappa_{11}^0 & \alpha_{11}^0 \\ \alpha_{11}^0 & \mu_{11}^0 \end{bmatrix}$ and $\begin{bmatrix} \kappa_{33}^0 & \alpha_{33}^0 \\ \alpha_{33}^0 & \mu_{33}^0 \end{bmatrix}$ are real and symmetric, it can be easily verified that the orthogonality conditions

$$\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \kappa_{33}^0 & \alpha_{33}^0 \\ \alpha_{33}^0 & \mu_{33}^0 \end{bmatrix} [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \kappa_{11}^0 & \alpha_{11}^0 \\ \alpha_{11}^0 & \mu_{11}^0 \end{bmatrix} [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} \lambda_1\delta_1 & 0 \\ 0 & \lambda_2\delta_2 \end{bmatrix} \tag{9}$$

are satisfied. In Eq. (9), the superscript T denotes matrix transpose, and

$$\delta_1 = \alpha_{11}^{02}\kappa_{33}^0 + \kappa_{11}^{02}\mu_{33}^0 - 2\alpha_{11}^0\alpha_{33}^0\kappa_{11}^0 + (\mu_{33}^0\kappa_{33}^0 - \alpha_{33}^{02}) \times (\lambda_1^2\kappa_{33}^0 - 2\lambda_1\kappa_{11}^0),$$

$$\delta_2 = \alpha_{11}^{02}\kappa_{33}^0 + \kappa_{11}^{02}\mu_{33}^0 - 2\alpha_{11}^0\alpha_{33}^0\kappa_{11}^0 + (\mu_{33}^0\kappa_{33}^0 - \alpha_{33}^{02}) \times (\lambda_2^2\kappa_{33}^0 - 2\lambda_2\kappa_{11}^0). \tag{10}$$

We now introduce two new functions f and g, which are related to ϕ and ψ through

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{\Phi} \begin{bmatrix} f \\ g \end{bmatrix}, \tag{11}$$

where $\mathbf{\Phi} = [\mathbf{v}_1 \ \mathbf{v}_2]$.

In view of Eqs. (5), (9), and (11), the two new functions f and g are required to satisfy the following two independent, inhomogeneous, and partial differential equations:

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{\lambda_1}\frac{\partial^2}{\partial x_3^2} + 2\beta_1\frac{\partial}{\partial x_1} + 2\beta_2\frac{\partial}{\partial x_2} + \frac{2\beta_3}{\lambda_1}\frac{\partial}{\partial x_3}\right) f = -\frac{(-\alpha_{11}^0 + \lambda_1\alpha_{33}^0)P + (\kappa_{11}^0 - \lambda_1\kappa_{33}^0)M}{\delta_1\lambda_1}\delta(x_1)\delta(x_2)\delta(x_3), \tag{12}$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{\lambda_2}\frac{\partial^2}{\partial x_3^2} + 2\beta_1\frac{\partial}{\partial x_1} + 2\beta_2\frac{\partial}{\partial x_2} + \frac{2\beta_3}{\lambda_2}\frac{\partial}{\partial x_3}\right) g = -\frac{(-\alpha_{11}^0 + \lambda_2\alpha_{33}^0)P + (\kappa_{11}^0 - \lambda_2\kappa_{33}^0)M}{\delta_2\lambda_2}\delta(x_1)\delta(x_2)\delta(x_3). \tag{13}$$

Equations (12) and (13) can be equivalently expressed as

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial(\sqrt{\lambda_1}x_3)^2} + 2\beta_1\frac{\partial}{\partial x_1} + 2\beta_2\frac{\partial}{\partial x_2} + \frac{2\beta_3}{\sqrt{\lambda_1}}\frac{\partial}{\partial(\sqrt{\lambda_1}x_3)}\right] f = -4\pi K_1\delta(x_1)\delta(x_2)\delta(\sqrt{\lambda_1}x_3), \tag{14}$$

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial(\sqrt{\lambda_2}x_3)^2} + 2\beta_1\frac{\partial}{\partial x_1} + 2\beta_2\frac{\partial}{\partial x_2} + \frac{2\beta_3}{\sqrt{\lambda_2}}\frac{\partial}{\partial(\sqrt{\lambda_2}x_3)}\right] g = -4\pi K_2\delta(x_1)\delta(x_2)\delta(\sqrt{\lambda_2}x_3), \tag{15}$$

where the two constants K_1 and K_2 are defined as

$$K_1 = \frac{(-\alpha_{11}^0 + \lambda_1\alpha_{33}^0)P + (\kappa_{11}^0 - \lambda_1\kappa_{33}^0)M}{4\pi\delta_1\sqrt{\lambda_1}},$$

$$K_2 = \frac{(-\alpha_{11}^0 + \lambda_2\alpha_{33}^0)P + (\kappa_{11}^0 - \lambda_2\kappa_{33}^0)M}{4\pi\delta_2\sqrt{\lambda_2}}. \tag{16}$$

Next we introduce another two new functions F and G defined by

$$f = \exp(-\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3)F, \quad g = \exp(-\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3)G. \tag{17}$$

Consequently, Eqs. (14) and (15) can now be changed into the following two independent inhomogeneous 3D Helmholtz equations:

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial(\sqrt{\lambda_1}x_3)^2} - \eta_1^2\right] F = -4\pi K_1\delta(x_1)\delta(x_2)\delta(\sqrt{\lambda_1}x_3), \tag{18}$$

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$$\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial(\sqrt{\lambda_2}x_3)^2}-\eta_2^2\right]G = -4\pi K_2\delta(x_1)\delta(x_2)\delta(\sqrt{\lambda_2}x_3), \tag{19}$$

where

$$\eta_1=\sqrt{\beta_1^2+\beta_2^2+\frac{\beta_3^2}{\lambda_1}},\quad \eta_2=\sqrt{\beta_1^2+\beta_2^2+\frac{\beta_3^2}{\lambda_2}}. \tag{20}$$

The solutions to Eqs. (18) and (19) can be expediently given by

$$F=K_1\frac{\exp(-\eta_1 r_1)}{r_1},\quad G=K_2\frac{\exp(-\eta_2 r_2)}{r_2}, \tag{21}$$

where $r_1=\sqrt{x_1^2+x_2^2+\lambda_1 x_3^2}$ and $r_2=\sqrt{x_1^2+x_2^2+\lambda_2 x_3^2}$.

In view of Eq. (17), the expressions of the two functions f and g are given by

$$f=K_1\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_1r_1)}{r_1},$$

$$g=K_2\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_2r_2)}{r_2}. \tag{22}$$

It follows from Eq. (11) that the electric and magnetic potentials can be now obtained as

$$\begin{bmatrix}\phi\\ \psi\end{bmatrix}=\mathbf{\Phi}\begin{bmatrix}K_1\dfrac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_1r_1)}{r_1}\\ K_2\dfrac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_2r_2)}{r_2}\end{bmatrix}. \tag{23}$$

Based on Eq. (23), the multiferroic Green's functions $G_{\alpha\beta}$ are then found to be

$$4\pi G_{\phi P}=\frac{(\alpha_{11}^0-\lambda_1\alpha_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_1r_1)}{r_1}+\frac{(\alpha_{11}^0-\lambda_2\alpha_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_2r_2)}{r_2}, \tag{24a}$$

$$4\pi G_{\phi M}=4\pi G_{\psi P}=\frac{(\lambda_1\alpha_{33}^0-\alpha_{11}^0)(\kappa_{11}^0-\lambda_1\kappa_{33}^0)}{\delta_1\sqrt{\lambda_1}}\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_1r_1)}{r_1}+\frac{(\lambda_2\alpha_{33}^0-\alpha_{11}^0)(\kappa_{11}^0-\lambda_2\kappa_{33}^0)}{\delta_2\sqrt{\lambda_2}}\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_2r_2)}{r_2}, \tag{24b}$$

$$4\pi G_{\psi M}=\frac{(\kappa_{11}^0-\lambda_1\kappa_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_1r_1)}{r_1}+\frac{(\kappa_{11}^0-\lambda_2\kappa_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}\frac{\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_2r_2)}{r_2}. \tag{24c}$$

In Eqs. (24a)–(24c), the definitions of Green's functions are $G_{\phi P}(x_i)$ is the electric potential ϕ at x_i due to a unit point electric charge $(P=1)$ at $x_i=0$, $G_{\phi M}(x_i)$ is the electric potential ϕ at x_i due to a unit point magnetic charge $(M=1)$ at $x_i=0$, $G_{\psi P}(x_i)$ is the magnetic potential ψ at x_i due to a unit point electric charge $(P=1)$ at $x_i=0$, and $G_{\psi M}(x_i)$ is the magnetic potential ψ at x_i due to a unit point magnetic charge $(M=1)$ at $x_i=0$. If one is interested in the electric and magnetic fields or the electric displacement and magnetic flux components induced by the point source, the derivatives of these Green's functions are essential as in the boundary integral equation formulation.[13] These derivatives of Green's functions and the corresponding boundary integral equation formulation for the FGM multiferroic are presented, respectively, in Appendices A and B.

Some interesting features can be observed from Green's function expression in Eqs. (24a)–(24c). (i) When $\beta_1=\beta_2=\beta_3=0$, Eqs. (24a)–(24c) reduce to Green's functions for a homogeneous multiferroic material,[6,7] as expected. (ii) The magnitudes of the FGM multiferroic Green's functions $G_{\alpha\beta}$ decay to zero $(G_{\alpha\beta}\to 0)$ faster than their counterparts for the homogeneous multiferroic material when the field point is away from the origin $(r=\sqrt{x_1^2+x_2^2+x_3^2}\to\infty)$. This is due to the appearance of the decaying exponential terms in these FGM Green's functions. (iii) Just like Green's functions in the corresponding homogeneous space, the FGM Green's functions in Eqs. (24a)–(24c) are also *symmetric* in their indices. In other words, the electric potential at x_i due to a unit point magnetic charge at the origin equals the magnetic potential at x_i due to a unit point electric charge at the origin. However, unlike the homogeneous Green's functions, the source and field points in the FGM Green's functions are *asymmetric*. Namely, the source and field points cannot be exchanged. (iv) Although the exact closed-form Green's functions are based on the special exponential variation, any smooth variation of the FGM in a small space domain can be piecewise approximated by the exponential function. In other words, the application of the present Green's function is not limited to the exponential variation, particularly when it is implemented into the corresponding boundary integral formulation (Appendix B).

We illustrate in Figs. 1–4 the distribution of the electromagnetic Green's functions $G_{\alpha\beta}$ along the x_3-axis for different gradient parameters β_3 (=−10, −3, −1, 0, 1, 3, and 10 m^{-1}) with $\beta_1=\beta_2=0$. This situation implies that the material is graded in the uniaxial x_3-direction.[11,12] The

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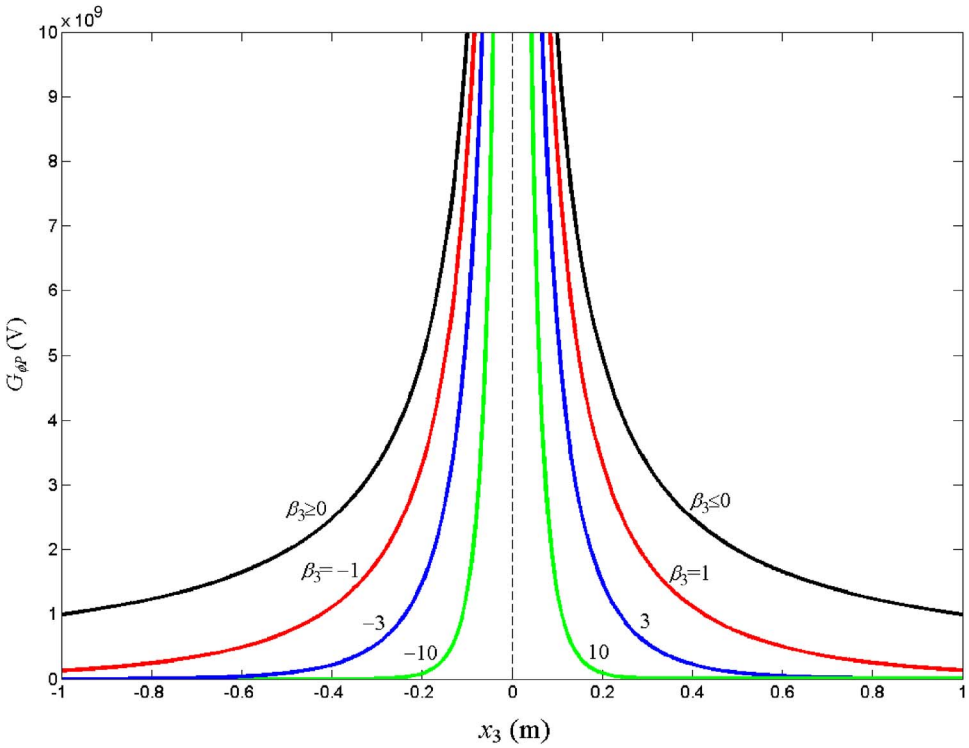

FIG. 1. (Color online) Distribution of the electromagnetic Green's function $G_{\phi P}$ along x_3-axis for different gradient parameters β_3 (=−10, −3, −1, 0, 1, 3, and 10 m^{-1}) with $\beta_1=\beta_2=0$.

coordinate-independent factors in the material coefficients are chosen to be (for a typical multiferroic composite)[6]

$$\alpha_{11}^0 = 5 \times 10^{-12}\ \text{N s/V C}, \quad \alpha_{33}^0 = 3 \times 10^{-12}\ \text{N s/V C},$$

$$\kappa_{11}^0 = 8 \times 10^{-11}\ \text{C}^2/\text{N m}^2, \quad \kappa_{33}^0 = 9.3 \times 10^{-11}\ \text{C}^2/\text{N m}^2,$$

$$\mu_{11}^0 = 5.9 \times 10^{-4}\ \text{N s}^2/\text{C}^2, \quad \mu_{33}^0 = 1.57 \times 10^{-4}\ \text{N s}^2/\text{C}^2.$$

As expected, the figures show that the gradient parameter β_3 has a significant influence on the distribution of Green's functions along the x_3-axis. In particular, the magnitudes of these Green's functions decrease with increasing magnitude of β_3. It is striking that for the special case where $\beta_1=\beta_2=0$ and $x_1=x_2=0$, the distribution of Green's functions in Eqs. (24a)–(24c) along the negative x_3-axis is the same as that in the corresponding homogeneous multiferroic material ($\beta_1=\beta_2=\beta_3=0$), independent of the positive values of β_3 ($\beta_3>0$). On the other hand, the distribution of Green's functions in Eq. (24a)–(24c) along the positive x_3-axis is also

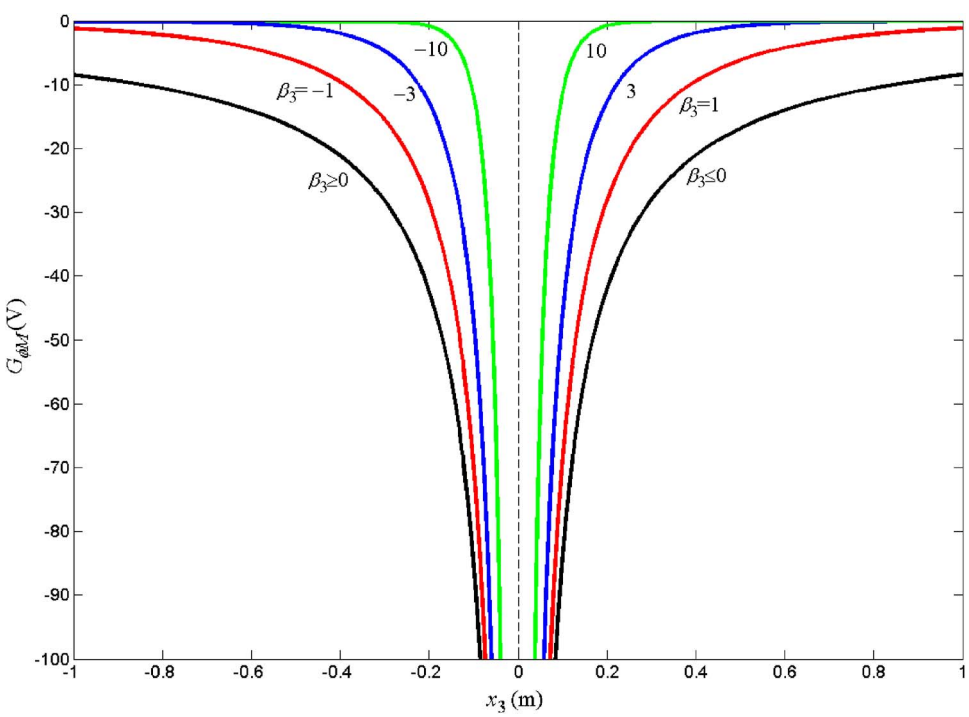

FIG. 2. (Color online) Distribution of the electromagnetic Green's function $G_{\phi M}$ along x_3-axis for different gradient parameters β_3 (=−10, −3, −1, 0, 1, 3, and 10 m^{-1}) with $\beta_1=\beta_2=0$.

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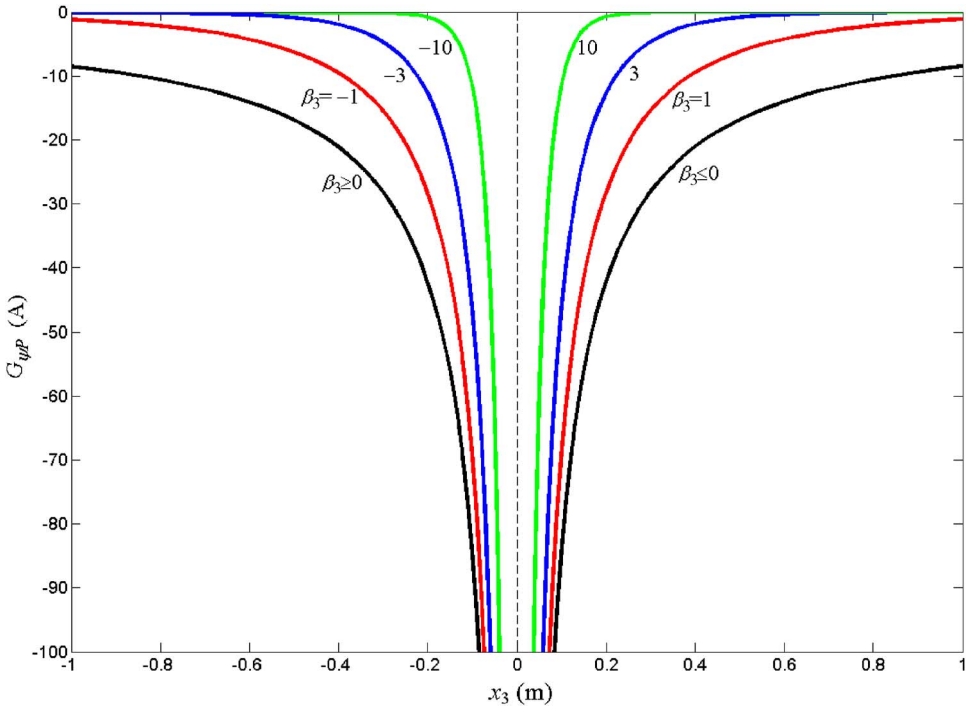

FIG. 3. (Color online) Distribution of the electromagnetic Green's function $G_{\psi P}$ along x_3-axis for different gradient parameters β_3 (=−10, −3, −1, 0, 1, 3, and 10 m^{-1}) with $\beta_1=\beta_2=0$

the same as that in the corresponding homogeneous multiferroic material, independent of the negative values of β_3 ($\beta_3<0$). These interesting behaviors of field responses can be also observed along the x_1-axis for positive and negative β_1 (with $\beta_2=\beta_3=0$) and along the x_2-axis for positive and negative β_2 (with $\beta_1=\beta_3=0$).

Finally, as an interesting application of the derived Green's function solutions, we study the electric and magnetic potentials induced by an electric dipole at the origin and along the uniaxial x_3-direction with the electric moment (or dipole moment) p. This is very important because charge pairs of opposite sign are the model for polarized and magnetized atoms and molecules.[17–19] To do so, we put an electric charge P at $(0,0,l/2)$, $(l>0)$ and another electric charge $-P$ at $(0,0,-l/2)$ in Green's function solutions [Eqs. (24a) and (24b)]. During the limiting process, we let $l\to 0$, while keep $p=Pl$ a constant. Here p is the electric moment (or dipole moment). The expressions of the electric and magnetic potentials induced by the electric dipole of moment p at the origin can then be given by

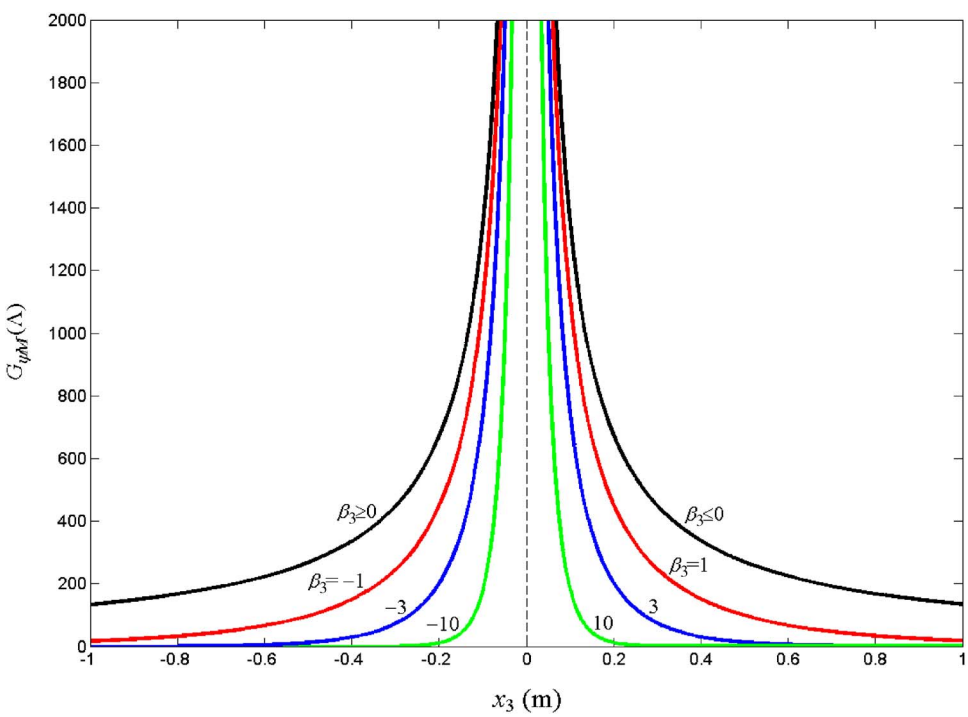

FIG. 4. (Color online) Distribution of the electromagnetic Green's function $G_{\psi M}$ along x_3-axis for different gradient parameters β_3 (=−10, −3, −1, 0, 1, 3, and 10 m^{-1}) with $\beta_1=\beta_2=0$.

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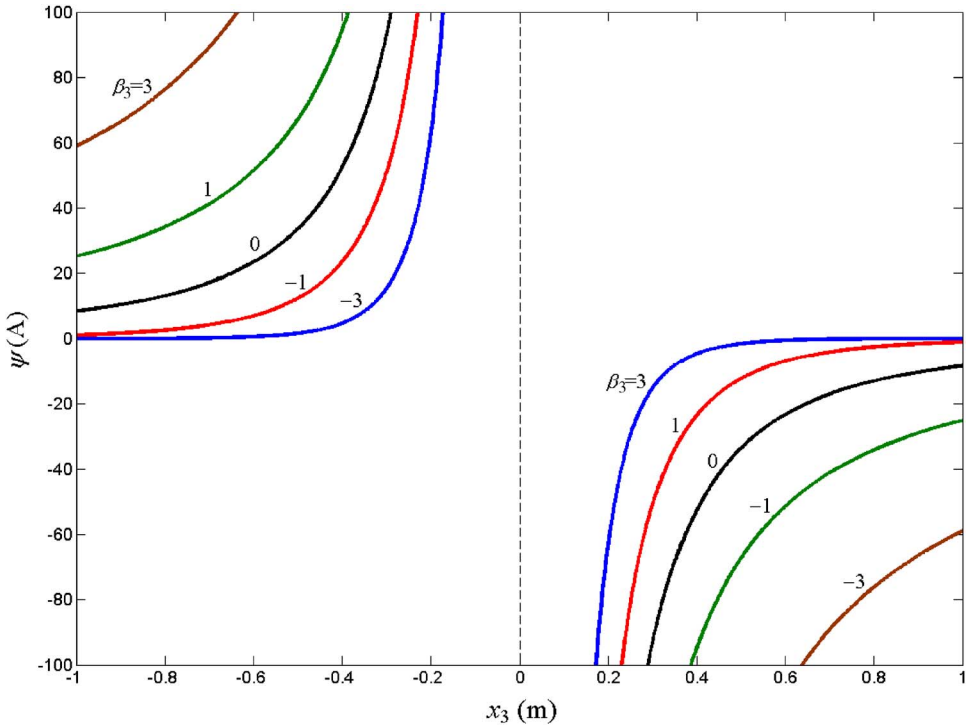

FIG. 5. (Color online) Distribution of the magnetic potential along x_3-axis induced by a unit electric dipole ($p=1$) at the origin and along the uniaxial x_3-direction for different gradient parameters β_3 (=−3, −1, 0, 1, and 3 m^{-1}) with $\beta_1=\beta_2=0$.

$$\phi = \frac{p\sqrt{\lambda_1}(\lambda_1\alpha_{33}^0-\alpha_{11}^0)^2\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_1r_1)}{4\pi\delta_1}\left(\frac{x_3}{r_1^3}+\frac{\eta_1x_3}{r_1^2}-\frac{\beta_3}{\lambda_1r_1}\right) + \frac{p\sqrt{\lambda_2}(\lambda_2\alpha_{33}^0-\alpha_{11}^0)^2\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_2r_2)}{4\pi\delta_2}\left(\frac{x_3}{r_2^3}+\frac{\eta_2x_3}{r_2^2}-\frac{\beta_3}{\lambda_2r_2}\right),$$

$$\psi = \frac{p\sqrt{\lambda_1}(\kappa_{11}^0-\lambda_1\kappa_{33}^0)(\lambda_1\alpha_{33}^0-\alpha_{11}^0)\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_1r_1)}{4\pi\delta_1}\left(\frac{x_3}{r_1^3}+\frac{\eta_1x_3}{r_1^2}-\frac{\beta_3}{\lambda_1r_1}\right) + \frac{p\sqrt{\lambda_2}(\kappa_{11}^0-\lambda_2\kappa_{33}^0)(\lambda_2\alpha_{33}^0-\alpha_{11}^0)\exp(-\beta_1x_1-\beta_2x_2-\beta_3x_3-\eta_2r_2)}{4\pi\delta_2}\left(\frac{x_3}{r_2^3}+\frac{\eta_2x_3}{r_2^2}-\frac{\beta_3}{\lambda_2r_2}\right). \tag{25}$$

It is observed from Eq. (25) that an electric dipole can also induce the magnetic field due to the magnetoelectric effect in multiferroic materials. To see this fact more clearly, we illustrate in Fig. 5 the distribution of the magnetic potential along the x_3-axis induced by a unit electric dipole ($p=1$) at the origin and along the uniaxial x_3-direction for different gradient parameters β_3 (=−3, −1, 0, 1, and 3 m^{-1}) with $\beta_1=\beta_2=0$. Comparing Fig. 5 to Green's function $G_{\psi M}$ in Fig. 4, it is interesting to observe that the distribution of the magnetic potential along the whole x_3-axis induced by the unit electric dipole always depends on the value of the gradient parameter β_3. It is further noticed from Fig. 5 that (1) when $\beta_3x_3>0$, the magnitude of the induced magnetic potential at any point is smaller than the corresponding one for the homogeneous material ($\beta_3=0$) at the same point and, (2) conversely when $\beta_3x_3<0$, the magnitude of the induced magnetic potential at any point is larger than the corresponding one for homogeneous material ($\beta_3=0$) at the same point.

III. CONCLUSIONS

In this paper, the 3D exact closed-form electromagnetic Green's functions for a FGM uniaxial multiferroic space are derived by assuming the material to be exponentially graded in an arbitrary direction. The explicit expressions of Green's functions are given in Eqs. (24a)–(24c), with their derivatives listed in Eqs. (A1)–(A3) of Appendix A. The corresponding boundary integral equations are also presented in Appendix B. Sharply different from the complicated Green's function expression in FGM transversely isotropic piezoelectric materials,[16] Green's functions in the FGM multiferroic space are expressed in terms of elementary functions only. Numerical examples of Green's functions and electric-dipole solutions have been presented to demonstrate the possibility of tailoring the material behavior. In particular, it is shown that along the gradient orientation, amplitudes of the field response in a FGM multiferroic space can be modulated by using different functional gradient factors, resulting in either reduced amplitude of the response or the same values as

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those in the corresponding homogeneous materials (for Green's functions). The derived Green's functions can be further employed to address inclusion problems in FGM multiferroic materials and can be implemented into the corresponding boundary element formulation to attack more complicated boundary value problems associated with FGM multiferroic materials. We point out that although the exact closed-form Green's functions are based on the special exponential variation, any smooth variation of the FGM can be treated within the boundary integral equation in a piecewise thin-layer manner.[16]

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APPENDIX A: DERIVATIVES OF GREEN'S FUNCTIONS

Derivatives of Green's functions are listed below. These field quantities are required to find the electric and magnetic fields and electric displacements and magnetic fluxes. They are also required in the FGM boundary integral formulation as presented in Appendix B.

$$4\pi\frac{\partial G_{\phi P}}{\partial x_1} = -\frac{(\alpha_{11}^0-\lambda_1\alpha_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}(\beta_1 r_1^{-1}+\eta_1 x_1 r_1^{-2}+x_1 r_1^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_1 r_1) - \frac{(\alpha_{11}^0-\lambda_2\alpha_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}(\beta_1 r_2^{-1}+\eta_2 x_1 r_2^{-2}+x_1 r_2^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_2 r_2),$$

$$4\pi\frac{\partial G_{\phi P}}{\partial x_2} = -\frac{(\alpha_{11}^0-\lambda_1\alpha_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}(\beta_2 r_1^{-1}+\eta_1 x_2 r_1^{-2}+x_2 r_1^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_1 r_1) - \frac{(\alpha_{11}^0-\lambda_2\alpha_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}(\beta_2 r_2^{-1}+\eta_2 x_2 r_2^{-2}+x_2 r_2^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_2 r_2),$$

$$4\pi\frac{\partial G_{\phi P}}{\partial x_3} = -\frac{(\alpha_{11}^0-\lambda_1\alpha_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}(\beta_3 r_1^{-1}+\eta_1\lambda_1 x_3 r_1^{-2}+\lambda_1 x_3 r_1^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_1 r_1) - \frac{(\alpha_{11}^0-\lambda_2\alpha_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}(\beta_3 r_2^{-1}+\eta_2\lambda_2 x_3 r_2^{-2}+\lambda_2 x_3 r_2^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_2 r_2), \quad \text{(A1)}$$

$$4\pi\frac{\partial G_{\phi M}}{\partial x_1} = 4\pi\frac{\partial G_{\psi P}}{\partial x_1} = \frac{(\alpha_{11}^0-\lambda_1\alpha_{33}^0)(\kappa_{11}^0-\lambda_1\kappa_{33}^0)}{\delta_1\sqrt{\lambda_1}}(\beta_1 r_1^{-1} + \eta_1 x_1 r_1^{-2}+x_1 r_1^{-3})\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3 - \eta_1 r_1) + \frac{(\alpha_{11}^0-\lambda_2\alpha_{33}^0)(\kappa_{11}^0-\lambda_2\kappa_{33}^0)}{\delta_2\sqrt{\lambda_2}}(\beta_1 r_2^{-1} + \eta_2 x_1 r_2^{-2}+x_1 r_2^{-3})\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3 - \eta_2 r_2),$$

$$4\pi\frac{\partial G_{\phi M}}{\partial x_2} = 4\pi\frac{\partial G_{\psi P}}{\partial x_2} = \frac{(\alpha_{11}^0-\lambda_1\alpha_{33}^0)(\kappa_{11}^0-\lambda_1\kappa_{33}^0)}{\delta_1\sqrt{\lambda_1}}(\beta_2 r_1^{-1} + \eta_1 x_2 r_1^{-2}+x_2 r_1^{-3})\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3 - \eta_1 r_1) + \frac{(\alpha_{11}^0-\lambda_2\alpha_{33}^0)(\kappa_{11}^0-\lambda_2\kappa_{33}^0)}{\delta_2\sqrt{\lambda_2}}(\beta_2 r_2^{-1} + \eta_2 x_2 r_2^{-2}+x_2 r_2^{-3})\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3 - \eta_2 r_2),$$

$$4\pi\frac{\partial G_{\phi M}}{\partial x_3} = 4\pi\frac{\partial G_{\psi P}}{\partial x_3} = \frac{(\alpha_{11}^0-\lambda_1\alpha_{33}^0)(\kappa_{11}^0-\lambda_1\kappa_{33}^0)}{\delta_1\sqrt{\lambda_1}}(\beta_3 r_1^{-1} + \eta_1\lambda_1 x_3 r_1^{-2}+\lambda_1 x_3 r_1^{-3})\exp(-\beta_1 x_1-\beta_2 x_2 - \beta_3 x_3-\eta_1 r_1) + \frac{(\alpha_{11}^0-\lambda_2\alpha_{33}^0)(\kappa_{11}^0-\lambda_2\kappa_{33}^0)}{\delta_2\sqrt{\lambda_2}}(\beta_3 r_2^{-1} + \eta_2\lambda_2 x_3 r_2^{-2}+\lambda_2 x_3 r_2^{-3})\exp(-\beta_1 x_1-\beta_2 x_2 - \beta_3 x_3-\eta_2 r_2), \quad \text{(A2)}$$

$$4\pi\frac{\partial G_{\psi M}}{\partial x_1} = -\frac{(\kappa_{11}^0-\lambda_1\kappa_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}(\beta_1 r_1^{-1}+\eta_1 x_1 r_1^{-2}+x_1 r_1^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_1 r_1) - \frac{(\kappa_{11}^0-\lambda_2\kappa_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}(\beta_1 r_2^{-1}+\eta_2 x_1 r_2^{-2}+x_1 r_2^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_2 r_2),$$

$$4\pi\frac{\partial G_{\psi M}}{\partial x_2} = -\frac{(\kappa_{11}^0-\lambda_1\kappa_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}(\beta_2 r_1^{-1}+\eta_1 x_2 r_1^{-2}+x_2 r_1^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_1 r_1) - \frac{(\kappa_{11}^0-\lambda_2\kappa_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}(\beta_2 r_2^{-1}+\eta_2 x_2 r_2^{-2}+x_2 r_2^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_2 r_2),$$

$$4\pi\frac{\partial G_{\psi M}}{\partial x_3} = -\frac{(\kappa_{11}^0-\lambda_1\kappa_{33}^0)^2}{\delta_1\sqrt{\lambda_1}}(\beta_3 r_1^{-1}+\eta_1\lambda_1 x_3 r_1^{-2}+\lambda_1 x_3 r_1^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_1 r_1) - \frac{(\kappa_{11}^0-\lambda_2\kappa_{33}^0)^2}{\delta_2\sqrt{\lambda_2}}(\beta_3 r_2^{-1}+\eta_2\lambda_2 x_3 r_2^{-2}+\lambda_2 x_3 r_2^{-3}) \times\exp(-\beta_1 x_1-\beta_2 x_2-\beta_3 x_3-\eta_2 r_2). \quad \text{(A3)}$$

We remark that in deriving the potential Green's functions and their derivatives, we have assumed that the point charge is located at the origin. For applications of these

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Green's functions to a boundary integral equation formulation as described in Appendix B, the source point has to be arbitrary.

Let us assume that the point charge is located at an arbitrary source point $\boldsymbol{x}_s=(x_1^s,x_2^s,x_3^s)$, and the field (or observation) point is at $\boldsymbol{x}_f=(x_1^f,x_2^f,x_3^f)$. We then can make the following coordinate translation:

$$\tilde{x}_1 = x_1^f - x_1^s, \quad \tilde{x}_2 = x_2^f - x_2^s, \quad \tilde{x}_3 = x_3^f - x_3^s, \tag{A4}$$

so that the point charge is located at the origin of the new coordinate system $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)$. Therefore, all Green's functions developed in this paper hold in the new coordinate system, with, however, the material coefficients vary exponentially in the new coordinates as follows:

$$[\kappa_{ii}(\tilde{\mathbf{x}})\ \alpha_{ii}(\tilde{\mathbf{x}})\ \mu_{ii}(\tilde{\mathbf{x}})] = [\tilde{\kappa}_{ii}^0\ \tilde{\alpha}_{ii}^0\ \tilde{\mu}_{ii}^0]\exp(2\beta_1\tilde{x}_1 + 2\beta_2\tilde{x}_2 + 2\beta_3\tilde{x}_3) \quad (i=1,3), \tag{A5}$$

where the three material constants $\tilde{\kappa}_{ii}^0$, $\tilde{\alpha}_{ii}^0$, and $\tilde{\mu}_{ii}^0$ are given by

$$\tilde{\kappa}_{ii}^0 = \exp(2\beta_1 x_1^s + 2\beta_2 x_2^s + 2\beta_3 x_3^s)\kappa_{ii}^0,$$

$$\tilde{\alpha}_{ii}^0 = \exp(2\beta_1 x_1^s + 2\beta_2 x_2^s + 2\beta_3 x_3^s)\alpha_{ii}^0,$$

$$\tilde{\mu}_{ii}^0 = \exp(2\beta_1 x_1^s + 2\beta_2 x_2^s + 2\beta_3 x_3^s)\mu_{ii}^0, \tag{A6}$$

$(i=1,3)$ with κ_{ii}^0, α_{ii}^0, and μ_{ii}^0 being the material coefficients at the origin of the old coordinate system (x_1,x_2,x_3).

APPENDIX B: BOUNDARY INTEGRAL EQUATION FORMULATION FOR A FGM MULTIFERROIC

For a uniaxial multiferroic material satisfying the constitutive relation (1), it can be easily shown that the following reciprocal property holds:[20]

$$D_i^{(1)}E_i^{(2)} + B_i^{(1)}H_i^{(2)} = D_i^{(2)}E_i^{(1)} + B_i^{(2)}H_i^{(1)}, \tag{B1}$$

where the superscripts "(1)" and "(2)" denote two independent systems of the field quantities under different loadings, and the dummy index i takes the summation from 1 to 3. We emphasize that Eq. (B1) holds for any spatial variation of the multiferroic material properties. Therefore, it also holds for the exponential variation as described by Eq. (2).

By integrating both sides of Eq. (B1) with respect to the problem domain and making use of the divergence theorem, we obtain the following integral relation:

$$\int_S [D_n^{(1)}\phi^{(2)} + B_n^{(1)}\psi^{(2)}]dS - \int_V [D_{i,i}^{(1)}\phi^{(2)} + B_{i,i}^{(1)}\psi^{(2)}]dV = \int_S [D_n^{(2)}\phi^{(1)} + B_n^{(2)}\psi^{(1)}]dS - \int_V [D_{i,i}^{(2)}\phi^{(1)} + B_{i,i}^{(2)}\psi^{(1)}]dV, \tag{B2}$$

where V is the problem domain and S is the boundary of domain V, the subscript, i denotes the derivative with respect to the coordinate x_i, and the subscript n means the normal component of the field vector. In other words, for example, $D_n=D_i n_i$ with n_i being the outward normal at the boundary point.

We assume now that system (2) corresponds to the real boundary value problem in a FGM multiferroic domain and (1) to the corresponding Green's function solution with $P=1$ and $M=0$ in Eq. (4). It can then be shown that the following integral equation expression holds for the electric potential ϕ at any point $\boldsymbol{x}_s$ within the problem domain:

$$\phi(\boldsymbol{x}_s) = \int_S [D_n^P(\boldsymbol{x}_f;\boldsymbol{x}_s)\phi(\boldsymbol{x}_f) + B_n^P(\boldsymbol{x}_f;\boldsymbol{x}_s)\psi(\boldsymbol{x}_s)]dS - \int_S [G_{\phi P}(\boldsymbol{x}_f;\boldsymbol{x}_s)D_n(\boldsymbol{x}_f) + G_{\psi P}(\boldsymbol{x}_f;\boldsymbol{x}_s)B_n(\boldsymbol{x}_f)]dS, \tag{B3}$$

here $G_{\phi P}$ and $G_{\psi P}$ are the (electric and magnetic) potential Green's functions at $\boldsymbol{x}_f$ (due to a point electric charge of unit amplitude at $\boldsymbol{x}_s$) given in Eqs. (24a)–(24c) and D_n^P and B_n^P are the normal components of the electric displacement and magnetic flux Green's functions at $\boldsymbol{x}_f$ (due to a point electric charge of unit amplitude at $\boldsymbol{x}_s$), defined as

$$D_n^P(\boldsymbol{x}_f;\boldsymbol{x}_s) = D_i^P(\boldsymbol{x}_f;\boldsymbol{x}_s)n_i(\boldsymbol{x}_f),$$

$$B_n^P(\boldsymbol{x}_f;\boldsymbol{x}_s) = B_i^P(\boldsymbol{x}_f;\boldsymbol{x}_s)n_i(\boldsymbol{x}_f), \tag{B4}$$

where $n_i(\boldsymbol{x}_f)$ is the outward normal at the boundary point $\boldsymbol{x}_f$ and D_i^P and B_i^P are related to the potential Green's functions through the following relations (for $i=1,2,3$; no summation on repeated index i):

$$\begin{bmatrix} D_i^P(\boldsymbol{x}_f;\boldsymbol{x}_s) \\ B_i^P(\boldsymbol{x}_f;\boldsymbol{x}_s) \end{bmatrix} = -\begin{bmatrix} \kappa_{ii}(\boldsymbol{x}_f-\boldsymbol{x}_s) & \alpha_{ii}(\boldsymbol{x}_f-\boldsymbol{x}_s) \\ \alpha_{ii}(\boldsymbol{x}_f-\boldsymbol{x}_s) & \mu_{ii}(\boldsymbol{x}_f-\boldsymbol{x}_s) \end{bmatrix} \times \begin{bmatrix} \dfrac{\partial G_{\phi P}(\boldsymbol{x}_f;\boldsymbol{x}_s)}{\partial x_i^f} \\ \dfrac{\partial G_{\psi P}(\boldsymbol{x}_f;\boldsymbol{x}_s)}{\partial x_i^f} \end{bmatrix}. \tag{B5}$$

It is noted that the multiferroic material property matrix for $i=2$ is the same as that for $i=1$ since the material is uniaxial. In Eq. (B5), we have also defined that $\boldsymbol{x}_f=(x_1^f,x_2^f,x_3^f)$.

We point out that in deriving Eq. (B3), the electric and magnetic charges within the problem domain are assumed to be zero. Should one of them or both of them are nonzero, the particular solution method[21] or dual reciprocity method[22] can be easily applied to take care of the contribution from these body charges.

Similarly, if we let system (1) be Green's function solution with $P=0$ and $M=1$ in Eq. (4), then we arrive at the integral equation expression for the magnetic potential ψ at any point $\boldsymbol{x}_s$ within the problem domain,

$$\psi(\boldsymbol{x}_s) = \int_S [D_n^M(\boldsymbol{x}_f;\boldsymbol{x}_s)\phi(\boldsymbol{x}_f) + B_n^M(\boldsymbol{x}_f;\boldsymbol{x}_s)\psi(\boldsymbol{x}_s)]dS - \int_S [G_{\phi M}(\boldsymbol{x}_f;\boldsymbol{x}_s)D_n(\boldsymbol{x}_f) + G_{\psi M}(\boldsymbol{x}_f;\boldsymbol{x}_s)B_n(\boldsymbol{x}_f)]dS, \tag{B6}$$

where the definitions for the involved Green's functions are similar to those in Eq. (B3), except for the fact that the

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sources are different [due to a point electric charge in Eq. (B3) and a point magnetic charge in Eq. (B6)].

Let the source point $\boldsymbol{x}_s$ approach a point on the boundary of the problem domain, then Eqs. (B3) and (B6) form a pair of boundary integral equations from which the unknown boundary values for ϕ, ψ, D_n, and B_n can be solved. Once these boundary values are solved, Eqs. (B3) and (B6) can be utilized to find the field quantities within the problem domain.[20]

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