Uniform Stresses Inside an Elliptical Inhomogeneity With an Imperfect Interface in Plane Elasticity

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We consider an elliptical inhomogeneity embedded in an infinite isotropic elastic matrix subjected to in-plane deformations under the assumption of remote uniform loading. The inhomogeneity-matrix interface is assumed to be imperfect, which is simulated by the spring-layer model with vanishing thickness. Its behavior is based on the assumption that tractions are continuous but displacements are discontinuous across the interface. We further assume that the same degree of imperfection on the interface is realized in both the normal and tangential directions. We find a form of interface function, which leads to uniform stress field within the elliptical inhomogeneity. The explicit expressions for the uniform stress field within the elliptical inhomogeneity are derived. The obtained results are verified by comparison with existing solutions. The condition under which the internal stress field is not only uniform but also hydrostatic is also presented. [DOI: 10.1115/1.2913045]

Keywords: elliptical inhomogeneity, imperfect interface, uniform stress, in-plane deformation

1 Introduction

Micromechanical analysis for elastic inhomogeneities with an imperfect interface has received much attention in literature (see, for example, Refs. [1–3]). Here, the widely used springlike model of the imperfect interface is based on the assumption that tractions are continuous but displacements are discontinuous across the interface. More precisely, the jumps in displacement components are proportional, in terms of the "spring-factor-type" interface functions (or interface parameters), to the respective interface traction components.

Some interesting phenomena have been observed for inhomogeneities with an imperfect interface. Hashin [1] examined a spherical inhomogeneity imperfectly bonded to an infinite matrix, and he found that the stress field inside the spherical inhomogeneity is intrinsically *nonuniform* under a remote uniform stress field. Gao [2] and Shen et al. [4] drew similar conclusions for the two-dimensional circular and elliptical inclusions under plane deformations. In sharp contrast to the above results, Ru and Schiavone [3] found that the stress field inside the inclusion is still *uniform* under remote uniform antiplane shear stresses when the inhomogeneity is circular and the interface is homogeneously imperfect. Antipov and Schiavone [5] developed a novel method to identify the shape of the inhomogeneity and the form of the corresponding interface function, which leads to a uniform interior stress field under antiplane shear deformation. It shall be mentioned that the practical significance of uniform stress field inside the inhomogeneity lies in the fact that a uniform stress distribution is optimal in the sense that it eliminates stress peaks within the inhomogeneity, which usually dominate the mechanical failure of the inhomogeneity [5–7].

This research is motivated by the interesting results in Refs. [5–7] for uniform stress field inside an elastic inhomogeneity with an imperfect interface or with an interphase layer. In this investigation, we confine our attention to the special kind of imperfect interface in which the same degree of imperfection is realized in both the normal and tangential directions along the interface [8,9]. In Sec. 2, we present the basic boundary value problem describing the in-plane deformation of an elastic elliptical inhomogeneity with an inhomogeneously imperfect interface. Here, the circumferentially inhomogeneous interface can reflect the more realistic scenario in which the damage varies along the interface [3,10]. In Sec. 3, we derive the explicit expressions for the uniform stress field inside the elliptical inhomogeneity with an inhomogeneously imperfect interface. In Sec. 4, we discuss several special cases to verify and to illustrate the obtained solution. It is verified that our result can reduce to that for a circular inhomogeneity with a homogeneously imperfect interface [8–10] and can also reduce to that for an elliptical inhomogeneity with perfect bonding conditions [11,12]. We also present the condition under which the internal stress field is not only uniform but also hydrostatic.

2 Basic Formulation

Consider a domain in R^2, infinite in extent, containing a single internal elastic inhomogeneity, with elastic properties different from those of the surrounding matrix. The linearly elastic materials occupying the inhomogeneity and the matrix are assumed to be homogeneous and isotropic with associated shear moduli μ_1 and μ_2, respectively. We represent the matrix by the domain $S_2: x^2/a^2+y^2/b^2 \geqslant 1$ and assume that the inhomogeneity occupies the elliptical region $S_1: x^2/a^2+y^2/b^2 \leqslant 1$. The ellipse Γ, whose semimajor and semiminor axes are, respectively, a and b, will denote the inhomogeneity-matrix interface. In what follows, the subscripts 1 and 2 (or the superscripts (1) and (2)) refer to the regions S_1 and S_2, respectively. At infinity, the matrix is subject to in-plane remote uniform stresses σ_{xx}^{∞}, σ_{xy}^{∞}, and σ_{yy}^{∞}. Without losing generality, it is further assumed that the rigid-body rotation at infinity is zero, i.e., $\varepsilon^{\infty}=0$.

For plane deformation, the stresses can be expressed in terms of the two Muskhelishvili's complex potentials $\phi(\zeta)$ and $\psi(\zeta)$ as (Muskhelishvili [13])

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$$\sigma_{xx}+\sigma_{yy}=4\,\mathrm{Re}\left[\frac{\phi'(\zeta)}{\omega'(\zeta)}\right]$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy}=2\frac{\overline{m(\zeta)}\left\{\frac{\phi'(\zeta)}{\omega'(\zeta)}\right\}'+\psi'(\zeta)}{\omega'(\zeta)} \tag{1}$$

$$\sigma_{rr}+\sigma_{\theta\theta}=\sigma_{xx}+\sigma_{yy}$$

$$\sigma_{\theta\theta}-\sigma_{rr}+2i\sigma_{r\theta}=\frac{\zeta^2\omega'(\zeta)}{|\zeta|^2\overline{\omega'(\zeta)}}(\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy})$$

The resultant force and displacements can be expressed in terms of $\phi(\zeta)$ and $\psi(\zeta)$ as

$$F_x+iF_y=(-i)\left[\phi(\zeta)+\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)}+\overline{\psi(\zeta)}\right] \tag{2}$$

$$2\mu(u_r+iu_\theta)=\frac{|\zeta\omega'(\zeta)|}{\zeta\omega'(\zeta)}\left[\kappa\phi(\zeta)-\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)}-\overline{\psi(\zeta)}\right] \tag{3}$$

where $\kappa=3-4\nu$ for plan strain (assumed henceforth in this research) and $\kappa=(3-\nu)/(1+\nu)$ for plane stress; μ and ν are the shear modulus and Poisson's ratio, respectively; u_r and u_θ are the normal and tangential displacement components in the curvilinear coordinate system expressed by the conformal mapping function $\omega(\zeta)$.

Here, we adopt the following conformal mapping function $\omega(\zeta)$, which maps the region S_2 (in the z-plane) onto the region $\sigma=\{|\zeta|\geqslant 1\}$ (in the ζ-plane):

$$\omega(\zeta)=d\left(\zeta+\frac{m}{\zeta}\right) \tag{4}$$

where

$$d=\frac{a+b}{2},\quad 0\leq m=\frac{a-b}{a+b}<1$$

It is assumed that the elliptical inhomogeneity is imperfectly bonded to the matrix along Γ by the spring-layer-type interface. The interface conditions are then given by

$$\sigma_{rr}^{(1)}+i\sigma_{r\theta}^{(1)}=\sigma_{rr}^{(2)}+i\sigma_{r\theta}^{(2)}=\beta(x,y)[(u_r^{(2)}+iu_\theta^{(2)})-(u_r^{(1)}+iu_\theta^{(1)})]\quad\text{on }\Gamma \tag{5}$$

where $\beta(x,y)$, which is non-negative, is the imperfect interface parameter. Equation (5) demonstrates that the same degree of imperfection is realized in both the normal and tangential directions [8,9]. When $\beta(x,y)\to+\infty$, the interface is perfect; while if $\beta(x,y)\to 0$, the interface becomes traction-free. Extending the results obtained by Antipov and Schiavone [5] for antiplane shear deformation, here, $\beta(x,y)$ is chosen to be

$$\beta(x,y)=\frac{2\mu_2}{\lambda|\omega'(\zeta)|}=\frac{2\mu_2}{\lambda b\sqrt{1+b^*\sin^2\theta}},\quad(\zeta=e^{i\theta}) \tag{6}$$

where λ ($\lambda>0$) is a dimensionless *constant* parameter and $b^*=(a^2-b^2)/b^2=4md^2/b^2$.

3 Uniform Stress Field Within the Elliptical Inhomogeneity

To simplify the expression for the boundary value problem, we introduce the following analytical continuation:

$$\phi_2(\zeta)=-\frac{\omega(\zeta)}{\overline{\omega}'(1/\zeta)}\overline{\phi_2'}(1/\zeta)-\overline{\psi}_2(1/\zeta),\quad|\zeta|<1 \tag{7}$$

Meanwhile, the two complex potentials $\phi_1(\zeta)$ and $\psi_1(\zeta)$ for the elliptical inhomogeneity must be assumed to take the following forms so as to ensure the uniform stress field within the elliptical inhomogeneity:

$$\phi_1(\zeta)=Ad\left(\zeta+\frac{m}{\zeta}\right),\quad\psi_1(\zeta)=Bd\left(\zeta+\frac{m}{\zeta}\right) \tag{8}$$

where A and B are two unknown complex constants to be determined.

In view of Eqs. (2), (3), and (6)–(8), the boundary condition (5) can be finally rewritten in terms of $\phi_2(\zeta)$ as

$$\begin{aligned}\phi_2^-(\zeta)-\phi_2^+(\zeta)&=d(A+\bar{A}+\bar{B}m)\zeta+d(Am+\bar{A}m+\bar{B})\zeta^{-1}\\ \kappa_2\phi_2^-(\zeta)+\phi_2^+(\zeta)&=d\left[A\left(\frac{\kappa_1\mu_2}{\mu_1}+\lambda\right)-\bar{A}\left(\frac{\mu_2}{\mu_1}-\lambda\right)-\bar{B}m\left(\frac{\mu_2}{\mu_1}-\lambda\right)\right]\zeta,\quad(|\zeta|=1)\\ &\quad+d\left[Am\left(\frac{\kappa_1\mu_2}{\mu_1}-\lambda\right)-\bar{A}m\left(\frac{\mu_2}{\mu_1}+\lambda\right)-\bar{B}\left(\frac{\mu_2}{\mu_1}+\lambda\right)\right]\zeta^{-1}\end{aligned} \tag{9}$$

where the superscripts "+" and "−" denote the limit values from the inner and outer sides of the circle $|\zeta|=1$.

Applying Liouville's theorem, we arrive at two expressions of $\phi_2(\zeta)$ as follows:

$$\begin{aligned}\phi_2(\zeta)&=d(Am+\bar{A}m+\bar{B}+\Gamma_2)\zeta^{-1}+d\Gamma_1\zeta,\quad(|\zeta|>1)\\ \phi_2(\zeta)&=d\Gamma_2\zeta^{-1}-d(A+\bar{A}+\bar{B}m-\Gamma_1)\zeta,\quad(|\zeta|<1)\end{aligned} \tag{10a}$$

$$\begin{aligned}\phi_2(\zeta)&=\frac{d}{\kappa_2}\left[Am\left(\frac{\kappa_1\mu_2}{\mu_1}-\lambda\right)-\bar{A}m\left(\frac{\mu_2}{\mu_1}+\lambda\right)-\bar{B}\left(\frac{\mu_2}{\mu_1}+\lambda\right)-\Gamma_2\right]\zeta^{-1}+d\Gamma_1\zeta,\quad(|\zeta|>1)\\ \phi_2(\zeta)&=d\Gamma_2\zeta^{-1}+d\left[A\left(\frac{\kappa_1\mu_2}{\mu_1}+\lambda\right)-\bar{A}\left(\frac{\mu_2}{\mu_1}-\lambda\right)-\bar{B}m\left(\frac{\mu_2}{\mu_1}-\lambda\right)-\kappa_2\Gamma_1\right]\zeta,\quad(|\zeta|<1)\end{aligned} \tag{10b}$$

where Γ_1 and Γ_2 are related to the remote loads σ_{xx}^∞, σ_{xy}^∞, and σ_{yy}^∞ through the following:

$$\Gamma_1=\frac{\sigma_{xx}^\infty+\sigma_{yy}^\infty}{4},\quad\Gamma_2=\frac{(2-m)\sigma_{xx}^\infty-(2+m)\sigma_{yy}^\infty}{4}+i\sigma_{xy}^\infty \tag{11}$$

In view of the fact that the two expressions of $\phi_2(\zeta)$ must be compatible, we can then uniquely determine the two unknowns A and B as

$$A=\frac{(1+\kappa_2)\left[(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})\left(\kappa_2-m^2+(1+m^2)\frac{\mu_2}{\mu_1}+(1-m^2)\lambda\right)+2m(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty})\left(1-\frac{\mu_2}{\mu_1}+\lambda\right)\right]}{4\left(2+\frac{(\kappa_1-1)\mu_2}{\mu_1}+2\lambda\right)\left(\kappa_2+\frac{\mu_2}{\mu_1}+\lambda\right)-4m^2\left(2\kappa_2+\frac{(1-\kappa_1)\mu_2}{\mu_1}+2\lambda\right)\left(1-\frac{\mu_2}{\mu_1}+\lambda\right)}$$

$$+i\frac{m(1+\kappa_2)\left(1-\frac{\mu_2}{\mu_1}+\lambda\right)\sigma_{xy}^{\infty}}{(1+\kappa_1)\frac{\mu_2}{\mu_1}\left[\kappa_2+m^2+(1-m^2)\frac{\mu_2}{\mu_1}+\lambda(1+m^2)\right]}$$

$$B=\frac{(1+\kappa_2)\left[m(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})\left((\kappa_2-1)+(1-\kappa_1)\frac{\mu_2}{\mu_1}\right)+(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty})\left(2+(\kappa_1-1)\frac{\mu_2}{\mu_1}+2\lambda\right)\right]}{2m^2\left(2\kappa_2+\frac{(1-\kappa_1)\mu_2}{\mu_1}+2\lambda\right)\left(1-\frac{\mu_2}{\mu_1}+\lambda\right)-2\left(2+\frac{(\kappa_1-1)\mu_2}{\mu_1}+2\lambda\right)\left(\kappa_2+\frac{\mu_2}{\mu_1}+\lambda\right)}$$

$$+i\frac{(1+\kappa_2)\sigma_{xy}^{\infty}}{\kappa_2+m^2+(1-m^2)\frac{\mu_2}{\mu_1}+\lambda(1+m^2)} \tag{12}$$

Now that the uniform stresses within the elliptical inhomogeneity can be explicitly given by

$$\sigma_{xx}=\frac{(1+\kappa_2)\left[\begin{array}{l}(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})\left(\kappa_2-m^2+m(\kappa_2-1)+(1-m^2)\lambda+(1+m^2+m-m\kappa_1)\frac{\mu_2}{\mu_1}\right)\\+(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty})\left(2(1+m)(1+\lambda)+(\kappa_1-1-2m)\frac{\mu_2}{\mu_1}\right)\end{array}\right]}{2\left(2+\frac{(\kappa_1-1)\mu_2}{\mu_1}+2\lambda\right)\left(\kappa_2+\frac{\mu_2}{\mu_1}+\lambda\right)-2m^2\left(2\kappa_2+\frac{(1-\kappa_1)\mu_2}{\mu_1}+2\lambda\right)\left(1-\frac{\mu_2}{\mu_1}+\lambda\right)} \tag{13a}$$

$$\sigma_{yy}=\frac{(1+\kappa_2)\left[\begin{array}{l}(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})\left(\kappa_2-m^2-m(\kappa_2-1)+(1-m^2)\lambda+(1+m^2-m+m\kappa_1)\frac{\mu_2}{\mu_1}\right)\\+(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty})\left(2(m-1)(1+\lambda)-(\kappa_1-1+2m)\frac{\mu_2}{\mu_1}\right)\end{array}\right]}{2\left(2+\frac{(\kappa_1-1)\mu_2}{\mu_1}+2\lambda\right)\left(\kappa_2+\frac{\mu_2}{\mu_1}+\lambda\right)-2m^2\left(2\kappa_2+\frac{(1-\kappa_1)\mu_2}{\mu_1}+2\lambda\right)\left(1-\frac{\mu_2}{\mu_1}+\lambda\right)} \tag{13b}$$

$$\sigma_{xy}=\frac{(1+\kappa_2)\sigma_{xy}^{\infty}}{\kappa_2+m^2+(1-m^2)\frac{\mu_2}{\mu_1}+\lambda(1+m^2)} \tag{13c}$$

and the rigid-body rotation ε of the elliptical inhomogeneity is given by

$$\varepsilon=\frac{1}{2}\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)=\frac{m\kappa_1(1+\kappa_2)\left(1-\frac{\mu_2}{\mu_1}+\lambda\right)\sigma_{xy}^{\infty}}{2\mu_2(1+\kappa_1)\left[\kappa_2+m^2+(1-m^2)\frac{\mu_2}{\mu_1}+\lambda(1+m^2)\right]} \tag{14}$$

which is a monotonic function of λ. Consequently, if $\sigma_{xy}^{\infty}>0$, we then obtain the following inequality for ε:

$$\frac{m\kappa_1(1+\kappa_2)\left(1-\frac{\mu_2}{\mu_1}\right)\sigma_{xy}^{\infty}}{2\mu_2(1+\kappa_1)\left[\kappa_2+m^2+(1-m^2)\frac{\mu_2}{\mu_1}\right]}\le\varepsilon\le\frac{m\kappa_1(1+\kappa_2)\sigma_{xy}^{\infty}}{2\mu_2(1+\kappa_1)(1+m^2)} \tag{15}$$

Furthermore, $\psi_2(\zeta)$ defined within the unbounded matrix can be determined from Eqs. (7) and (10*a*) as follows:

$$\psi_2(\zeta)=-d\bar{\Gamma}_2\zeta+\frac{d(A+\bar{A}+Bm-\Gamma_1)}{\zeta}-\frac{d(m\zeta^2+1)[\Gamma_1\zeta^2-(Am+\bar{A}m+\bar{B}+\Gamma_2)]}{\zeta(\zeta^2-m)},\quad |\zeta|>1 \tag{16}$$

Now that the two complex potentials $\phi_2(\zeta)$ and $\psi_2(\zeta)$, ($|\zeta|$

>1) defined in the unbounded matrix have been completely determined, it is not difficult to derive the stress and displacement fields in the unbounded matrix by using Eqs. (1) and (3).

4 Discussions

In this section, we will discuss several special cases to verify and to illustrate the obtained solution.

4.1 Circular Inhomogeneity With an Imperfect Interface. For a circular inclusion ($a=b$), we have $m=0$, then it follows from Eqs. (4) and (6) that $\beta(x,y)=2\mu_2/\lambda a$, which means that the imperfection must be circumferentially homogeneous along the circular interface so as to get uniform internal stress field. Furthermore, it follows from Eqs. (13*a*) and (13*b*) that

$$\sigma_{xx}=\frac{(1+\kappa_2)\left[(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})\left(\kappa_2+\lambda+\frac{\mu_2}{\mu_1}\right)+(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty})\left(2(1+\lambda)+(\kappa_1-1)\frac{\mu_2}{\mu_1}\right)\right]}{2\left(2+\frac{(\kappa_1-1)\mu_2}{\mu_1}+2\lambda\right)\left(\kappa_2+\frac{\mu_2}{\mu_1}+\lambda\right)} \tag{17a}$$

$$\sigma_{yy}=\frac{(1+\kappa_2)\left[(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})\left(\kappa_2+\lambda+\frac{\mu_2}{\mu_1}\right)+(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})\left(2(1+\lambda)+(\kappa_1-1)\frac{\mu_2}{\mu_1}\right)\right]}{2\left(2+\frac{(\kappa_1-1)\mu_2}{\mu_1}+2\lambda\right)\left(\kappa_2+\frac{\mu_2}{\mu_1}+\lambda\right)} \tag{17b}$$

$$\sigma_{xy}=\frac{(1+\kappa_2)\sigma_{xy}^{\infty}}{\kappa_2+\frac{\mu_2}{\mu_1}+\lambda} \tag{17c}$$

which are just the results obtained in Refs. [8–10]. In addition, it is found from Eq. (14) that the rigid-body rotation ε of the imperfectly bonded circular inhomogeneity is always zero. It is added that the stress field within a circular inhomogeneity is nonuniform for a homogeneously imperfect interface on which the degree of imperfection in the normal direction and that in the tangential direction are not equal [2,10].

4.2 Elliptical Inhomogeneity With a Perfect Interface. When $\lambda=0$ for a perfect interface, we have carefully checked that Eqs. (13*a*) and (13*b*) for this case will just reduce to that derived by Hardiman [11] and Sendeckyj [12] for an elliptical inhomogeneity with a perfect interface.

4.3 Materials Comprising the Matrix and the Inhomogeneity are Identical ($\mu_1=\mu_2=\mu$, $\kappa_1=\kappa_2=\kappa$). In this case, it follows from Eq. (13*a*) that the stress component σ_{xx} is uniformly distributed within the inhomogeneity as

$$\sigma_{xx}=\frac{(1+\kappa)[\sigma_{xx}^{\infty}(2+2\kappa+3\lambda+2\lambda m-\lambda m^2)-\lambda\sigma_{yy}^{\infty}(1+m)^2]}{2(1+\kappa+2\lambda)(1+\kappa+\lambda-\lambda m^2)} \tag{18}$$

Some interesting phenomena can be observed from the above internal stress expression. For example, if $\sigma_{xx}^{\infty}=0$ while $\sigma_{yy}^{\infty}>0$ (i.e., the matrix is subjected to uniaxial tension along the y-direction), then the internal stress σ_{xx} is given by

$$\sigma_{xx}=-\frac{\lambda\sigma_{yy}^{\infty}(1+\kappa)(1+m)^2}{2(1+\kappa+2\lambda)(1+\kappa+\lambda-\lambda m^2)}\leq 0 \tag{19}$$

which means that σ_{xx} is compressive. It is found from the above expression that $\sigma_{xx}=0$ when the interface is perfect ($\lambda=0$) or when the interface is completely debonded ($\lambda=\infty$). Particularly, when the dimensionless imperfect parameter λ attains the following value:

$$\lambda=\frac{1+\kappa}{\sqrt{2(1-m^2)}} \tag{20}$$

the internal compressive stress component σ_{xx} will get its maximum magnitude of

$$|\sigma_{xx}|_{\max}=\frac{(1+m)^2\sigma_{yy}^{\infty}}{2[\sqrt{2}+\sqrt{1-m^2}]^2} \tag{21}$$

We have checked that the above phenomenon is also valid for the more general case in which the elastic properties of the elliptical inhomogeneity and those of the surrounding matrix are distinct, i.e., $\mu_1\neq\mu_2$, $\kappa_1\neq\kappa_2$.

4.4 Condition for Internal Uniform Hydrostatic Stresses. Here, the uniform hydrostatic stress state within the elliptical inhomogeneity is especially preferred because it achieves both uniform normal stress and vanishing tangential stress along the entire interface [7]. It is observed from Eqs. (1) and (12) that the uniform stresses within the elliptical inhomogeneity is also hydrostatic when $B=0$, i.e.,

$$\frac{\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}}{\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}}=\frac{m[\mu_1(\kappa_2-1)-\mu_2(\kappa_1-1)]}{2\mu_1(1+\lambda)+\mu_2(\kappa_1-1)},\quad \sigma_{xy}^{\infty}=0 \tag{22}$$

Hence, it follows from Eqs. (12) and (22) that

$$\sigma_{rr}=\sigma_{\theta\theta}=\frac{\mu_1(1+\kappa_2)(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})}{2[2\mu_1(1+\lambda)+\mu_2(\kappa_1-1)]} \tag{23}$$

within the elliptical inhomogeneity.

When $\lambda=0$, Eq. (22) reduces to the condition of uniform hydrostatic stress state within a perfectly bonded elliptical inhomogeneity (Ref. [7], Eq. 4.1). It is also observed from Eq. (22) that the two remote principal stresses must have the same sign to ensure the existence of uniform hydrostatic stresses within the imperfectly bonded elliptical inhomogeneity.

5 Conclusions

In this research, we find that uniform stress field can still be retained for an elliptical inhomogeneity with an inhomogeneously

imperfect interface under remote uniform in-plane stresses. The conditions for the uniform stress state within the elliptical inhomogeneity are as follows: (i) the same degree of imperfection is realized in both the normal and tangential directions along the interface; and (ii) the imperfect interface parameter $\beta(x,y)$ is inversely proportional to $|\omega'(\zeta)|$, $(\zeta=e^{i\theta})$ in which $z=\omega(\zeta)$ maps the elliptical interface in the z-plane onto a unit circle in the ζ-plane. Finally, it shall be mentioned that the conclusion for the uniform stress field also holds when uniform eigenstrains are imposed on the elliptical inhomogeneity.

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