# Mathematical modeling and numerical computation for the vibration of a magnetostrictive actuator 

Xinchun Shang ${ }^{1,4}$, Ernie Pan ${ }^{1,2}$ and Liping Qin ${ }^{3}$<br>${ }^{1}$ Department of Mathematics and Mechanics, University of Science and Technology Beijing, Beijing 100083, People's Republic of China<br>${ }^{2}$ Department of Civil Engineering, University of Akron, Akron, OH 44325, USA<br>${ }^{3}$ Department of Civil Engineering, University of Science and Technology Beijing,<br>Beijing 100083, People's Republic of China<br>E-mail: shangxc@public3.bta.net.cn

Received 17 August 2007, in final form 25 March 2008
Published 10 July 2008
Online at stacks.iop.org/SMS/17/045026


#### Abstract

Mathematical modeling to analyze the vibration problem of a Terfenol-D actuator is presented, using Hamilton's variational principle. The total kinetic, strain and magnetic energy stored in the rod is expressed as a function of the displacement. The material constitutive relation of the magnetostrictive rod is assumed to be cubic nonlinear. Both the governing equation and the boundary condition derived are of time variable coefficient. A numerical approach which combines the finite difference method with the transfer matrix method is proposed for solving the excited vibration problem of a magnetostrictive rod. By discretizing the displacement in space and making the finite difference formulation, the nodal displacement is obtained in terms of a system of linear second-order ordinary differential equations (ODEs), which can be subsequently transformed into a system of linear first-order ODEs in the time domain. The time domain within a period is then discretized with the numerical solution being expressed by the transfer matrix method. As a numerical example, the vibration of a Terfenol-D rod excited by a harmonic current is analyzed. The numerical results show that the induced displacement in the rod is periodic, with its frequency being roughly twice that of the exciting current. Such a double frequency effect has been observed in experiments. The stress behavior and the peak displacement within the rod are also numerically analyzed, with an emphasis on the effect of the involved magnetostrictive and magnetoelastic parameters. The good agreement between the numerical results and the experimental data available in the literature verifies the validity of the present modeling and method.


(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Magnetostriction, also called the Joule effect, is the shape change of a material under the influence of an external magnetic field. As early as the 1930s to 1950s, magnetostrictive transducers made from nickel and its alloys were routinely used in sonars for military and civil applications. However, the low magnetostriction effect in nickel (typically of the order of 50 ppm ) limited the scope

[^0]of new applications [1, 2]. In the last 30 years, the discovery of Terfenol-D has led to the development of magnetostrictive actuators. Terfenol-D is a kind of giant magnetostrictive material (of the order of 1500-2000 ppm) containing a rare earth compound [3]. Terfenol-D has many merits as compared to the piezoelectric materials. These include large strain, high power density, low working voltage and less performance degradation at room temperature. Therefore, Terfenol-D has become a promising candidate of materials for sensors and actuators. Recently, the electromagnetic-


Figure 1. Sketch of a typical Terfenol-D actuator.
mechanical coupling problems in magnetostrictive materials and structures have attracted considerable attention of engineering and science researchers. For instance, nonlinear constitutive relations for magnetostrictive materials have been studied in [4-9]. Dynamic characteristics of Terfenol-D actuators have been observed in experiments [10-12, 31]. Theoretical analysis models of Terfenol-D rods in mechanical devices have been developed [12-16]. However, in the aspect of numerical computation, only a few works have so far appeared. For example, finite element methods have been used to simulate numerically the performance of Terfenol-D actuators [7, 17-19]. The vibration of a beam with a magnetostrictive layer was analyzed by the finite element method [20]. Vibrations in plates made of piezoelectric and magnetostrictive materials were also investigated recently. These studies have shown the important role of the magnetostrictive layer on the natural frequency and mode shape of the plate [21, 22]. Composites made of magnetostrictive and piezoelectric materials possess certain merits over the corresponding single phase materials. Among them, the magnetoelectric coupling has been particularly attractive to the design of various novel devices, as it is also associated with the new materials called multiferroics [23, 24].

The strains generated by magnetostrictive actuators in response to applied input current can be modeled as the vibration of a magnetostrictive rod in an actuator. The aim of this paper is to present both a mathematical modeling and a numerical approach to solve the vibration problem of a Terfenol-D rod, which is employed to model an actuator. On the basis of the nonlinear constitutive relation of magnetostrictive materials and Ampère's law for the magnetic field, a mathematical modeling of the vibration problem is constructed by using Hamilton's variational principle. The governing equation and boundary condition are of time variable coefficient. To simulate the vibration behavior of the actuator, numerical solutions are obtained by employing the finite difference method in the space domain and the transfer matrix method in the time domain. Numerical examples are also presented, which show clearly the so-called double frequency effect. In other words, the induced displacement is periodic, with its frequency being roughly twice that of the periodic exciting current; such an effect has been observed in experiments [11]. The presented numerical results also show the effect of the magnetostrictive and magnetoelastic parameters on the response of the Terfenol-D rod.

This paper is organized as follows. In section 2, the basic formulation, including the Hamilton variational principle and


Figure 2. Simplified mechanical model for the vibration of the Terfenol-D rod.
the boundary/initial value problems for the nonlinear TerfenolD rod, is presented. In section 3, the detailed numerical approach is discussed. These contain the finite difference discretization in the space domain, the transfer matrix method in the time domain, the periodicity of the solution, and the double frequency effect. Numerical results and comparison with experimental data from [31] are given in section 4, and conclusions are drawn in section 5 .

## 2. Formulation

### 2.1. Problem description

Let us consider the dynamic problem of a typical Terfenol-D actuator. The sectional view of the actuator is shown in figure 1 [1,25]. The primary components consist of a cylindrical magnetostrictive Terfenol-D rod in the middle, a surrounding solenoid coil, and a prestress spring washer near one end of the structure. Since Terfenol-D is much more brittle in tension than in compression, a prestressed spring is usually added in the design to provide an initial static compressive stress (prestress) $\sigma_{0}$ so that tensile stresses in the Terfenol-D rod can be avoided. The mechanical behavior of the actuator under consideration can be simplified as the vibration of the TerfenolD rod as shown in figure 2. The vibration is excited by an input alternating current running through a solenoid around the rod. The time-dependent current generates a varying magnetic field within the rod. Because of the expansion or contraction of the magnetostrictive material in response to the variation of the applied magnetic field, vibration of the Terfenol-D rod occurs.

We now denote the length of the rod as $l$, the cross-section area of the rod as $A$, the Young's modulus of the rod as $E$, and the stiffness of the prestressed spring as $K_{0}$. The space coordinate $x$-axis is aligned with the axis of the rod. The rod is fixed at $x=0$ and restricted at the end $x=l$ by the spring (figure 2). The initial static deformation of the rod and the spring are linear elastic. Using Hooke's law and the balance relation between the internal force of the rod and the restoring force of the spring, we can obtain the static displacement in the $\operatorname{rod}$ as $u_{0}(x)=\sigma_{0} x / E$ and the static compression of the spring as $\Delta=\sigma_{0} A / K_{0}$. Evidently, the initial static deformation of the rod is homogeneous and is in a state of self-equilibrium. Therefore, for the vibration analysis of the rod, we can choose the static deformation as the reference state.

### 2.2. Basic equation

For the longitudinal vibration of a Terfenol-D rod, the displacement $u(x, t)$ is a function of the time $t$ and the space
coordinate $x$ measured from the statically deformed position. The corresponding strain in the longitudinal direction is

$$
\begin{equation*}
\varepsilon=\frac{\partial u}{\partial x} . \tag{1}
\end{equation*}
$$

According to the experimental results, the behavior of magnetostrictive materials is nonlinear. For the onedimensional problem the constitutive relations can be described in general as [7]

$$
\begin{equation*}
\varepsilon=f(\sigma, H), \quad B=g(\sigma, H) \tag{2}
\end{equation*}
$$

where $\sigma$ is the longitudinal stress, $H$ the magnetic field, and $B$ the magnetic flux (also called the magnetic induction). The constitutive functions $f$ and $g$ may be obtained by measuring the magnetostriction and the magnetization versus the applied magnetic field and the external stress [4]. Recent experimental measurements showed that, for a given stress, the strain is independent of the direction (or orientation) of the applied magnetic field, and its magnitude increases with increasing magnitude of the applied magnetic field [8]. Thus, the function $f$ should satisfy the following constitutive restrictions:

$$
\begin{gather*}
f(\sigma,-H)=f(\sigma, H), \\
\frac{\partial f(\sigma, H)}{\partial H}>0 \quad(H>0) . \tag{3}
\end{gather*}
$$

A few constitutive relations for $g$ as well as for $f$ have been proposed in the literature $[6,8]$ to fit the experimental curves of the measured magnetostriction. In this paper, the standard one-dimensional nonlinear constitutive law [6, 8] is employed for the magnetostrictive material:

$$
\begin{gather*}
\varepsilon=\left(\frac{1}{E}+r H^{2}\right) \sigma+m H^{2}  \tag{4}\\
B=\left(\mu+m \sigma+r \sigma^{2}\right) H \tag{5}
\end{gather*}
$$

where $\mu$ is the magnetic permeability, $m$ the magnetostrictive modulus, and $r$ the magnetoelastic coefficient.

We remark that, in the constitutive law (4) and (5), the hysteresis effect of magnetostriction is not included as the dissipation energy is small in the complete magnetization loop for Terfenol-D. We further mention that the constitutive law (4) and (5) agrees well with the experimental curves under applied magnetic fields of low and moderate magnitude [8]. Physically, the constitutive law (4) can be regarded as the extended Hooke's law for magnetostrictive materials: the total strain is the summation of the elastic strain and the magnetostrictive strain resulting from the magnetic field. That is, $\varepsilon=\sigma / \bar{E}+$ $m H^{2}$ with the effective elastic compliance, $1 / \bar{E}=1 / E+r H^{2}$, being an even function of the magnetic field $H$. Also, the constitutive law (5) is an extension of the classical relation in the theory of magnetism. Namely, the magnetic flux $B$ is proportional to the magnetic field $H: B=\bar{\mu} H$. However, the effective magnetic permeability, $\bar{\mu}=\mu+m \sigma+r \sigma^{2}$, depends quadratically on the stress $\sigma$.

Inversion of the constitutive relations (4) and (5) gives the following equivalent relations:

$$
\begin{equation*}
\sigma=\frac{E \varepsilon-E m H^{2}}{1+E r H^{2}} \tag{6}
\end{equation*}
$$

$B=\mu H+m H \frac{E \varepsilon-E m H^{2}}{1+E r H^{2}}+r H\left(\frac{E \varepsilon-E m H^{2}}{1+E r H^{2}}\right)^{2}$.
Therefore, both the stress and magnetic flux are functions of the strain and magnetic field.

For the magnetostrictive rod shown in figure 1, the magnetic field $H$ can be assumed to be uniform along the rod. As the impedance of the coil, housing, and end caps are much smaller than that of the magnetostrictive rod, all the impedance effects of the coil, housing and end caps on the magnetic field are negligible. Thus, the magnetic field $H$ is connected to the applied current by Ampère's law [1]. In other words,

$$
\begin{equation*}
H=\frac{n}{l} i(t) \tag{8}
\end{equation*}
$$

where $n$ is the number of turns in the coil and $i(t)$ is the given input current passing through the coil.

Therefore, the problem is reduced to the solution for the elastic displacement or strain, subjected to the boundary and initial conditions described in detail in the next section.

### 2.3. Boundary and initial conditions

In order to obtain the equation of vibration and the boundary/initial conditions for the Terfenol-D rod, the variational approach based on the extended Hamilton's principle [7] is employed. The Hamilton energy functional of the Terfenol-D rod from time $t_{1}$ to $t_{2}$ is defined as

$$
\begin{equation*}
I=\int_{t_{1}}^{t_{2}}\left(E_{c}+U_{\mathrm{M}}-U_{\mathrm{E}}\right) \mathrm{d} t \tag{9}
\end{equation*}
$$

The kinetic energy in the $\operatorname{rod}$ from $x=0$ to $l$ is

$$
\begin{equation*}
E_{c}=\frac{A}{2} \int_{0}^{l} \rho\left(\frac{\partial u}{\partial t}\right)^{2} \mathrm{~d} x \tag{10}
\end{equation*}
$$

with $\rho$ being the mass density of the rod. The magnetic energy is

$$
\begin{equation*}
U_{\mathrm{M}}=\frac{A}{2} \int_{0}^{1} B H \mathrm{~d} x \tag{11}
\end{equation*}
$$

The strain energy plus the energy due to the end spring is

$$
\begin{equation*}
U_{\mathrm{E}}=\frac{A}{2} \int_{0}^{l} \sigma \varepsilon \mathrm{~d} x-\frac{1}{2} K_{0} u^{2}(l, t) . \tag{12}
\end{equation*}
$$

Substitution of the constitutive equations (4)-(6) into (9) leads to the Hamilton energy functional:

$$
\begin{align*}
I[u] & =\int_{t_{1}}^{t_{2}}\left[\frac { A } { 2 } \int _ { 0 } ^ { l } \left(\rho\left(\frac{\partial u}{\partial t}\right)^{2}-\frac{E}{\left(1+E r H^{2}(t)\right)^{2}}\right.\right. \\
& \left.\left.\times\left(\frac{\partial u}{\partial x}-m H^{2}(t)\right)^{2}+\mu H^{2}\right) \mathrm{~d} x-\frac{K_{0}}{2} u^{2}(l, t)\right] \mathrm{d} t . \tag{13}
\end{align*}
$$

Since the magnetic field $H(t)$ is a given function from equation (8), the Hamilton energy $I[u]$ is a functional with respect to the magnetoelastic displacement $u(x, t)$. This variational functional (13) has been used for developing an extended finite element formulation [18].

Using Hamilton's principle, i.e., the variational $\delta I[u]=0$, we find the governing equation for the longitudinal vibration of the Terfenol-D rod as [18]
$\frac{\partial^{2} u}{\partial t^{2}}=\frac{E}{\rho} \frac{1}{\left[1+r E H^{2}(t)\right]^{2}} \frac{\partial^{2} u}{\partial x^{2}}$
$(0 \leqslant x \leqslant l, t \geqslant 0)$
with the natural boundary condition at the spring end of the Terfenol-D $\operatorname{rod}(x=l)$ as
$\left[\frac{\partial u}{\partial x}+\frac{K_{0}}{E A}\left(1+r E H^{2}(t)\right)^{2} u\right]_{x=l}=m H^{2}(t)$
$(t \geqslant 0)$.
At the fixed end $(x=0)$ the boundary condition is

$$
\begin{equation*}
u(0, t)=0 \quad(t \geqslant 0) \tag{16}
\end{equation*}
$$

For practical applications of the actuator, a periodic varying input current is usually applied, which thus induces a periodic vibration of the Terfenol-D rod. Therefore, the periodicity condition in time can be described as follows:
$u(x, 0)=u(x, T), \quad \frac{\partial u}{\partial t}(x, 0)=\frac{\partial u}{\partial t}(x, T)$

$$
\begin{equation*}
(0 \leqslant x \leqslant l) \tag{17}
\end{equation*}
$$

where $T=2 \pi / \omega$ is the vibration period of the rod.
In summary, the vibration problem of the Terfenol-D rod is mathematically reduced to the solution to the boundary value problem with the periodicity condition in time (14)-(17). Note that the equation of vibration (14) has variable coefficient with time $t$, and as such, we propose, in the next section, a numerical method to solve the problem.

## 3. Numerical computation method

### 3.1. Finite difference discretization in the space domain

The first step in obtaining a numerical solution of the initial and boundary value problem (14)-(17) is to divide the $x$ domain $[0, l]$ into $N$ subintervals (or a mesh) with nodes: $0=x_{0}<x_{1}<\cdots<x_{N-1}<x_{N}=l$. The mesh spacing is assumed to be uniform, in other words, $x_{k+1}-x_{k}=h=$ $l / N(k=0,1, \ldots, N-1)$. We then denote the corresponding displacement at node $x_{k}$ for any time $t \in[0, \infty]$ as

$$
\begin{equation*}
u_{k}=u_{k}(t) \cong u\left(x_{k}, t\right) \quad(k=1,2, \ldots, N) . \tag{18}
\end{equation*}
$$

At each interior point $x_{k}(k=1,2, \ldots, N-1)$, we use the second-order difference approximation to the second derivative $\partial^{2} u / \partial x^{2}$ in the space domain. Thus, the wave equation (14) is approximated by
$\ddot{u}_{k}=\frac{E}{\rho h^{2}} c_{1}(t)\left(u_{k-1}-2 u_{k}+u_{k+1}\right) \quad(k=1,2, \ldots, N-1)$
where $\ddot{u}_{k}=\mathrm{d}^{2} u_{k} / \mathrm{d} t^{2}$ and $u_{0}(t)=0$ from the boundary condition (16). Also in equation (19), $c_{1}(t)$ is a given function defined as

$$
\begin{equation*}
c_{1}(t)=\left[1+r E H^{2}(t)\right]^{-2} . \tag{20}
\end{equation*}
$$

At the boundary point $x=x_{N}$, we apply the first-order difference approximation to equation (14) twice, which yields

$$
\begin{aligned}
\ddot{u}_{N} & =\frac{E}{\rho h} c_{1}(t)\left[\left.\frac{\partial u}{\partial x}\right|_{x=x_{N}}-\left.\frac{\partial u}{\partial x}\right|_{x=x_{N-1}}\right] \\
& =\frac{E}{\rho h} c_{1}(t)\left[\left.\frac{\partial u}{\partial x}\right|_{x=x_{N}}-\frac{1}{h}\left(u_{N}-u_{N-1}\right)\right] .
\end{aligned}
$$

Moreover, inserting the boundary condition (5) into the above equation, we obtain

$$
\begin{equation*}
\ddot{u}_{N}=\frac{E}{\rho h^{2}}\left[c_{1}(t)\left(u_{N-1}-\left(1+\frac{c_{0}}{c_{1}(t)}\right) u_{N}\right)+h c_{1}(t)\right] \tag{21}
\end{equation*}
$$

where the constant $c_{0}$ and the function $c_{2}(t)$ are given by

$$
\begin{equation*}
c_{0}=K_{0} h / E A, \quad c_{2}(t)=m H^{2}(t) c_{1}(t) \tag{22}
\end{equation*}
$$

In summary, equations (19) and (21) can be expressed in a compact matrix form as

$$
\begin{equation*}
\ddot{\mathbf{u}}+\mathbf{B}(t) \mathbf{u}=\mathbf{q}(t) \quad(t \geqslant 0) \tag{23}
\end{equation*}
$$

This is the vibration equation for the node displacement $\mathbf{u}=\left[u_{1}(t), u_{2}(t), \ldots, u_{N}(t)\right]^{\mathrm{T}}$, with $\ddot{\mathbf{u}}=\mathrm{d}^{2} \mathbf{u} / \mathrm{d} t^{2}$ being the node acceleration vectors. The coefficient matrix and the external excitation vector in (23) are, respectively, given by
$\mathbf{B}(t)=\frac{E}{\rho h^{2}} c_{1}(t)\left[\begin{array}{ccccc}2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1+\frac{c_{0}}{c_{1}(t)}\end{array}\right]_{N \times N}$

$$
\begin{equation*}
\text { and } \quad \mathbf{q}(t)=\frac{E}{\rho h} c_{2}(t) \underbrace{[0,0, \ldots, 0,1]}_{N}]^{\mathrm{T}} \text {. } \tag{24}
\end{equation*}
$$

The periodicity condition (17) can be written as

$$
\begin{equation*}
\mathbf{u}(0)=\mathbf{u}(T), \quad \dot{\mathbf{u}}(0)=\dot{\mathbf{u}}(T) \tag{25}
\end{equation*}
$$

### 3.2. General solutions in the time domain

In order to solve the periodic problem described by equations (23) and (25), we introduce the state vector

$$
\begin{equation*}
\mathbf{y}(t)=\left[y_{1}(t), y_{2}(t), \ldots, y_{2 N}(t)\right]^{\mathrm{T}} \equiv[\mathbf{u}(t), \dot{\mathbf{u}}(t)]^{\mathrm{T}} \tag{26}
\end{equation*}
$$

where $\dot{\mathbf{u}}(t)=\mathrm{d} \mathbf{u}(t) / \mathrm{d} t$ is the node velocity vector.
The problem described by equations (23) and (25) can now be equivalently transformed into a periodic problem for a first-order system of ordinary differential equations (ODEs) as follows:

$$
\begin{gather*}
\frac{\mathrm{d} \mathbf{y}}{\mathrm{~d} t}=\mathbf{A}(t) \mathbf{y}+\mathbf{f}(t) \quad(t>0)  \tag{27}\\
\mathbf{y}(0)=\mathbf{y}(T) \tag{28}
\end{gather*}
$$

where the coefficient matrix and the inhomogeneous term vector are, respectively, given by

$$
\begin{gather*}
\mathbf{A}(t)=\left[\begin{array}{cc}
\mathbf{0} & \mathbf{I}_{N} \\
-\mathbf{B}(t) & \mathbf{0}
\end{array}\right] \quad \text { and } \\
\mathbf{f}(t)=[\mathbf{0}, \mathbf{q}(t)]^{\mathrm{T}}=\frac{E}{\rho h} c_{2}(t) \underbrace{[0, \ldots, 0,1}_{2 N}]^{\mathrm{T}} \tag{29}
\end{gather*}
$$

in which $\mathbf{I}_{N}$ is an $N \times N$ identity matrix.
Following the basic theory of first-order ODEs [26], the solution to equation (27) can be expressed by the fundamental solution matrix $\mathbf{X}(t)$ and the initial value $\mathbf{y}(0)$ as

$$
\begin{equation*}
\mathbf{y}(t)=\mathbf{X}(t)\left[\mathbf{y}(0)+\int_{0}^{t} \mathbf{X}^{-1}(\tau) \mathbf{f}(\tau) \mathrm{d} \tau\right] \quad(t>0) \tag{30}
\end{equation*}
$$

where the matrix $\mathbf{X}(t)$ has a size of $2 N \times 2 N$, and is the socalled fundamental matrix of equation (27). Namely, it is the solution to the following initial value problem:

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{X}}{\mathrm{~d} t}=\mathbf{A}(t) \mathbf{X}(t) \quad(t>0), \quad \mathbf{X}(0)=\mathbf{I}_{2 N} \tag{31}
\end{equation*}
$$

Letting $t=T$ in equation (30) and making use of the periodicity condition (26), we find the initial value as

$$
\begin{equation*}
\mathbf{y}_{0}=\left[\mathbf{I}_{2 N}-\mathbf{X}(T)\right]^{-1} \int_{0}^{\mathrm{T}} \mathbf{X}^{-1}(\tau) \mathbf{f}(\tau) \mathrm{d} \tau \quad(t>0) \tag{32}
\end{equation*}
$$

Thus, the periodic problem described by equations (27) and (22) is equivalent to the initial value problem given as

$$
\begin{gather*}
\frac{\mathrm{d} \mathbf{y}}{\mathrm{~d} t}=\mathbf{A}(t) \mathbf{y}+\mathbf{f}(t) \quad(t>0)  \tag{33}\\
\mathbf{y}(0)=\mathbf{y}_{0} \tag{34}
\end{gather*}
$$

where the initial value $\mathbf{y}_{0}$ is given by equation (32).
Clearly, once the fundamental solution matrix $\mathbf{X}(t)$ is solved from the initial value problem (31), the solution to the periodic problem (27) and (28) can be obtained from equations (30) and (32).

### 3.3. Periodicity of the solution

3.3.1. Existence condition of the periodic solution. The periodic problem (27) and (28), or equivalently, the initial value problem (32)-(34), needs to be solved in the infinite-time domain $[0, \infty)$. However, if the solution is periodic, then it is enough to find the solution in the finite-time periodic domain $[0, T]$ with the period $T$. To solve the problem in the periodic domain only, we need to first discuss the periodicity of solution for the initial value problem (33) and (34).

We assume that the input current is alternating and periodic:

$$
\begin{equation*}
i(t)=i(t+T) \tag{35}
\end{equation*}
$$

where the period of input current is $T=2 \pi / \omega$, with $\omega$ being the circular frequency of the input current. Thus, from equation (8), we observed that the magnetic field $H(t)$ is also $T$-periodic (i.e., periodic with period $T$ ). Consequently, both
the coefficient matrix $\mathbf{A}(t)$ and the inhomogeneous term vector $f(t)$ in equation (33) are also $T$-periodic. Namely,

$$
\begin{equation*}
\boldsymbol{A}(t)=\boldsymbol{A}(t+T), \quad \boldsymbol{f}(t)=\boldsymbol{f}(t+T) \tag{36}
\end{equation*}
$$

According to the general theory of a linear ODE system with periodic coefficients [27], we conclude that equation (27) has a unique $T$-periodic solution if the frequency $\omega$ of the excitation current satisfies

$$
\begin{equation*}
F(\omega)=\operatorname{det}\left[\mathbf{I}_{2 N}-\mathbf{X}(2 \pi / \omega)\right] \neq 0 \tag{37}
\end{equation*}
$$

In fact, if condition (37) holds then the initial value $\mathbf{y}_{0}$ exists (as from (32)). Moreover, based on the existence and uniqueness of the solution for the initial value problem of a first-order ODE system, equation (27) with the periodicity condition (28) has a unique $T$-periodic solution:

$$
\begin{equation*}
\mathbf{y}(t)=\mathbf{y}(t+T) \tag{38}
\end{equation*}
$$

In the following two subsections, instead of discussing the periodicity solution to equation (27), we discuss its equivalent initial value problem (33) and (34) for the special case with magnetoelastic coefficient $r=0$ and for the general case where $r \neq 0$. The latter case is analyzed by employing the perturbation approach.
3.3.2. Periodicity of the solution for the case $r=0$. When the magnetoelastic coefficient $r=0$, we have $c_{1}(t) \equiv 1$ from equation (20). Thus, we observe from equations (24) and (29) that the coefficient matrix in equation (27) is reduced to a constant matrix:

$$
\mathbf{A}(t)=\mathbf{A}_{0}=\left[\begin{array}{cc}
\mathbf{0} & \mathbf{I}_{N}  \tag{39a}\\
-\mathbf{B}_{0} & \mathbf{0}
\end{array}\right]
$$

with

$$
\mathbf{B}(t)=\mathbf{B}_{0}=\frac{E}{\rho h^{2}}\left[\begin{array}{ccccc}
2 & -1 & & &  \tag{39b}\\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -1 & 1+c_{0}
\end{array}\right]_{N \times N}
$$

and the inhomogeneous vector becomes

$$
\begin{equation*}
\mathbf{f}(t)=\mathbf{f}_{0}(t)=\frac{E m}{\rho h}\left(\frac{n}{l} i(t)\right)^{2} \underbrace{[0, \ldots, 0,1]^{\mathrm{T}}}_{N} . \tag{40}
\end{equation*}
$$

Moreover, the fundamental solution matrix to equation (27) can be written as a matrix exponential function:

$$
\begin{equation*}
\mathbf{X}(t)=\exp \left(t \mathbf{A}_{0}\right) \tag{41}
\end{equation*}
$$

Substituting equations (40) and (41) into (30) and (32), we find that the solution to the initial value problem (33) and (34) can be expressed as

$$
\begin{align*}
& \mathbf{y}(t)=\mathbf{z}_{0}(t)=\exp \left(t \mathbf{A}_{0}\right)\left[\mathbf{y}_{0}+\int_{0}^{t} \exp \left[-s \mathbf{A}_{0}\right] \mathbf{f}_{0}(s) \mathrm{d} s\right] \\
& \quad(t>0) \tag{42}
\end{align*}
$$

where the initial value
$\begin{aligned} \mathbf{y}_{0}= & {\left[\mathbf{I}_{2 N}-\exp \left(T \mathbf{A}_{0}\right)\right]^{-1} \int_{0}^{\mathrm{T}} \exp \left(-s \mathbf{A}_{0}\right) \mathbf{f}_{0}(s) \mathrm{d} s } \\ & (t>0) .\end{aligned}$
Furthermore, the existence condition for a periodic solution, i.e., equation (37), is equivalent to the following condition (see appendix A in detail):

$$
\begin{equation*}
\Delta(\omega)=\operatorname{det}\left[\mathbf{B}_{0}-(\alpha \omega)^{2} \mathbf{I}_{N}\right] \neq 0 \quad(\alpha=1,2, \ldots) \tag{44}
\end{equation*}
$$

Therefore, we conclude that the solution $z_{0}(t)$ given by equation (42) and (43) is $T$-periodic if the condition (44) holds.
3.3.3. Periodicity of the solution for the case $r \neq 0$. To discuss the periodicity of the solution for the case $r \neq 0$, we first expand the given function $c_{1}(t)$ in equation (20) into a power series of $r$ :

$$
\begin{gather*}
c_{1}(t)=1+\sum_{s=1}^{\infty} a_{k}(t) r^{s} \\
a_{s}(t)=(-1)^{s}(s+1)\left(E \frac{n^{2}}{l^{2}} i^{2}(t)\right)^{s} \quad(s=1,2, \ldots) . \tag{45}
\end{gather*}
$$

Thus, the coefficient matrix $\boldsymbol{A}(t)$ and the inhomogeneous term $\boldsymbol{f}(t)$ in equation (33) can be also represented in a power series of $r$ :

$$
\begin{gather*}
\mathbf{A}(t)=\mathbf{A}_{0}+\overline{\mathbf{A}} \sum_{s=1}^{\infty} a_{s}(t) r^{s}, \\
\mathbf{f}(t)=\mathbf{f}_{0}(t)\left(1+\sum_{s=1}^{\infty} a_{s}(t) r^{s}\right) \tag{46}
\end{gather*}
$$

where

$$
\overline{\mathbf{A}}=\left[\begin{array}{cc}
\mathbf{0} & \mathbf{o}  \tag{47}\\
\overline{\mathbf{B}} & \mathbf{o}
\end{array}\right], \quad \overline{\mathbf{B}}=\mathbf{B}_{0}-\frac{E}{\rho h^{2}} \operatorname{diag}(\underbrace{0, \ldots, 0, c_{0}}_{N}) .
$$

The radius of convergence for the power series (45) can be determined by d'Alembert's ratio test formula: $R=$ $\lim _{s \rightarrow \infty}\left|a_{s}(t) / a_{s+1}(t)\right|=l^{2} /\left(E n^{2} i^{2}(t)\right)$. This means that the power series of $\boldsymbol{A}(t)$ and $\boldsymbol{f}(t)$ in equation (46) are convergent when the parameter $r$ satisfies

$$
\begin{equation*}
r<\frac{l^{2}}{E n^{2} i_{\max }^{2}}, \quad i_{\max }=\max _{0 \leqslant t \leqslant 2 \pi / \omega}|i(t)| . \tag{48}
\end{equation*}
$$

Moreover, the solution of the initial value problem (33) and (34) can be expanded by a convergent power series:

$$
\begin{equation*}
\mathbf{y}(t)=\mathbf{z}_{0}(t)+\sum_{s=1}^{\infty} \mathbf{z}_{s}(t) r^{s} \tag{49}
\end{equation*}
$$

where $\mathbf{z}_{0}(t)$ is given by equation (42), with the other expansion coefficients $\mathbf{z}_{s}(t)(s=1,2, \ldots)$ being determined below.

Substituting the expanding power series (45) and (46) into equation (27) and the periodicity condition (28), and comparing the coefficients of the like powers of $r$ on both sides
of equations (27) and (28), we obtain the perturbation problems of the original periodic solution as follows:

$$
\begin{align*}
& r^{s}: \frac{\mathrm{d} \mathbf{z}_{s}(t)}{\mathrm{d} t}=\mathbf{A}_{0} \mathbf{z}_{s}(t)+\mathbf{g}_{s}(t), \quad \mathbf{z}_{s}(0)=\mathbf{z}_{s}(T) \\
& \quad(s=1,2, \ldots) \tag{50}
\end{align*}
$$

where the inhomogeneous vector is given by

$$
\begin{equation*}
\mathbf{g}_{s}(t)=\mathbf{f}_{0}(t) a_{s}(t)+\sum_{i=0}^{s-1} a_{s-i}(t) \overline{\mathbf{A}} \mathbf{z}_{i}(t) \tag{51}
\end{equation*}
$$

It is observed from equations (35), (40), and (45) that both $a_{s}(t)(s=1,2, \ldots)$ and $f_{0}(t)$ are $T$-periodic. If condition (44) holds, then the solution $z_{0}(t)$ is $T$-periodic as we have shown in section 3.3.3. Thus the inhomogeneous vector $\mathbf{g}_{1}(t)$ is also $T$-periodic. Therefore, the solution $z_{1}(t)$ is $T$-periodic due to the same reason as the solution $z_{0}(t)$ is. Analogously, the solutions $\mathbf{z}_{s}(t)(s=2,3, \ldots)$ are all $T$-periodic as well.

In summary, we conclude that when the input current is $T$-periodic, the solution $\mathbf{y}(t)$ for the initial value problem (33) and (34) is also $T$-periodic if conditions (44) and (48) hold.
3.3.4. Double frequency effect. We observe from the constitutive law (4) that the strain for the magnetostrictive material is the same for both positively and negatively applied magnetic field of same magnitude. This implies that the vibration frequency, in physics, will be twice that of the applied field. This is the so-called double frequency effect [8, 11]. The double frequency effect has been observed for the TerfenolD actuator. Experimental results reveal that the frequency of the magnetostrictive output is doubled for sinusoidal input current and in the absence of a bias magnet field (no permanent magnets) [11]. To demonstrate theoretically that the responses of the displacement and the velocity at every spacial node have such a feature, we assume that the input current is sinusoidal alternating with the period $T=2 \pi / \omega$. That is,

$$
\begin{equation*}
i(t)=i_{\max } \sin \omega t \tag{52}
\end{equation*}
$$

For this case, equations (40) and (45) are reduced to

$$
\begin{gathered}
a_{s}(t)=(-1)^{s}(s+1)(\phi(t))^{s} \quad(s=1,2, \ldots), \\
\mathbf{f}_{0}(t)=\frac{m}{\rho h} \phi(t) \underbrace{[0, \ldots, 0,1]^{\mathrm{T}}}_{N}
\end{gathered}
$$

where $\phi(t)=\frac{E}{2}\left(\frac{n I_{m}}{l}\right)^{2}(1-\cos 2 \omega t)$. Obviously, $a_{s}(t)$ ( $s=1,2, \ldots$ ) and $f_{0}(t)$ are $T / 2$-periodic. Analogously to the discussion in sections 3.3.2 and 3.3.3, we conclude that when the input current is $T$-periodic sinusoidal alternating, the solution $\mathbf{y}(t)$ for the initial value problem (33) and (34) is $T / 2$ periodic if the conditions (44) and (48) hold. This means that the responses of the displacement and the velocity at every spacial node have a frequency which is twice that of the input exciting current. Hence, the effect of double frequency on the response of the vibration is demonstrated.

### 3.4. Time-domain propagation via transfer matrix method

In the previous section we have demonstrated that the periodic problem (27) and (28) or equivalently the initial value problem (32)-(34) has a $T$-periodic solution if the input exciting current is $T$ - periodic, under certain additional conditions satisfied. Thus, we need only to solve the initial value problem (32)-(34) in a single periodic time interval $[0, T]$. To find the numerical solution for the initial value problem (32)-(34), the transfer matrix method is employed, which is presented below.

First, we partition the time interval $[0, T]$ into $M$ uniform segments, the temporal nodes being
$t_{k-1}=(k-1) \Delta t \quad(k=1,2, \ldots, M), \quad \Delta t=T / M$.
We let $t=t_{k}$ and $t=t_{k-1}$ in the general solution (30) to eliminate the initial value $\mathbf{y}(0)$. This yields

$$
\begin{equation*}
\mathbf{y}\left(t_{k}\right)=\mathbf{X}\left(t_{k}\right)\left[\mathbf{X}^{-1}\left(t_{k-1}\right) \mathbf{y}\left(t_{k-1}\right)+\int_{t_{k-1}}^{t_{k}} \mathbf{X}^{-1}(\tau) \mathbf{f}(\tau) \mathrm{d} \tau\right] \tag{54}
\end{equation*}
$$

Next, to determine the fundamental solution matrix $\mathbf{X}(t)$ from the initial value problem (31), we make the following approximation in each time segment $\left[t_{k-1}, t_{k}\right)(k=$ $1,2, \ldots, M)$ :

$$
\begin{align*}
& \mathbf{A}(t) \cong \mathbf{A}_{k}=\mathbf{A}\left(t_{k-1}+\Delta t / 2\right) \\
& \quad \mathbf{f}(t) \cong \mathbf{f}_{k}=\mathbf{f}\left(t_{k-1}+\Delta t / 2\right) \quad\left(t_{k-1} \leqslant t \leqslant t_{k}\right) \tag{55}
\end{align*}
$$

In other words, in each segment $\boldsymbol{f}(t)$ and $\boldsymbol{A}(t)$ are approximately replaced, respectively, by their corresponding middle-point values in the segment. Thus, in the $k$ th segment we have the approximate initial value problem for equation (31):

$$
\begin{align*}
& \frac{\mathrm{d} \mathbf{X}(t)}{\mathrm{d} t}=\mathbf{A}_{k} \mathbf{X}(t), \quad \mathbf{X}\left(t_{k-1}\right)=\mathbf{X}_{k-1}  \tag{56}\\
& \quad\left(t_{k-1} \leqslant t<t_{k}\right)
\end{align*}
$$

The solution of the initial value problem (56) can be expressed in terms of the matrix exponential function:

$$
\begin{equation*}
\mathbf{X}(t)=\exp \left[\left(t-t_{k-1}\right) \mathbf{A}_{k}\right] \mathbf{X}_{k-1} \quad\left(t_{k-1} \leqslant t<t_{k}\right) \tag{57}
\end{equation*}
$$

Substituting the approximate expressions (55) and (57) into equation (54), we can obtain (see appendix B for detail)

$$
\begin{equation*}
\mathbf{y}\left(t_{k}\right)=\mathbf{T}_{k} \mathbf{y}\left(t_{k-1}\right)+\left(\mathbf{T}_{k}-\mathbf{I}_{2 N}\right) \mathbf{A}_{k}^{-1} \mathbf{f}_{k} \quad(k=1,2, \ldots, M) \tag{58}
\end{equation*}
$$

where the segment transfer matrix is defined as

$$
\begin{equation*}
\mathbf{T}_{k}=\exp \left[\Delta t \mathbf{A}_{k}\right] \tag{59}
\end{equation*}
$$

Equation (58) gives the transfer relation for the solution $\mathbf{y}(t)$ between the left and right nodes of the $k$ th segment. The transfer relation for the fundamental solution matrix is

$$
\begin{equation*}
\mathbf{X}_{k}=\mathbf{T}_{k} \mathbf{X}_{k-1}=\mathbf{T}_{k} \mathbf{T}_{k-1} \ldots \mathbf{T}_{1} \quad(k=1,2, \ldots, M) \tag{60}
\end{equation*}
$$

The segment transfer matrix $\mathbf{T}_{k}$ can be calculated by using a fourth-order Padé approximation for the exponential of a matrix [28]:
$\mathbf{T}_{k}=\exp \left(\Delta t \mathbf{A}_{k}\right)=\left[\mathbf{I}-\frac{1}{2} \Delta t \mathbf{A}_{k}+\frac{1}{12} \Delta t^{2} \mathbf{A}_{k}^{2}\right]^{-1}$
$\times\left[\mathbf{I}+\frac{1}{2} \Delta t \mathbf{A}_{k}+\frac{1}{12} \Delta t^{2} \mathbf{A}_{k}^{2}\right]+\mathbf{o}\left(\Delta t^{4}\right)$.
Moreover, from equation (32) the initial value $\mathbf{y}_{0}$ can be approximately calculated as follows:

$$
\begin{equation*}
\mathbf{y}_{0}=\left(\mathbf{I}_{2 N}-\mathbf{X}_{\mathrm{M}}\right)^{-1} \sum_{k=1}^{M} \mathbf{X}_{k}^{-1}\left(\mathbf{T}_{k}-\mathbf{I}\right) \mathbf{A}_{k}^{-1} \mathbf{f}_{k} \tag{62}
\end{equation*}
$$

where the matrix $\mathbf{X}_{k}^{-1}=\mathbf{T}_{1}^{-1} \mathbf{T}_{2}^{-1} \ldots \mathbf{T}_{k}^{-1}$ and $\mathbf{T}_{k}^{-1}=$ $\exp \left[-\Delta t \mathbf{A}_{k}\right]$.

Finally, we can carry out the numerical calculation for the state vectors $\mathbf{y}\left(t_{k}\right)(k=1,2, \ldots, M)$ at the temporal nodes by using the recursive relation (58) starting from the initial condition (62), in which the calculation for the matrix exponential $\exp \left( \pm \Delta t \boldsymbol{A}_{k}\right)$ on the basic of the Padé approximation (61) is the key step. Once the components of the state vectors $y_{j}\left(t_{k}\right)(j=1,2, \ldots, 2 N ; k=1,2, \ldots, M)$ is solved, the displacements and velocities in each time and space node for the vibration of the rod can be given by $u\left(x_{i}, t_{k}\right)=y_{i}\left(t_{k}\right)(i=1,2, \ldots, N)$ and $\partial u\left(x_{i}, t_{k}\right) / \partial t=y_{i}\left(t_{k}\right)$ $(i=N+1, N+2, \ldots, 2 N)$.

## 4. Numerical results and discussion

To illustrate the efficiency of the proposed method, the vibration of a magnetostrictive (Terfenol-D) rod is considered. The length of the rod $l=0.25 \mathrm{~m}$ and the area cross-section $A=3.14 \times 10^{-4} \mathrm{~m}^{2}$ (the radius $R=0.01 \mathrm{~m}$ ). For Terfenol-D material, the Young's modulus $E=26.5 \mathrm{GPa}$ and the mass density $\rho=9250 \mathrm{~kg} \mathrm{~m}^{-3}$. The magnetostrictive modulus $m=0.09 \times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$ and the magnetoelastic coefficient $r=-2.77 \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$, with a prestress $\sigma_{0}=6.9 \mathrm{MPa}$. All material parameters are typical values taken from [8]. The spring stiffness $K_{0}=3 \times 10^{7} \mathrm{~N} \mathrm{~m}^{-1}$ is chosen to be moderately compliant as in [11]. The number of turns in the coil $n=120$ and the alternating current passing through the solenoid is harmonic:

$$
\begin{equation*}
i(t)=i_{\max } \sin \omega t \tag{63}
\end{equation*}
$$

with the peak current $i_{\text {max }}=10 \mathrm{~A}$ and the circle frequency $\omega=4.4 \mathrm{kHz}$ (the frequency $f=\omega / 2 \pi=700 \mathrm{~Hz}$ ). Figure 3 shows the input current curve in one period $T=2 \pi / \omega=$ $1.43 \times 10^{-3} \mathrm{~s}$.

In the following, we present some numerical results for the vibration of a magnetostrictive (Terfenol-D) rod based on the method discussed above. The numerical implementation is only for a period of the exciting current. The time $t$-domain [ $0, T$ ] is partitioned into $M$ segments with a uniform time step $\Delta t=2 \pi /(M \omega)$. The space $x$-domain $[0, l]$ is divided into $N$ elements, with the length of each element being $h=l / N$.

First, in order to examine the convergence of the numerical method, the displacement response at the pusher end $u(l, t)$ is calculated for different numbers of time segments $M$ and


Figure 3. Variation of input current in one period $T=1.43 \mathrm{~ms}$ $\left(1.43 \times 10^{-3} \mathrm{~s}\right)$ with peak $i_{\max }=10 \mathrm{~A}$.


Figure 4. Response of the displacement at the pusher end $u(l, t)$ for different numbers of time segments $M$ and space elements $N$.
space elements $N$. Excellent convergence is observed when $N \geqslant 12$ and $M \geqslant 40$, as shown in figure 4 , where the displacement curves converge gradually into a smooth one with increasing $M$ and $N$. Moreover, figure 4 exhibits that the frequency of the displacement response is twice that of the exciting current. This is a numerical confirmation of the double frequency effect which was observed experimentally for a Terfenol-D actuator without a bias magnetic field [11] and demonstrated theoretically in section 3.3.4 of this paper.

Figures 5(a)-(c) show, respectively, the responses of displacement, stress, and the coupling part of the magnetic field and stress in the magnetic flux $B_{0}(x, t)$ at the pusher end $(x=l)$ and in the midpoint $(x=l / 2)$. Here, $B_{0}=$ $\mu H-B=\left(m \sigma+r \sigma^{2}\right) H$ and the permeability of the material $\mu=9.2 \times 4 \pi \times 10^{-7} \mathrm{H} / \mathrm{m}$. All these results show that the vibration at different points of the rod is synchronous. It is also interesting that while the responses of displacement and stress are of double frequency, the response of the magnetic flux is not.

Next, we investigate the effect of changing material parameters on the performance of the actuator. The displacements and stresses at the pusher end of $\operatorname{rod}(x=l)$ are calculated for three different values of the magnetostrictive modulus: $m=0.07 \times 10^{-12}, 0.09 \times 10^{-12}$ and $0.12 \times$ $10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$, while the magnetoelastic coefficient is fixed at $r=-2.77 \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$. It is observed from figures 6(a) and (b) that with increasing magnetostrictive


Figure 5. Response curves at the end $(x=l)$ and at the midpoint ( $x=l / 2$ ) of the rod: (a) for displacement, (b) for stress, and (c) for the coupling part of the magnetic field and stress in magnetic flux.
modulus, the amplitude of the displacement and stress increases. Figures 7(a) and (b) present the response curves for different values of the magnetoelastic coefficient: $r=-2.77 \times$ $10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}, r=0$ and $5.00 \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$, while the magnetostrictive modulus is fixed at $m=0.09 \times$ $10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$. We observed from figures 7(a) and (b) that, in contrast to the effect of the magnetostrictive modulus on the response, the effect of the magnetoelastic coefficient on the output displacement and stress is tiny.

The effect of the stiffness of the prestressed spring $K_{0}$ on the displacement and stress at the pusher end is presented in figures 8(a) and (b) for fixed magnetostrictive modulus $m=0.09 \times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$ and magnetoelastic coefficient $r=-2.77 \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$. Three different spring values are selected: $K_{0}=3 \times 10^{7} \mathrm{~N} \mathrm{~m}^{-1}, K_{0}=12 \times 10^{7}$, and $30 \times 10^{7} \mathrm{~N} \mathrm{~m}^{-1}$. It is observed from figure 8(a) that decreasing the spring stiffness will increase the amplitude of the displacement. However, the stress response is more complicated, as shown in figure 8(b). In figure 9, the peak displacement output (amplitude) at the pusher end of the rod


Figure 6. Response curves at the end $(x=l)$ of the rod for various values of magnetostrictive modulus $m=(0.07,0.09,0.12)$ $\times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$, with fixed magnetoelastic coefficient $r=$ $-2.77 \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$ : (a) for displacement, and (b) for stress.


Figure 7. Response curves at the end $(x=l)$ of the rod for various magnetoelastic coefficient $r=(-2.77,0,5) \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$, with fixed $m=0.09 \times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$ : (a) for displacement, and (b) for stress.
$\max |u(l, t)|$ is examined. Figure 9 indicates that the amplitude $\max |u(l, t)|$ is nonlinearly dependent on the stiffness of the prestressed spring $K_{0}$.


Figure 8. Response curves at the end $(x=l)$ of the rod for various values of prestressed spring stiffness $K_{0}(=3,12,30) \times 10^{7} \mathrm{~N} \mathrm{~m}^{-1}$ $\left(m=0.09 \times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}\right.$ and $\left.r=-2.77 \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}\right)$ : (a) for displacement, and (b) for stress.


Figure 9. Amplitude at the end $(x=l)$ of the rod versus prestressed spring stiffness $K_{0}\left(m=0.09 \times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}\right.$ and $r=-2.77$ $\times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$ ).

For control applications with the magnetostrictive actuator, it is necessary to numerically simulate the peak displacement output responses (amplitude) at the pusher end of rod max $|u(l, t)|$ on the peak current input of the solenoid $i_{\max }$ and the frequency $f$, respectively. Figure 10 presents the three groups of relationship curves corresponding to different magnetoelastic coefficient $m$ and magnetoelastic coefficient $r$. The curves shown in figure 11 predict that the peak displacement outputs remain constant within range of lower current frequency; however, the figure reveals nonlinear characteristics within the range of higher frequency.


Figure 10. Peak displacement output at the end $(x=l)$ versus peak current input for various material parameters $m(=0.07,0.09,0.12)$ $\times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$ and $r(=-2.77,0,0.1) \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$.


Figure 11. Peak displacement output at the end $(x=l)$ versus frequency for various peak current values $i_{\max }$ ( $m=0.09$ $\times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$ and $r=-2.77 \times 10^{-20} \mathrm{~m}^{2} \mathrm{~A}^{-2} \mathrm{~Pa}^{-1}$ ).

Finally, to verify the validity of the present mathematic modeling and numeric method, as a verification example, the present numeric results are compared with experimental data available for the vibration of a magnetostrictive actuator in [31]. In this example, the Young's modulus of the material and the mass density are taken as $E=26.5 \mathrm{GPa}$ and $\rho=$ $9250 \mathrm{~kg} \mathrm{~m}^{-3}$, respectively. The length of the $\operatorname{rod} l=0.1 \mathrm{~m}$ and the radius $R=0.0125 \mathrm{~mm}$. The prestress $\sigma_{0}=7 \mathrm{MPa}$ and the bias magnetic field $H_{\mathrm{b}}=20 \mathrm{kA} \mathrm{m}^{-1}$. The prestressed spring stiffness $K_{0}=3.24 \times 10^{5} \mathrm{~N} \mathrm{~m}^{-1}$ and the number of coil turns $n=700$. The piezomagnetic coefficient of the material $d=1.37 \times 10^{-8} \mathrm{~m} \mathrm{~A}^{-1}$, which is calculated from the testing data [31]. Thus, the magnetostrictive modulus $m=d /\left(2 H_{\mathrm{b}}\right)=0.34 \times 10^{-12} \mathrm{~m}^{2} \mathrm{~A}^{-2}$. The magnetoelastic coefficient is taken as $r=0$. Under direct current (DC) diving conditions, the current is constant, $i_{0}$. For the inducedstrain actuator displacement at the end $(x=l) u_{\mathrm{ISA}}$, excluding the displacement generated by the bias magnetic field $H_{\mathrm{b}}$, the present numeric results shown in figure 12 are in a good agreement with the measured data from the magnetostrictive actuator experiment in [31]. Figure 13 displays a comparison between the present numeric results and the experimental data available in [31]. The present numeric results agree also well


Figure 12. Comparison of present numeric results and measured experimental data [31] for induced-strain displacement at the end $(x=l)$ under DC diving conditions. The line is the numeric result and the dots represent experimental data.


Figure 13. Comparison of present numeric results and measured experimental data [31] for induced-strain displacement at the end ( $x=l$ ) versus frequency under AC diving conditions, for various peak current $i_{\max }(=0.5,1.0,1.5,2.0,2.5,3.0) \mathrm{A}$. All lines are numeric results and all symbols represent experimental data.
with the experimental data of $u_{\text {ISA }}$ under different alternating current (AC) diving conditions.

## 5. Conclusions

A new mathematical modeling technique and a computational method have been developed to investigate the excited vibration of a nonlinear Terfenol-D rod. The space domain is discretized via the finite difference method, and the transfer matrix method is introduced for the time-domain propagation. The periodicity of the solution to the vibration has also been discussed in detail, including the so-called double frequency phenomenon. The numerical results should be useful for the design of Terfenol-D actuators. The good agreement between the numerical results and the experimental data available in the literature verifies the validity of the present modeling and method. The presented approach can be extended to analyze the corresponding two-dimensional plate vibration for nonlinear Terfenol-D materials.

## Appendix A. Derivation of the equivalent condition (44)

Since the tridiagonal matrix $\mathbf{B}_{0}$ in equation (39b) is symmetric and positive definite, all its eigenvalues $\Omega_{k}^{2}$ are therefore positive, i.e., $\Omega_{k}^{2}>0(k=1,2, \ldots, N)$ ([29], p228). In addition, since there is a nonzero element in its subdiagonal line of the tridiagonal matrix $\mathbf{B}_{0}$, all its eigenvalues are also distinct ([30], p112). Therefore, the eigenvalues of equation (39b) satisfy the following condition:

$$
\begin{align*}
& \operatorname{det}\left[\mathbf{A}_{0}-\left( \pm \mathrm{j} \Omega_{k}\right) \mathbf{I}_{2 N}\right]=\operatorname{det}\left[\begin{array}{cc}
-\left( \pm \mathrm{j} \Omega_{k}\right) \mathbf{I}_{N} & \mathbf{I}_{N} \\
\mathbf{B}_{0} & -\left( \pm \mathrm{j} \Omega_{k}\right) \mathbf{I}_{N}
\end{array}\right] \\
& \quad=-\operatorname{det}\left[\mathbf{B}_{0}-\Omega_{k}^{2} \mathbf{I}_{N}\right]=0 \tag{A.1}
\end{align*}
$$

with $\mathrm{j}=\sqrt{ }(-1)$. In other words, all the eigenvalues of the matrix $\mathbf{A}_{0}$ are distinct and purely imaginary, equal to $\pm \mathrm{j} \Omega_{k}$ $(k=1,2, \ldots, N)$.

According to the theory of similarity matrices, there exists a nonsingular matrix $\mathbf{Q}$ such that

$$
\begin{equation*}
\mathbf{Q}^{-1} \mathbf{A}_{0} \mathbf{Q}=\Lambda \equiv \operatorname{diag}\left[+\mathrm{j} \Omega_{1},-\mathrm{j} \Omega_{1}, \ldots,+\mathrm{j} \Omega_{N},-\mathrm{j} \Omega_{N}\right] \tag{A.2}
\end{equation*}
$$

Furthermore, we have
$\mathbf{Q}^{-1} \exp \left(\frac{2 \pi}{\omega} \mathbf{A}_{0}\right) \mathbf{Q}=\exp \left(\frac{2 \pi}{\omega} \mathbf{Q}^{-1} \mathbf{A}_{0} \mathbf{Q}\right)=\exp \left(\frac{2 \pi}{\omega} \boldsymbol{\Lambda}\right)$.
Thus, for the case $r=0$ the condition (37) can be equivalently expressed as

$$
\begin{align*}
& F(\omega)=\operatorname{det}\left[\exp \left(\frac{2 \pi}{\omega} \mathbf{A}_{0}\right)-\mathbf{I}_{2 N}\right] \\
& =\operatorname{det}\left(\mathbf{Q}^{-1}\left[\exp \left(\frac{2 \pi}{\omega} \mathbf{A}_{0}\right)-\mathbf{I}_{2 N}\right] \mathbf{Q}\right) \\
& =\operatorname{det}\left[\exp \left(\frac{2 \pi}{\omega} \boldsymbol{\Lambda}\right)-\mathbf{I}_{2 N}\right]=\prod_{k=1}^{N}\left[\exp \left(2 \pi \mathrm{j} \frac{\Omega_{k}}{\omega}\right)-1\right] \\
& \quad \times\left[\exp \left(-2 \pi \mathrm{j} \frac{\Omega_{k}}{\omega}\right)-1\right] \neq 0 . \tag{A.4}
\end{align*}
$$

Therefore, it is required that $\omega \neq \Omega_{k} / \alpha(k=1,2, \ldots, N ; \alpha=$ $1,2, \ldots$ ), and thus the equivalent condition (44) is obtained.

## Appendix B. Derivation of equation (58)

On the basis of the differential formula for the exponential matrix function,

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} s} \exp \left[-\left(\tau-t_{k-1}\right) \mathbf{A}_{k}\right]=-\exp \left[-\left(\tau-t_{k-1}\right) \mathbf{A}_{k}\right] \mathbf{A}_{k} \\
& \quad\left(t_{k-1} \leqslant t<t_{k}\right) \tag{B.1}
\end{align*}
$$

and making use of the approximate expressions (55) and (57), the integral in equation (54) can be expressed as

$$
\begin{align*}
& \int_{t_{k-1}}^{t_{k}} \mathbf{X}^{-1}(\tau) \mathbf{f}(\tau) \mathrm{d} \tau=\mathbf{X}_{k-1}^{-1}\left(\int_{t_{k-1}}^{t_{k}} \exp \left[-\left(\tau-t_{k-1}\right) \mathbf{A}_{k}\right] \mathrm{d} \tau\right) \mathbf{f}_{k} \\
& \quad=\mathbf{X}_{k-1}^{-1}\left[\mathbf{I}_{2 N}-\exp \left(-\Delta t \boldsymbol{A}_{k}\right)\right] \mathbf{A}_{k}^{-1} \mathbf{f}_{k} . \tag{B.2}
\end{align*}
$$

From solution (57) we obtain $\mathbf{X}_{k} \mathbf{X}_{k-1}^{-1}=\exp \left(\Delta t \mathbf{A}_{k}\right)=\mathbf{T}_{k}$, and therefore we arrive at equation (58) from equation (54).

## References

[1] Busch-Vishniac H J 1997 Electromechanical Sensors and Actuators (New York: Springer-Verlag)
[2] Pons J L 2005 Emerging Actuator Technologies (New York: Wiley-Interscience)
[3] Clark A E and Belson H S 1972 Giant room-temperature magnetostrictions in $\mathrm{TbFe}_{2}$ and $\mathrm{DyFe}_{2}$ Phys. Rev. B 5 3642-4
[4] Moffet M B, Clark A and Wun-Fogle M 1991 Characterization of Terfenol-D for magnetostrictive transducers J. Acoust. Soc. Am. 89 1448-55
[5] Lacheisserie E D 1993 Magnetostrictions: Theory and Applications (New York: CRC Press)
[6] Carman G P and Mitrovic M 1996 Nonlinear constitutive relations for magnetostrictive materials with applications to 1D problems J. Intell. Mater. Syst. Struct. 6 673-83
[7] Claeyssen F, Lhermet N, Letty R L and Bouchilloux P 1997 Actuators, transducers and motors based on giant magnetostrictive materials J. Alloys Compounds 258 61-73
[8] Wan Y P, Fang D and Hwang K C 2003 Non-linear constitutive relations for magnetostrictive materials Int. J. Non-linear Mech. 38 1053-65
[9] Smith R C, Seelecke S, Dapiono M and Ounaies Z 2006 A unified framework for modeling hysteresis in ferroic materials J. Mech. Phys. Solids 54 46-85
[10] Hiller M W, Bryant M D and Umegaki J 1989 Attenuation and transformation of vibration through active control of magnetostrictive Terfenol J. Sound Vib. 134 507-19
[11] Aston M G, Greenough R D, Jenner A G I, Metheringham W J and Prajapati K 1997 Controlled high power actuation utilizing Terfenol-D J. Alloys Compounds 258 97-100
[12] Engdahl E and Svensson L 1988 Simulation of the magnetostrictive performance of Terfenol-D in mechanical devices J. Appl. Phys. 63 3924-6
[13] Anjanappa M and Bi J 1994 A theoretical and experimental study of magnetostrictive mini-actuators Smart Mater. Struct. 3 83-91
[14] Ackerman A E, Liang C and Rogers C A 1996 Dynamic transduction characterization of magnetostrictive actuators Smart Mater. Struct. 5115-20
[15] Dapino M J, Smith R C and Flatau A B 2000 Structural magnetic strain model for magnetostrictive transducers IEEE Trans. Magn. 36 545-56
[16] Wan Y P and Zhong Z 2004 Vibration analysis of Tb-Dy-Fe magnetostriction actuator and transducer Int. J. Mech. Mater. Des. 1 95-107
[17] Kannan K S and Dasgupta A 1997 A nonlinear Galerkin finite-element theory for modeling magnetostrictive smart structures Smart Mater. Struct. 6 341-50
[18] Shang X C, Jin M Y and Han X 2004 A numeric approach combining finite element with transfer matrix for vibration of magnetostrictive rod WCCM VI: Computational Mechanics (Beijing)
[19] Kim J and Jung E 2005 Finite element analysis for acoustic characteristics of a magnetostrictive transducer Smart Mater. Struct. 14 1273-80
[20] Kumar J S, Ganesan N, Sarnamani S and Padmanabhan C 2003 Active control of beam with magnetostrictive layer Comput. Struct. 81 1375-82
[21] Pan E and Heyliger P 2002 Free vibrations of simply supported and multilayered magneto-electro-elastic plates J. Sound Vib. 253 429-43
[22] Chen J, Chen H, Pan E and Heyliger P 2007 Modal analysis of magneto-electro-elastic plates using the state-vector approach J. Sound Vib. 304 722-34
[23] Eerenstein W, Mathur N D and Scott J F 2006 Multiferroic and magnetoelectric materials Nature 442 759-65
[24] Ramesh R and Spaldin N A 2007 Multiferroics: progress and prospects in thin films Nat. Mater. 621-9
[25] Goodfriend M J, Shoop K M and McMasters O D 1992 Characteristics of the magnetostrictive alloy Terfenol-D produced for the manufacture of devices Conf. on Recent Advances in Adaptive and Sensory Materials and their Applications (Blacksburg, VA)
[26] Walter W 1998 Ordinary Differential Equations (New York: Springer)
[27] Yakubovich Y A and Starzhinskii V M 1975 Linear Differential Equation with Periodic Coefficients vol 1 (New York: Wiley)
[28] Moler C B and Van Loan C F 1978 Nineteen dubious ways to compute the exponential of a matrix SIAM Rev. 20 801-36
[29] Phillips G M and Taylor P J 1973 Theory and Applications of Numerical Analysis (London: Academic)
[30] Jain M K, Iyengar S R K and Jain R K 1985 Numerical Methods for Scientific and Engineering Computation (New York: Wiley)
[31] Moon S J, Lim C W, Kim B H and Park Y 2007 Structural vibration control using linear magnetostrictive actuators J. Sound Vib. 302 875-91


[^0]:    4 Author to whom any correspondence should be addressed.

