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# Two-dimensional Green's functions in anisotropic multiferroic bimaterials with a viscous interface 

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## ARTICLE INFO

## Article history:

Received 24 January 2008
Received in revised form
17 April 2008
Accepted 22 April 2008

## Keywords:

Viscous interface
Multiferroic material
Stroh formalism
Image singularity
Image force


#### Abstract

We derive, by virtue of the unified Stroh formalism, the extremely concise and elegant solutions for two-dimensional and (quasi-static) time-dependent Green's functions in anisotropic magnetoelectroelastic multiferroic bimaterials with a viscous interface subjected to an extended line force and an extended line dislocation located in the upper half-plane. It is found for the first time that, in the multiferroic bimaterial Green's functions, there are 25 static image singularities and 50 moving image singularities in the form of the extended line force and extended line dislocation in the upper or lower halfplane. It is further observed that, as time evolves, the moving image singularities, which originate from the locations of the static image singularities, will move further away from the viscous interface with explicit time-dependent locations. Moreover, explicit expression of the time-dependent image force on the extended line dislocation due to its interaction with the viscous interface is derived, which is also valid for mathematically degenerate materials. Several special cases are discussed in detail for the image force expression to illustrate the influence of the viscous interface on the mobility of the extended line dislocation, and various interesting features are observed. These Green's functions can not only be directly applied to the study of dislocation mobility in the novel multiferroic bimaterials, they can also be utilized as kernel functions in a boundary integral formulation to investigate more complicated boundary value problems where multiferroic materials/composites are involved.


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## 1. Introduction

Artificial multiferroic composites made of ferromagnetic and ferroelectric phases can exhibit a magnetoelectric (ME) effect, which is absent in the constituents and which can be several orders larger than that observed in natural single-phase multiferroic materials (Benveniste, 1995; Ryu et al., 2001; Nan et al., 2003; Fiebig, 2005). The ME effect in the multiferroic composite is achieved through the product property: a magnetic field applied to the multiferroic composite will induce a strain in the ferromagnetic phase which is passed through the interface to the ferroelectric phase, where it induces an electric polarization. Thus the interface in multiferroic composites is critical in achieving the ME effect. In fact it has been

[^0]found that any imperfection or non-ideal coupling at the interface will always cause a reduction in the ME effect in multiferroic laminated or fibrous composites (Bichurin et al., 2003; Nan et al., 2003; Wang and Pan, 2007).

At elevated working temperatures exceeding about one-third of the homologous temperature, mass transport becomes important along high diffusivity path such as interface or grain boundary (Chen et al., 1998). Raj and Ashby (1971), Ashby (1972) and Suo (1997) suggested that the microscopically mass diffusion-controlled mechanism can be macroscopically described by the linear law for a viscous interface: $\dot{\delta}=\tau / \eta$, where $\dot{\delta}$ is the sliding velocity (i.e., the differentiation of the relative sliding with respect to time $t$ ), $\tau$ is the interfacial shear stress and $\eta$ is the interfacial viscosity which can be determined experimentally and theoretically (Raj and Ashby, 1971; Suo, 1997; Funn and Dutta, 1999; He and Lim, 2001, 2003). Furthermore, the viscous interface has been utilized to model incoherent interfaces between metal films and amorphous substrates to quantitatively study dislocation core spreading in thin films (Gao et al., 2002). There are also plenty of cases in which the interface in multiferroic composites should be considered as viscous. The direct bonding of PZT and Terfenol-D disks with conductive epoxy is one of the most effective methods to achieve a giant ME response (Ryu et al., 2001; Nan et al., 2003). The melting temperatures of PZT and Terfenol-D are higher than 1100 K, while the melting temperature of epoxy is only around $340-380 \mathrm{~K}$ (Fan and Wang, 2003). If the multiferroic composite works at a room temperature (say 300 K ), then the interfacial bonding should be considered as viscous. Therefore, there is an urgent need to understand the potential effect of viscous interface on the multiphase field quantities in multiferroic composites. This motivates us to introduce the viscous interface into the anisotropic multiferroic composites and subsequently to study the multi-field response in terms of the powerful and elegant Green's function method.

This paper is structured as follows: in Section 2, the Stroh formalism suitable for two-dimensional problems in generally anisotropic multiferroic materials in the presence of viscous interface is presented. In Section 3, based on a novel approach, we present, in terms of a unified formalism, the elegant time-dependent (quasi-static) Green's function solutions for an anisotropic magnetoelectroelastic multiferroic bimaterial with a viscous interface subjected to an extended line force and an extended line dislocation located in the upper half-plane. It is emphasized that the method proposed in this section is very simple and concise, and it is technically more attractive than previous approaches. In Section 4, the derived exact closed-form Green's functions are physically interpreted in terms of the static and moving image singularities in the form of an extended line force and an extended line dislocation. We derive in Section 5 the time-dependent image force for the extended line dislocation due to its interaction with the nearby viscous interface. In this section, some special cases of the image force expressions are discussed in detail to demonstrate the influence of the viscous interface on the mobility of the dislocation, and certain important features are observed. We draw our conclusions in Section 6.

## 2. Basic formulations

The basic equations for an anisotropic and linearly multiferroic material are (Pan, 2001)

$$
\begin{align*}
& \sigma_{i j}=C_{i j k l} u_{k, l}+e_{k i j} \phi_{, k}+q_{k i j} \varphi_{, k}, \quad D_{k}=e_{k i j} u_{i, j}-\varepsilon_{k l} \phi_{, l}-\alpha_{l k} \varphi_{l, l}, \\
& B_{k}=q_{k j} u_{i, j}-\alpha_{k l} \phi_{, l}-\mu_{k l} \varphi_{, l}, \quad \sigma_{i j, j}=0, \quad D_{i, i}=0, \quad B_{i, i}=0, \quad i, j, k, l=1,2,3, \tag{2.1}
\end{align*}
$$

where repeated indices mean summation, a comma follows by $i(i=1,2,3)$ stands for the derivative with respect to the $i$ th spatial coordinates; $u_{i}, \phi$ and $\varphi$ are the elastic displacement, electric potential and magnetic potential; $\sigma_{i j}, D_{i}$ and $B_{i}$ are the stress, electric displacement and magnetic induction; $C_{i j k l}, \varepsilon_{i j}$ and $\mu_{i j}$ are the elastic, dielectric and magnetic permeability coefficients, respectively; $e_{i j k}, q_{i j k}$ and $\alpha_{i j}$ are the piezoelectric, piezomagnetic and magnetoelectric coefficients, respectively.

For two-dimensional problems in which all quantities depend only on $x_{1}$ and $x_{2}$, one can seek the solution in the form of

$$
\mathbf{u}=\left[\begin{array}{lllll}
u_{1} & u_{2} & u_{3} & \phi & \varphi \tag{2.2}
\end{array}\right]^{\mathrm{T}}=\mathbf{a} f\left(x_{1}+p x_{2}, t\right),
$$

where ( $u_{1}, u_{2}$ ) are the elastic displacements in the ( $x_{1}, x_{2}$ )-plane and $u_{3}$ is the anti-plane elastic displacement perpendicular to the ( $x_{1}, x_{2}$ )-plane; $\mathbf{a}$ is a constant vector; $p$ is a complex number or the Stroh eigenvalue; $\left.f f^{*}, t\right)$ is an analytic function of the complex variable * and the real-time variable $t$. The appearance of the time $t$ comes from the influence of the viscous interface under quasi-static deformation. It can be verified that all equations in Eq. (2.1) are satisfied for an arbitrary analytic function $f\left(^{*}, t\right)$ if

$$
\begin{equation*}
\left[\mathbf{Q}+p\left(\mathbf{R}+\mathbf{R}^{\mathrm{T}}\right)+p^{2} \mathbf{T}\right] \mathbf{a}=\mathbf{0} \tag{2.3}
\end{equation*}
$$

where the $5 \times 5$ real matrix $\mathbf{R}$ and the two $5 \times 5$ symmetric real matrices $\mathbf{Q}$ and $\mathbf{T}$ are defined by

$$
\mathbf{Q}=\left[\begin{array}{ccc}
\mathbf{Q}^{\mathrm{E}} & \mathbf{e}_{11} & \mathbf{q}_{11}  \tag{2.4}\\
\mathbf{e}_{11}^{\mathrm{T}} & -\varepsilon_{11} & -\alpha_{11} \\
\mathbf{q}_{11}^{\mathrm{T}} & -\alpha_{11} & -\mu_{11}
\end{array}\right], \quad \mathbf{R}=\left[\begin{array}{ccc}
\mathbf{R}^{\mathrm{E}} & \mathbf{e}_{21} & \mathbf{q}_{21} \\
\mathbf{e}_{12}^{\mathrm{T}} & -\varepsilon_{12} & -\alpha_{21} \\
\mathbf{q}_{12}^{\mathrm{T}} & -\alpha_{12} & -\mu_{12}
\end{array}\right], \quad \mathbf{T}=\left[\begin{array}{ccc}
\mathbf{T}^{\mathrm{E}} & \mathbf{e}_{22} & \mathbf{q}_{22} \\
\mathbf{e}_{22}^{\mathrm{T}} & -\varepsilon_{22} & -\alpha_{22} \\
\mathbf{q}_{22}^{\mathrm{T}} & -\alpha_{22} & -\mu_{22}
\end{array}\right],
$$

where

$$
\begin{equation*}
\left(\mathbf{Q}^{\mathrm{E}}\right)_{i k}=C_{i 1 k 1}, \quad\left(\mathbf{R}^{\mathrm{E}}\right)_{i k}=C_{i 1 k 2}, \quad\left(\mathbf{T}^{\mathrm{E}}\right)_{i k}=C_{i 2 k 2}, \quad\left(\mathbf{e}_{i j}\right)_{m}=e_{i j m}, \quad\left(\mathbf{q}_{i j}\right)_{m}=q_{i j m} \tag{2.5}
\end{equation*}
$$

For a stable material with positive-definite energy density (elastic strain energy and electromagnetic energy), the 10 roots of Eq. (2.3) form five distinct conjugate pairs with non-zero imaginary parts (see Appendix A). Let $p_{i}$, ( $i=1-5$ ) be the five distinct roots with positive imaginary parts and $\mathbf{a}_{i}$ the associated eigenvectors, then the general solution is given by

$$
\begin{align*}
& \mathbf{u}=\left[\begin{array}{lllll}
u_{1} & u_{2} & u_{3} & \phi & \varphi
\end{array}\right]^{\mathrm{T}}=\mathbf{A} \mathbf{f}(z, t)+\overline{\mathbf{A}} \overline{\mathbf{f}(z, t)}, \\
& \mathbf{\Phi}=\left[\begin{array}{lllll}
\Phi_{1} & \Phi_{2} & \Phi_{3} & \Phi_{4} & \Phi_{5}
\end{array}\right]^{\mathrm{T}}=\mathbf{B f}(z, t)+\overline{\mathbf{B} \mathbf{f}(z, t),} \\
& \mathbf{b}_{i}=\left(\begin{array}{lllll}
\left.\mathbf{R}^{\mathrm{T}}+p_{i} \mathbf{T}\right) \mathbf{a}_{i}=\frac{-1}{p_{i}}\left(\mathbf{Q}+p_{i} \mathbf{R}\right) \mathbf{a}_{i} & (i=1-5) \\
\mathbf{A}=\left[\begin{array}{lllll}
\mathbf{a}_{1} & \mathbf{a}_{2} & \mathbf{a}_{3} & \mathbf{a}_{4} & \mathbf{a}_{5}
\end{array}\right], \quad \mathbf{B}=\left[\begin{array}{llll}
\mathbf{b}_{1} & \mathbf{b}_{2} & \mathbf{b}_{3} & \mathbf{b}_{4}
\end{array} \mathbf{b}_{5}\right.
\end{array}\right], \\
& \mathbf{f}(z, t)=\left[\begin{array}{lllll}
f_{1}\left(z_{1}, t\right) & f_{2}\left(z_{2}, t\right) & f_{3}\left(z_{3}, t\right) & f_{4}\left(z_{4}, t\right) & f_{5}\left(z_{5}, t\right)
\end{array}\right]^{\mathrm{T}}, \\
& z_{i}=x_{1}+p_{i} x_{2}, \\
& \operatorname{Im}\left\{p_{i}\right\}>0 \tag{2.6}
\end{align*}(i=1-5) . \quad .
$$

where the overbar denotes complex conjugate, and the extended stress function vector $\boldsymbol{\Phi}$ is defined, in terms of the stresses, electric displacements and magnetic inductions, as follows:

$$
\begin{array}{ll}
\sigma_{i 1}=-\Phi_{i, 2}, & \sigma_{i 2}=\Phi_{i, 1} \quad(i=1-3) \\
D_{1}=-\Phi_{4,2}, & D_{2}=\Phi_{4,1} \\
B_{1}=-\Phi_{5,2}, & B_{2}=\Phi_{5,1} . \tag{2.7}
\end{array}
$$

In addition, the two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following normalized orthogonal relationship (Ting, 1996):

$$
\left[\begin{array}{cc}
\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}  \tag{2.8}\\
\overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{A} & \overline{\mathbf{A}} \\
\mathbf{B} & \overline{\mathbf{B}}
\end{array}\right]=\mathbf{I}
$$

Furthermore, the following three real matrices $\mathbf{S}, \mathbf{H}$ and $\mathbf{L}$, which are called the Barnett-Lothe tensors, can be introduced (Ting, 1996):

$$
\begin{equation*}
\mathbf{S}=\mathrm{i}\left(2 \mathbf{A B}^{\mathrm{T}}-\mathbf{I}\right), \quad \mathbf{H}=2 \mathrm{i} \mathbf{A} \mathbf{A}^{\mathrm{T}}, \quad \mathbf{L}=-2 \mathrm{i} \mathbf{B} \mathbf{B}^{\mathrm{T}} \tag{2.9}
\end{equation*}
$$

with $\mathbf{H}$ and $\mathbf{L}$ being symmetric, and $\mathbf{S H}, \mathbf{L S}, \mathbf{H}^{-1} \mathbf{S}, \mathbf{S L}^{-1}$ being anti-symmetric.

## 3. General solution for the Green's functions

Let us assume that the anisotropic multiferroic materials 1 and 2 occupy, respectively, the half-planes $x_{2}>0$ and $x_{2}<0$. At the initial moment we introduce at the location $\left[\hat{x}_{1}, \hat{x}_{2}\right],\left(\hat{x}_{2}>0\right)$ in the upper half-plane an extended line force $\widehat{\mathbf{f}}=$ $\left[\begin{array}{lllll}f_{1} & f_{2} & f_{3} & -f_{e} & -f_{m}\end{array}\right]^{\mathrm{T}}$ and an extended line dislocation $\widehat{\mathbf{b}}=\left[\begin{array}{lllll}b_{1} & b_{2} & b_{3} & \Delta \phi & \Delta \varphi\end{array}\right]^{\mathrm{T}}$ where $b_{i}$, ( $i=1-3$ ) are three displacement jumps across the slip plane while $\Delta \phi$ and $\Delta \varphi$ are jumps in electric potential and magnetic potential. In the following, the superscripts (1) and (2) (or the subscripts 1 and 2 to the bold-face vectors and matrices) will be used to identify the quantities in the upper and lower half-planes, respectively. The two anisotropic multiferroic half-planes are bonded together through a viscous interface at $x_{2}=0$. The boundary conditions on the viscous interface can be expressed as (see He and Jiang, 2003; Chen and Lee, 2004; Wang et al., 2007)

$$
\begin{align*}
& \left.\begin{array}{llll}
\sigma_{12}^{(1)}=\sigma_{12}^{(2)}, & \sigma_{22}^{(1)}=\sigma_{22}^{(2)}, & \sigma_{32}^{(1)}=\sigma_{32}^{(2)}, & D_{2}^{(1)}=D_{2}^{(2)}, \\
B_{2}^{(1)}=B_{2}^{(2)} \\
u_{1}^{(1)}=u_{1}^{(2)}, & u_{2}^{(1)}=u_{2}^{(2)}, & u_{3}^{(1)}=u_{3}^{(2)}, & \phi^{(1)}=\phi^{(2)},
\end{array} \varphi^{(1)}=\varphi^{(2)}\right\}, \quad x_{2}=0 \quad \text { and } \quad t=0,  \tag{3.1}\\
& \left.\begin{array}{l}
\sigma_{12}^{(1)}=\sigma_{12}^{(2)}, \quad \sigma_{22}^{(1)}=\sigma_{22}^{(2)}, \quad \sigma_{32}^{(1)}=\sigma_{32}^{(2)}, \quad D_{2}^{(1)}=D_{2}^{(2)}, \quad B_{2}^{(1)}=B_{2}^{(2)} \\
u_{2}^{(1)}=u_{2}^{(2)}, \quad \phi^{(1)}=\phi^{(2)}, \quad \varphi^{(1)}=\varphi^{(2)} \\
\sigma_{12}^{(2)}=\eta_{1}\left(\dot{u}_{1}^{(1)}-\dot{u}_{1}^{(2)}\right), \quad \sigma_{32}^{(2)}=\eta_{3}\left(\dot{u}_{3}^{(1)}-\dot{u}_{3}^{(2)}\right)
\end{array}\right\}, \quad x_{2}=0 \quad \text { and } \quad t>0, \tag{3.2}
\end{align*}
$$

where an overdot denotes the derivative with respect to the time $t ; \eta_{1}$ and $\eta_{3}$ are the viscous coefficients in the $x_{1}$ and $x_{3}$ directions, respectively. Due to the fact that on the interface $x_{2}=0$ we have $z_{1}=z_{2}=z_{3}=z_{4}=z_{5}=z,\left(z=x_{1}+\mathrm{i} x_{2}\right)$, then during the analysis we can first replace $z_{k},(k=1-5)$ by the common complex variable $z$ (Clements, 1971; Suo, 1990; Wang et al., 2007). After the analysis is finished, we can then change $z$ back to the corresponding complex variables.

The above boundary conditions can also be concisely and equivalently expressed in terms of the extended displacement and extended stress function vectors as

$$
\begin{align*}
& \boldsymbol{\Phi}_{1}=\boldsymbol{\Phi}_{2}, \quad \mathbf{u}_{1}=\mathbf{u}_{2}, \quad x_{2}=0 \quad \text { and } \quad t=0,  \tag{3.3}\\
& \boldsymbol{\Phi}_{1}=\boldsymbol{\Phi}_{2}, \quad \dot{\mathbf{u}}_{1}-\dot{\mathbf{u}}_{2}=\boldsymbol{\Lambda} \boldsymbol{\Phi}_{2,1}, \quad x_{2}=0 \quad \text { and } \quad t>0, \tag{3.4}
\end{align*}
$$

where $\Lambda$ is a $5 \times 5$ real and diagonal matrix defined by

$$
\boldsymbol{\Lambda}=\operatorname{diag}\left[\begin{array}{lllll}
\eta_{1}^{-1} & 0 & \eta_{3}^{-1} & 0 & 0 \tag{3.5}
\end{array}\right]
$$

The boundary conditions in Eq. (3.4) can further be expressed in terms of the analytic function vectors $\mathbf{f}_{1}(z, t)$ and $\mathbf{f}_{2}(z, t)$ as

$$
\begin{equation*}
\mathbf{B}_{1} \mathbf{f}_{1}^{+}\left(x_{1}, t\right)+\overline{\mathbf{B}}_{1} \overline{\mathbf{f}}_{1}^{-}\left(x_{1}, t\right)=\mathbf{B}_{2} \mathbf{f}_{2}^{-}\left(x_{1}, t\right)+\overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}, t\right), \quad x_{2}=0 \quad \text { and } \quad t>0 \tag{3.6}
\end{equation*}
$$

$$
\begin{align*}
& \mathbf{A}_{1} \dot{\mathbf{f}}_{1}^{+}\left(x_{1}, t\right)+\overline{\mathbf{A}}_{1} \dot{\mathbf{f}}_{1}^{-}\left(x_{1}, t\right)-\mathbf{A}_{2} \dot{\mathbf{f}}_{2}^{-}\left(x_{1}, t\right)-\overline{\mathbf{A}}_{2} \dot{\mathbf{f}}_{2}^{+}\left(x_{1}, t\right) \\
& \quad=\boldsymbol{\Lambda}\left[\mathbf{B}_{2} \mathbf{f}_{2}^{\prime}\left(x_{1}, t\right)+\overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}, t\right)\right], \quad x_{2}=0 \quad \text { and } \quad t>0 . \tag{3.7}
\end{align*}
$$

It follows from Eq. (3.6) that

$$
\begin{align*}
& \mathbf{f}_{1}(z, t)=\mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}(z, t)+\mathbf{f}_{0}(z)-\mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1} \overline{\mathbf{f}}_{0}(z), \\
& \overline{\mathbf{f}}_{1}(z, t)=\overline{\mathbf{B}}_{1}^{-1} \mathbf{B}_{2} \mathbf{f}_{2}(z, t)+\overline{\mathbf{f}}_{0}(z)-\overline{\mathbf{B}}_{1}^{-1} \mathbf{B}_{1} \mathbf{f}_{0}(z), \tag{3.8}
\end{align*}
$$

where $\mathbf{f}_{0}(z)$ is the analytic function vector in a homogeneous plane occupied by material 1 given by

$$
\begin{equation*}
\mathbf{f}_{0}(z)=\frac{1}{2 \pi \mathrm{i}}\left\langle\ln \left(z-\hat{z}_{\alpha}\right)\right\rangle\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right) \tag{3.9}
\end{equation*}
$$

with $\hat{z}_{\alpha}=\hat{x}_{1}+p_{\alpha} \hat{x}_{2}$, and $\left\langle^{*}\right\rangle$ a $5 \times 5$ diagonal matrix in which each component is varied according to the Greek index $\alpha$ (from 1 to 5).

Substituting the above expressions into Eq. (3.7) and eliminating $\mathbf{f}_{1}^{+}\left(x_{1}, t\right), \overline{\mathbf{f}}_{1}^{-}\left(x_{1}, t\right)$, we finally obtain

$$
\begin{align*}
& \overline{\mathbf{N}}_{2} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}, t\right)-\mathrm{i} \boldsymbol{\Lambda} \overline{\mathbf{B}}_{\mathbf{2}} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}, t\right) \\
& \quad=\mathbf{N B}_{2} \dot{\mathbf{f}}_{2}^{-}\left(x_{1}, t\right)+\mathrm{i} \boldsymbol{\Lambda} \mathbf{B}_{2} \mathbf{f}_{2}^{-}\left(x_{1}, t\right), \quad x_{2}=0 \quad \text { and } \quad t>0 \tag{3.10}
\end{align*}
$$

where $\mathbf{N}$ is a $5 \times 5$ Hermitian matrix given by

$$
\begin{align*}
& \mathbf{N}=\overline{\mathbf{M}}_{1}^{-1}+\mathbf{M}_{2}^{-1}=\mathbf{L}_{1}^{-1}+\mathbf{L}_{2}^{-1}+\mathrm{i}\left(\mathbf{S}_{1} \mathbf{L}_{1}^{-1}-\mathbf{S}_{2} \mathbf{L}_{2}^{-1}\right), \\
& \mathbf{M}_{k}^{-1}=\mathrm{i} \mathbf{A}_{k} \mathbf{B}_{k}^{-1}=\left(\mathbf{I}-\mathrm{i} \mathbf{S}_{k}\right) \mathbf{L}_{k}^{-1} \quad(k=1,2) \tag{3.11}
\end{align*}
$$

Due to the fact that $\mathbf{N}$ is a $5 \times 5$ Hermitian matrix, we can further write $\mathbf{N}$ as

$$
\mathbf{N}=\overline{\mathbf{N}}^{\mathrm{T}}=\left[\begin{array}{lllll}
N_{11} & N_{12} & N_{13} & N_{14} & N_{15}  \tag{3.12}\\
\bar{N}_{12} & N_{22} & N_{23} & N_{24} & N_{25} \\
\bar{N}_{13} & \bar{N}_{23} & N_{33} & N_{34} & N_{35} \\
\bar{N}_{14} & \bar{N}_{24} & \bar{N}_{34} & N_{44} & N_{45} \\
\bar{N}_{15} & \bar{N}_{25} & \bar{N}_{35} & \bar{N}_{45} & N_{55}
\end{array}\right]
$$

It is apparent that the left-hand side of Eq. (3.10) is analytic in the upper half-plane, while the right-hand side of Eq. (3.10) is analytic in the lower half-plane. Consequently the continuity condition in Eq. (3.10) implies that the left- and right-hand sides of Eq. (3.10) are identically zero in the upper and lower half-planes, respectively. It then follows that

$$
\begin{equation*}
\mathbf{N B}_{2} \dot{\mathbf{f}}_{2}(z, t)+\mathrm{i} \boldsymbol{\Lambda} \mathbf{B}_{2} \mathbf{f}^{\prime}{ }_{2}(z, t)=\mathbf{0}, \quad \operatorname{Im}\{z\}<0 \tag{3.13}
\end{equation*}
$$

We next consider the following eigenvalue problem:

$$
\begin{equation*}
(\boldsymbol{\Lambda}-\lambda \mathbf{N}) \mathbf{v}=\mathbf{0} \tag{3.14}
\end{equation*}
$$

It is observed that in total there exist five eigenvalues to the above eigenvalue problem. Furthermore, these five eigenvalues $\lambda_{i},(i=1-5)$ can be explicitly determined as

$$
\begin{align*}
& \lambda_{1}=\frac{a_{1}+\sqrt{a_{1}^{2}-4 a_{0} a_{2}}}{2 a_{2}}>0 \\
& \lambda_{2}=\frac{a_{1}-\sqrt{a_{1}^{2}-4 a_{0} a_{2}}}{2 a_{2}}>0 \\
& \lambda_{3}=\lambda_{4}=\lambda_{5}=0 \tag{3.15}
\end{align*}
$$

where

$$
a_{2}=|\mathbf{N}|, \quad a_{1}=\frac{\widehat{N}_{11}}{\eta_{1}}+\frac{\widehat{N}_{33}}{\eta_{3}}, \quad a_{0}=\frac{1}{\eta_{1} \eta_{3}}\left|\begin{array}{lll}
N_{22} & N_{24} & N_{25}  \tag{3.16}\\
\bar{N}_{24} & N_{44} & N_{45} \\
\bar{N}_{25} & \bar{N}_{45} & N_{55}
\end{array}\right| \text {, }
$$

with $\widehat{N}_{i j}$ denoting the cofactors of the matrix $\mathbf{N}$.
We specially choose the eigenvectors associated with the three zero eigenvalues $\lambda_{3}=\lambda_{4}=\lambda_{5}=0$ as

$$
\mathbf{v}_{3}=\left[\begin{array}{l}
0  \tag{3.17}\\
1 \\
0 \\
0 \\
0
\end{array}\right], \quad \mathbf{v}_{4}=\left[\begin{array}{c}
0 \\
-N_{24} \\
0 \\
N_{22} \\
0
\end{array}\right], \quad \mathbf{v}_{5}=\left[\begin{array}{c}
0 \\
N_{24} N_{45}-N_{25} N_{44} \\
0 \\
N_{25} \bar{N}_{24}-N_{45} N_{22} \\
N_{22} N_{44}-N_{24} \bar{N}_{24}
\end{array}\right],
$$

so that the following orthogonal relationships with respect to the Hermitian matrix $\mathbf{N}$ and to the real and diagonal matrix $\boldsymbol{\Lambda}$ hold:

$$
\begin{align*}
& \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \mathbf{N} \boldsymbol{\Psi}=\boldsymbol{\Lambda}_{0}=\operatorname{diag}\left[\begin{array}{lllll}
\delta_{1} & \delta_{2} & \delta_{3} & \delta_{4} & \delta_{5}
\end{array}\right] \\
& \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \boldsymbol{\Lambda} \boldsymbol{\Psi}=\operatorname{diag}\left[\begin{array}{lllll}
\lambda_{1} \delta_{1} & \lambda_{2} \delta_{2} & \lambda_{3} \delta_{3} & \lambda_{4} \delta_{4} & \lambda_{5} \delta_{5}
\end{array}\right], \tag{3.18}
\end{align*}
$$

where $\delta_{k}=\overline{\mathbf{v}}_{k}^{\mathrm{T}} \mathbf{N} \mathbf{v}_{k}(k=1-5)$ are non-zero real values and

$$
\boldsymbol{\Psi}=\left[\begin{array}{lllll}
\mathbf{v}_{1} & \mathbf{v}_{2} & \mathbf{v}_{3} & \mathbf{v}_{4} & \mathbf{v}_{5} \tag{3.19}
\end{array}\right] .
$$

In addition, due to the fact that $\overline{\mathbf{v}}_{i}^{\mathrm{T}} \boldsymbol{\Lambda} \mathbf{v}_{i}=\lambda_{i} \delta_{i}>0(i=1,2)$, then $\delta_{1}$ and $\delta_{2}$ are positive.
Next we introduce the following new analytic function vector $\boldsymbol{\Omega}(z, t)$ :

$$
\begin{equation*}
\mathbf{B}_{2} \mathbf{f}_{2}(z, t)=\boldsymbol{\Psi} \boldsymbol{\Omega}(z, t) \tag{3.20}
\end{equation*}
$$

Employing the orthogonal relationship in Eq. (3.18), then Eq. (3.13) can be decoupled into

$$
\begin{equation*}
\dot{\Omega}_{k}(z, t)+\mathrm{i} \lambda_{k} \Omega_{k}^{\prime}(z, t)=0, \quad k=1-5, \quad \operatorname{Im}\{z\}<0 \tag{3.21}
\end{equation*}
$$

whose solutions can be conveniently given by

$$
\begin{equation*}
\Omega_{k}(z, t)=\Omega_{k}\left(z-\mathrm{i} \lambda_{k} t, 0\right), \quad k=1-5, \quad \operatorname{Im}\{z\}<0 \tag{3.22}
\end{equation*}
$$

The above expression indicates that once the initial state $\Omega_{k}(z, 0)$ is known, then one only needs to replace the complex variable $z$ by $z-\mathrm{i} \lambda_{k} t$ to arrive at the expression of $\Omega_{k}(z, t)$. In view of the fact that at the initial moment $t=0$, the interface is a perfect one, then we arrive at the following:

$$
\begin{equation*}
\mathbf{f}_{2}(z, 0)=2 \mathbf{B}_{2}^{-1} \mathbf{N}^{-1} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{f}_{0}(z), \quad \operatorname{Im}\{z\}<0 \tag{3.23}
\end{equation*}
$$

Consequently, it follows from Eq. (3.20) and the above expression that

$$
\begin{align*}
& \boldsymbol{\Omega}(z, 0)=2 \boldsymbol{\Psi}^{-1} \mathbf{N}^{-1} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{f}_{0}(z)=\frac{1}{\pi \mathrm{i}} \boldsymbol{\Psi}^{-1} \mathbf{N}^{-1} \mathbf{L}_{1}^{-1} \mathbf{B}_{1}<\ln \left(z-\hat{z}_{\alpha}\right)>\left(\mathbf{A}_{1}^{T} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{T} \widehat{\mathbf{b}}\right), \\
& \operatorname{Im}\{z\}<0 \tag{3.24}
\end{align*}
$$

Therefore, we can conveniently write down the expression of $\boldsymbol{\Omega}(z, t)$ as

$$
\begin{equation*}
\boldsymbol{\Omega}(z, t)=\frac{1}{\pi \mathrm{i}} \sum_{k=1}^{5}<\ln \left(z-\mathrm{i} \lambda_{\alpha} t-\hat{z}_{k}\right)>\boldsymbol{\Psi}^{-1} \mathbf{N}^{-1} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k}\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad \operatorname{Im}\{z\}<0 \tag{3.25}
\end{equation*}
$$

where

$$
\left.\begin{array}{rllll}
\mathbf{I}_{1} & =\operatorname{diag}\left[\begin{array}{lllll}
1 & 0 & 0 & 0 & 0
\end{array}\right], & \quad \mathbf{I}_{2}=\operatorname{diag}\left[\begin{array}{lllll}
0 & 1 & 0 & 0 & 0
\end{array}\right], \\
\mathbf{I}_{3} & =\operatorname{diag}\left[\begin{array}{llll}
0 & 0 & 1 & 0
\end{array} 0\right.
\end{array}\right], \quad \mathbf{I}_{4}=\operatorname{diag}\left[\begin{array}{lllll}
0 & 0 & 0 & 1 & 0
\end{array}\right],
$$

Substituting the above into Eq. (3.20), and then the results into Eq. (3.8), we can obtain the two analytic function vectors $\mathbf{f}_{1}(z, t)$ and $\mathbf{f}_{2}(z, t)$ as

$$
\begin{equation*}
\mathbf{f}_{2}(z, t)=\frac{1}{\pi \mathrm{i}} \sum_{k=1}^{5} \mathbf{B}_{2}^{-1} \Psi\left\langle\ln \left(z-\mathrm{i} \lambda_{\alpha} t-\hat{z}_{k}\right)\right\rangle \boldsymbol{\Psi}^{-1} \mathbf{N}^{-1} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k}\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad \operatorname{Im}\{z\}<0 \tag{3.27}
\end{equation*}
$$

$$
\begin{align*}
\mathbf{f}_{1}(z, t)= & -\frac{1}{\pi \mathrm{i}} \sum_{k=1}^{5} \mathbf{B}_{1}^{-1} \overline{\mathbf{\Psi}}\left\langle\ln \left(z+\mathrm{i} \lambda_{\alpha} t+\overline{\hat{z}}_{k}\right)\right\rangle \overline{\mathbf{Y}}^{-1} \overline{\mathbf{N}}^{-1} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k}\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right) \\
& +\frac{1}{2 \pi \mathrm{i}} \mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1}\left\langle\ln \left(z-\overline{\hat{z}}_{\alpha}\right)\right\rangle\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)+\frac{1}{2 \pi \mathrm{i}}\left\langle\ln \left(z-\hat{z}_{\alpha}\right)\right\rangle\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad \operatorname{Im}\{z\}>0 \tag{3.28}
\end{align*}
$$

The above expressions are in fact only valid along the $x_{1}$-axis. We can further write down the full-field expressions of $\mathbf{f}_{1}(z, t)$ and $\mathbf{f}_{2}(z, t)$ as

$$
\begin{align*}
\mathbf{f}_{1}(z, t)= & -\frac{1}{\pi \mathrm{i}} \sum_{m=1}^{5} \sum_{k=1}^{5}\left\langle\ln \left(z_{\alpha}+\mathrm{i} \lambda_{m} t-\bar{z}_{k}\right)\right\rangle \mathbf{B}_{1}^{-1} \overline{\mathbf{\Psi}} \overline{\mathbf{I}}_{m} \mathbf{\Lambda}_{0}^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k}\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right) \\
& +\frac{1}{2 \pi \mathrm{i}} \sum_{k=1}^{5}\left\langle\ln \left(z_{\alpha}-\overline{\hat{z}}_{k}\right)\right\rangle \mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k}\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)+\frac{1}{2 \pi \mathrm{i}}\left\langle\ln \left(z_{\alpha}-\hat{z}_{\alpha}\right)\right\rangle \\
& \times\left(\mathbf{A}_{1}^{\mathrm{T}} \mathbf{f}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathrm{~b}}\right), \quad x_{2}>0,  \tag{3.29}\\
\mathbf{f}_{2}(z, t)= & \frac{1}{\pi \mathrm{i}} \sum_{m=1}^{5} \sum_{k=1}^{5}\left\langle\ln \left(z_{\alpha}^{*}-\mathrm{i} \lambda_{m} t-\hat{z}_{k}\right)\right\rangle \mathbf{B}_{2}^{-1} \Psi \mathbf{I}_{m} \mathbf{\Lambda}_{0}^{-1} \overline{\mathbf{\Psi}}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k}\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad x_{2}<0 \tag{3.30}
\end{align*}
$$

where the superscript ${ }^{* *}$ ' is utilized to distinguish the Stroh eignvalues associated with the lower half-plane ( $z_{\alpha}^{*}$ ) from those associated with the upper half-plane $\left(z_{\alpha}\right)$. One can observe that the derived time-dependent multiferroic Green's function solutions (3.29) and (3.30) are even more concise and elegant than previously obtained ones for the purely elastic bimaterial (Wang et al., 2007). Substitution of Eqs. (3.29) and (3.30) into Eq. (2.6) will yield the expressions of $\mathbf{u}$ and $\boldsymbol{\Phi}$. For example, the tractions, normal electric displacement and normal magnetic induction are distributed along the interface $x_{2}=0$ as

$$
\begin{align*}
& {\left[\begin{array}{lllll}
\sigma_{12} & \sigma_{22} & \sigma_{32} & D_{2} & B_{2}
\end{array}\right]^{\mathrm{T}}} \\
& =\frac{2}{\pi} \operatorname{Im}\left\{\sum_{m=1}^{5} \boldsymbol{\Psi} \mathbf{I}_{m} \boldsymbol{\Lambda}_{0}^{-1} \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1}\left\langle\frac{1}{x_{1}-\mathrm{i} \lambda_{m} t-\hat{z}_{\alpha}}\right\rangle\left(\mathbf{A}_{1}^{\mathrm{T}} \hat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \hat{\mathbf{b}}\right)\right\}, \\
& -\infty<x_{1}<+\infty, \quad t \geqslant 0 \tag{3.31}
\end{align*}
$$

It also follows from Eqs. (3.29) and (3.30) that at the initial moment $t=0$ :

$$
\begin{align*}
\mathbf{f}_{1}(z, 0)= & \frac{1}{2 \pi \mathrm{i}} \sum_{k=1}^{5}\left\langle\ln \left(z_{\alpha}-\bar{z}_{k}\right)\right\rangle \mathbf{B}_{1}^{-1}\left(\mathbf{I}-2 \overline{\mathbf{N}}^{-1} \mathbf{L}_{1}^{-1}\right) \overline{\mathbf{B}}_{1} \mathbf{I}_{k}\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right) \\
& +\frac{1}{2 \pi \mathrm{i}}\left\langle\ln \left(z_{\alpha}-\hat{z}_{\alpha}\right)\right\rangle\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad x_{2}>0,  \tag{3.32}\\
\mathbf{f}_{2}(z, 0)= & \frac{1}{\pi \mathrm{i}} \sum_{k=1}^{5}\left\langle\ln \left(z_{\alpha}^{*}-\hat{z}_{k}\right)\right\rangle \mathbf{B}_{2}^{-1} \mathbf{N}^{-1} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k}\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad x_{2}<0, \tag{3.33}
\end{align*}
$$

which are just the bimaterial Green's functions with a perfect interface derived by Jiang and Pan (2004). On the other extreme case when $t \rightarrow \infty$, we have

$$
\begin{align*}
\mathbf{f}_{1}(z, \infty)= & \frac{1}{2 \pi \mathrm{i}} \sum_{k=1}^{5}\left\langle\ln \left(z_{\alpha}-\bar{z}_{k}\right)\right\rangle \mathbf{B}_{1}^{-1}\left[\mathbf{I}-2 \overline{\mathbf{\Psi}}\left(\mathbf{I}_{3}+\mathbf{I}_{4}+\mathbf{I}_{5}\right) \boldsymbol{\Lambda}_{0}^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1}\right] \overline{\mathbf{B}}_{1} \mathbf{I}_{k}\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right) \\
& +\frac{1}{2 \pi \mathrm{i}}\left\langle\ln \left(z_{\alpha}-\hat{z}_{\alpha}\right)\right\rangle\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad x_{2}>0,  \tag{3.34}\\
\mathbf{f}_{2}(z, \infty)= & \frac{1}{\pi \mathrm{i}} \sum_{k=1}^{5}\left\langle\ln \left(z_{\alpha}^{*}-\hat{z}_{k}\right)\right\rangle \mathbf{B}_{2}^{-1} \boldsymbol{\Psi}\left(\mathbf{I}_{3}+\mathbf{I}_{4}+\mathbf{I}_{5}\right) \boldsymbol{\Lambda}_{0}^{-1} \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k}\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad x_{2}<0, \tag{3.35}
\end{align*}
$$

which are the results for a free-sliding interface on which the two shear stresses are zero.
We add that the method presented in this paper is not limited to the Green's functions for an extended line force and an extended line dislocation. It can also be easily adopted to derive Green's functions for other types of singularities, such as concentrated couples and line heat sources once the Green's functions due to these singularities for a perfect bimaterial interface are known.

## 4. Image singularities

Here it is of particular interest to look into the physical meanings of the obtained Green's function solutions (3.29) and (3.30). The last term on the right-hand side of Eq. (3.29) represents the Green's function for an infinite multiferroic space with singularities in the form of an extended line force $\hat{\mathbf{f}}$ and an extended line dislocation $\hat{\mathbf{b}}$ located at $\left[x_{1}, x_{2}\right]=\left[\hat{x}_{1}, \hat{x}_{2}\right]$. In
the following we will demonstrate that: (i) the first and second terms on the right-hand side of Eq. (3.29) represent 50 time-dependent and 25 time-independent Green's functions for the infinite space occupied by material 1 whose singularities are also in the form of an extended line force and an extended line dislocation located in $x_{2}<0$; (ii) Eq. (3.30) represents 50 time-dependent and 25 time-independent Green's functions for the infinite space occupied by material 2 whose singularities are also in the form of an extended line force and an extended line dislocation located in $x_{2}>0$.

The moving singularities of the first term in Eq. (3.29) are located at

$$
\begin{equation*}
z_{n}+\mathrm{i} \lambda_{k} t-\overline{\hat{z}}_{m}=x_{1}+p_{n} x_{2}+\mathrm{i} \lambda_{k} t-\hat{x}_{1}-\bar{p}_{m} \hat{x}_{2}=0 \quad(n, m=1-5, \quad k=1,2) \tag{4.1}
\end{equation*}
$$

while the static singularities of the first and second terms in Eq. (3.29) are located at

$$
\begin{equation*}
z_{n}-\overline{\hat{z}}_{m}=x_{1}+p_{n} x_{2}-\hat{x}_{1}-\bar{p}_{m} \hat{x}_{2}=0 \quad(n, m=1-5) \tag{4.2}
\end{equation*}
$$

We let $p^{\prime}, p^{\prime \prime}$ be, respectively, the real and imaginary parts of $p$. If we equal the real and imaginary parts of Eqs. (4.1) and (4.2), then the locations $\left[x_{1}^{n m k}(t), x_{2}^{n m k}(t)\right]$ of the moving singularities and the locations $\left[x_{1}^{n m}, x_{2}^{n m}\right]$ of the static singularities for the upper half-plane are found to be

$$
\begin{align*}
& x_{1}^{n m k}(t)=\hat{x}_{1}+\frac{\lambda_{k} p_{n}^{\prime} t+\left(p_{n}^{\prime} p_{m}^{\prime \prime}+p_{m}^{\prime} p_{n}^{\prime \prime}\right) \hat{x}_{2}}{p_{n}^{\prime \prime}}, \quad x_{2}^{n m k}(t)=-\frac{\lambda_{k} t+p_{m}^{\prime \prime} \hat{x}_{2}}{p_{n}^{\prime \prime}} \\
& (n, m=1-5, k=1,2)  \tag{4.3}\\
& x_{1}^{n m}=\hat{x}_{1}+\frac{\left(p_{n}^{\prime} p_{m}^{\prime \prime}+p_{m}^{\prime} p_{n}^{\prime \prime}\right) \hat{x}_{2}}{p_{n}^{\prime \prime}}, \quad x_{2}^{n m}=-\frac{p_{m}^{\prime \prime} \hat{x}_{2}}{p_{n}^{\prime \prime}} \quad(n, m=1-5) \tag{4.4}
\end{align*}
$$

Due to the fact that $p_{n}^{\prime \prime}, p_{m}^{\prime \prime}, \lambda_{k}>0$, then $x_{1}^{n m k}(t), x_{2}^{n m k}(t)$ and $x_{1}^{n m}, x_{2}^{n m}$ exist and $x_{2}^{n m k}(t), x_{2}^{n m}<0$, which means that the moving and static image singularities for the upper half-plane are always located in the lower half-plane. In addition it is observed from Eq. (4.3) that the moving image singularities for the upper half-plane move further away from the interface as the time evolves. The moving image singularities in Eq. (3.30) are located at

$$
\begin{equation*}
z_{n}^{*}-\mathrm{i} \lambda_{k} t-\hat{z}_{m}=x_{1}+p_{n}^{*} x_{2}-\mathrm{i} \lambda_{k} t-\hat{x}_{1}-p_{m} \hat{x}_{2}=0 \quad(n, m=1-5, \quad k=1,2) \tag{4.5}
\end{equation*}
$$

while the static singularities in Eq. (3.30) are located at

$$
\begin{equation*}
z_{n}^{*}-\hat{z}_{m}=x_{1}+p_{n}^{*} x_{2}-\hat{x}_{1}-p_{m} \hat{x}_{2}=0 \quad(n, m=1-5) \tag{4.6}
\end{equation*}
$$

Equating the real and imaginary parts of Eqs. (4.5) and (4.6), the locations $\left[x_{1}^{* n m k}(t), x_{2}^{* n m k}(t)\right]$ of the moving singularities and the locations $\left[x_{1}^{* n m}, x_{2}^{* n m}\right.$ ] of the static singularities for the lower half-plane are

$$
\begin{align*}
& x_{1}^{* n m k}(t)=\hat{x}_{1}-\frac{\lambda_{k} p_{n}^{\prime *} t+\left(p_{n}^{\prime *} p_{m}^{\prime \prime}-p_{m}^{\prime} p_{n}^{\prime \prime *}\right) \hat{x}_{2}}{p^{\prime \prime *}}, \quad x_{n}^{* n m k}(t)=\frac{\lambda_{k} t+p_{m}^{\prime \prime} \hat{x}_{2}}{p_{n}^{\prime *}} \\
& (n, m=1-5, \quad k=1,2),  \tag{4.7}\\
& x_{1}^{* n m}=\hat{x}_{1}-\frac{\left(p_{n}^{\prime *} p_{m}^{\prime \prime}-p_{m}^{\prime} p_{n}^{\prime * *}\right) \hat{x}_{2}}{p_{n}^{\prime \prime *}}, \quad x_{2}^{* n m}=\frac{p_{m}^{\prime \prime} \hat{x}_{2}}{p_{n}^{\prime *}} \quad(n, m=1-5) \tag{4.8}
\end{align*}
$$

Due to the fact that $p^{\prime \prime *}{ }_{n}, p^{\prime \prime}{ }_{m}, \lambda_{k}>0$, then $x_{1}^{* n m k}(t), x_{2}^{* n m k}(t)$ and $x_{1}^{* n m}, x_{2}^{* n m}$ exist and $x_{2}^{* n m k}(t), x_{2}^{* n m}>0$, which means that the moving and static image singularities for the lower half-plane are always located in the upper half-plane. In addition it is observed from Eq. (4.7) that the moving image singularities for the lower half-plane also move further away from the interface as the time evolves.

Based on the previous results, we can further write Eqs. (3.29) and (3.30) into the following equivalent forms:

$$
\begin{align*}
\mathbf{f}_{1}(z, t)= & \frac{1}{2 \pi \mathrm{i} \mathrm{i}} \sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2}\left\langle\ln \left[z_{\alpha}-z_{\alpha}^{n m k}(t)\right]\right\rangle\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}_{n m k}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}_{n m k}\right) \\
& +\frac{1}{2 \pi \mathrm{i}} \sum_{n=1}^{5} \sum_{m=1}^{5}\left\langle\ln \left(z_{\alpha}-z_{\alpha}^{n m}\right)\right\rangle\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}_{n m}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}_{n m}\right)+\frac{1}{2 \pi \mathrm{i}}\left\langle\ln \left(z_{\alpha}-\hat{z}_{\alpha}\right)\right\rangle \\
& \times\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right), \quad x_{2}>0,  \tag{4.9}\\
\mathbf{f}_{2}(z, t)= & \frac{1}{2 \pi \mathrm{i} \mathrm{i}} \sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2}\left\langle\ln \left[z_{\alpha}^{*}-z_{\alpha}^{* n m k}(t)\right]\right\rangle\left(\mathbf{A}_{2}^{\mathrm{T}} \widehat{\mathbf{f}}_{n m k}^{*}+\mathbf{B}_{2}^{\mathrm{T}} \widehat{\mathbf{b}}_{n m k}^{*}\right) \\
& +\frac{1}{2 \pi \mathrm{i}} \sum_{n=1}^{5} \sum_{m=1}^{5}\left\langle\ln \left(z_{\alpha}^{*}-z_{\alpha}^{* n m}\right)\right\rangle\left(\mathbf{A}_{2}^{\mathrm{T}} \widehat{\mathbf{f}}_{n m}^{*}+\mathbf{B}_{2}^{\mathrm{T}} \widehat{\mathbf{b}}_{n m}^{*}\right), \quad x_{2}<0 \tag{4.10}
\end{align*}
$$

where

$$
\begin{align*}
& z_{\alpha}^{n m k}(t)=x_{1}^{n m k}(t)+p_{\alpha} x_{2}^{n m k}(t), \quad z_{\alpha}^{n m}=x_{1}^{n m}+p_{\alpha} x_{2}^{n m} \quad(n, m=1-5, \quad k=1,2)  \tag{4.11}\\
& z_{\alpha}^{* n m k}(t)=x_{1}^{* n m k}(t)+p_{\alpha}^{*} x_{2}^{* n m k}(t), \quad z_{\alpha}^{* n m}=x_{1}^{* n m}+p_{\alpha}^{*} x_{2}^{* n m} \quad(n, m=1-5, \quad k=1,2) \tag{4.12}
\end{align*}
$$

and

$$
\begin{align*}
& \widehat{\mathbf{f}}_{n m k}=-4 \operatorname{Re}\left\{\mathbf{B}_{1} \mathbf{I}_{n} \mathbf{B}_{1}^{-1} \overline{\mathbf{\Psi}}_{k} \boldsymbol{\Lambda}_{0}^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{m}\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)\right\}, \\
& \widehat{\mathbf{b}}_{n m k}=-4 \operatorname{Re}\left\{\mathbf{A}_{1} \mathbf{I}_{n} \mathbf{B}_{1}^{-1} \overline{\mathbf{\Psi}}_{k} \boldsymbol{\Lambda}_{0}^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{m}\left(\overline{\mathbf{A}}_{1}^{-\mathrm{f}} \hat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)\right\}, \\
& \widehat{\mathbf{f}}_{n m}=2 \operatorname{Re}\left\{\mathbf{B}_{1} \mathbf{I}_{n} \mathbf{B}_{1}^{-1}\left[\mathbf{I}-2 \overline{\boldsymbol{\Psi}}\left(\mathbf{I}_{3}+\mathbf{I}_{4}+\mathbf{I}_{5}\right) \mathbf{\Lambda}_{0}^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1}\right] \overline{\mathbf{B}}_{1} \mathbf{I}_{m}\left(\overline{\mathbf{A}}_{1}^{-\mathrm{f}} \hat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)\right\}, \\
& \widehat{\mathbf{b}}_{n m}=2 \operatorname{Re}\left\{\mathbf{A}_{1} \mathbf{I}_{n} \mathbf{B}_{1}^{-1}\left[\mathbf{I}-2 \overline{\boldsymbol{\Psi}}\left(\mathbf{I}_{3}+\mathbf{I}_{4}+\mathbf{I}_{5}\right) \boldsymbol{\Lambda}_{0}^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1}\right] \overline{\mathbf{B}}_{1} \mathbf{I}_{m}\left(\overline{\mathbf{A}}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\overline{\mathbf{B}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)\right\},  \tag{4.13}\\
& \widehat{\mathbf{f}}_{n m k}^{*}=4 \operatorname{Re}\left\{\mathbf{B}_{2} \mathbf{I}_{n} \mathbf{B}_{2}^{-1} \boldsymbol{\Psi} \mathbf{I}_{k} \boldsymbol{\Lambda}_{0}^{-1} \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{m}\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)\right\}, \\
& \widehat{\mathbf{b}}_{n m k}^{*}=4 \operatorname{Re}\left\{\mathbf{A}_{2} \mathbf{I}_{n} \mathbf{B}_{2}^{-1} \boldsymbol{\Psi} \mathbf{I}_{k} \boldsymbol{\Lambda}_{0}^{-1} \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{m}\left(\mathbf{A}_{1}^{\mathrm{f}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)\right\} \text {, } \\
& \widehat{\mathbf{f}}_{n m}^{*}=4 \operatorname{Re}\left\{\mathbf{B}_{2} \mathbf{I}_{n} \mathbf{B}_{2}^{-1} \boldsymbol{\Psi}\left(\mathbf{I}_{3}+\mathbf{I}_{4}+\mathbf{I}_{5}\right) \boldsymbol{\Lambda}_{0}^{-1} \overline{\mathbf{\Psi}}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{m}\left(\mathbf{A}_{1}^{\mathrm{T}} \widehat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)\right\} \text {, } \\
& \widehat{\mathbf{b}}_{n m}^{*}=4 \operatorname{Re}\left\{\mathbf{A}_{2} \mathbf{I}_{n} \mathbf{B}_{2}^{-1} \boldsymbol{\Psi}\left(\mathbf{I}_{3}+\mathbf{I}_{4}+\mathbf{I}_{5}\right) \boldsymbol{\Lambda}_{0}^{-1} \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{m}\left(\mathbf{A}_{1}^{\mathrm{T}} \hat{\mathbf{f}}+\mathbf{B}_{1}^{\mathrm{T}} \hat{\mathbf{b}}\right)\right\} . \tag{4.14}
\end{align*}
$$

The first term in Eq. (4.9) represents 50 time-dependent Green's functions with singularities located at $\left[x_{1}^{n m k}(t), x_{2}^{n m k}(t)\right]$. These moving image singularities consist of an extended line force $\widehat{\mathbf{f}}_{n m k}$ and an extended line dislocation $\widehat{\mathbf{b}}_{n m k}$. The second term in Eq. (4.9) represents 25 time-independent Green's functions with singularities located at $\left[x_{1}^{n m}, x_{2}^{n m}\right]$. These static image singularities consist of an extended line force $\widehat{\mathbf{f}}_{n m}$ and an extended line dislocation $\widehat{\mathbf{b}}_{n m}$. The first term in Eq. (4.10) represents 50 time-dependent Green's functions with singularities located at $\left[x_{1}^{* n m k}(t), x_{2}^{* n m k}(t)\right]$. These moving image singularities consist of an extended line force $\hat{\mathbf{f}}_{n m k}^{*}$ and an extended line dislocation $\hat{\mathbf{b}}_{n m k}^{*}$. The second term in Eq. (4.10) represents 25 time-independent Green's functions with singularities located at $\left[x_{1}^{* n m}, x_{2}^{* n m}\right]$. These static image singularities consist of an extended line force $\hat{\mathbf{f}}_{n m}^{*}$ and an extended line dislocation $\hat{\mathbf{b}}_{n m}^{*}$. In addition all the moving singularities originate from the locations of the static singularities due to the fact that $x_{1}^{n m k}(0)=x_{1}^{n m}, x_{2}^{n m k}(0)=x_{2}^{n m}$ and $x_{1}^{* m k}(0)=x_{1}^{* n m}, x_{2}^{* n m k}(0)=$ $x_{2}^{* n m}$ by noticing Eqs. (4.3), (4.4), (4.7) and (4.8). It can be easily checked from Eq. (4.13) that the total extended force due to the moving and static image singularities $\sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \hat{\mathbf{f}}_{n m k}+\sum_{n=1}^{5} \sum_{m=1}^{5} \hat{\mathbf{f}}_{n m}$ and the total extended dislocation due to the moving and static image singularities $\sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \hat{\mathbf{b}}_{m k}+\sum_{n=1}^{5} \sum_{m=1}^{5} \hat{\mathbf{b}}_{n m}$ for the upper half-plane are exactly the same as those for a perfect interface. Similarly it can be easily checked from Eq. (4.14) that the total extended force due to the moving and static image singularities $\sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \hat{\mathbf{f}}_{n m k}^{*}+\sum_{n=1}^{5} \sum_{m=1}^{5} \hat{\mathbf{f}}_{n m}^{*}$ and the total extended dislocation due to the moving and static image singularities $\sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \hat{\mathbf{b}}_{n m k}^{*}+\sum_{n=1}^{5} \sum_{m=1}^{5} \hat{\mathbf{b}}_{n m}^{*}$ for the lower half-plane are also exactly the same as those for a perfect interface.

In summary, 25 static image singularities and 50 moving image singularities, which originate from the locations of the static image singularities, in the form of an extended line force and an extended line dislocation for the upper or the lower half-plane are needed to exactly satisfy the boundary conditions on a viscous interface. For a perfect interface only the 25 static image singularities are needed [in the context of pure elasticity, the number is reduced to nine (Ting, 1992)]. In addition it is observed from Eqs. (4.3) and (4.4) that the locations of the 25 image singularities for the upper half-plane are independent of the property of the lower half-plane, while the locations of the 50 moving image singularities are reliant on the properties of both half-planes as well as the viscous coefficients. It is found from Eqs. (4.7) and (4.8) that the locations of all the static and moving image singularities for the lower half-plane are dependent on the properties of both half-planes as well as the viscous coefficients.

The discussions presented in this section are restricted to mathematically non-degenerate materials. It is expected that for degenerate materials such as the isotropic material the static and moving image singularities are not simply concentrated forces and dislocations. For the mathematically degenerate materials, some of the 25 static image singularities may coalesce into one static singularity and some of the 50 moving singularities may also converge into one moving singularity, resulting in static and moving double-forces, concentrated couples and other higher-order singularities. It should be pointed out that even though the image singularity discussions for an isotropic elastic half-plane and for an isotropic elastic bimaterial with a perfect interface have been carried out (see for example, Dundurs, 1969; Ma, 2001), the corresponding discussions for an isotropic elastic bimaterial with a viscous interface is still unavailable.

## 5. Time-dependent image force on an extended line dislocation

Here we are also much interested in the image force on the extended line dislocation $\hat{\mathbf{b}}$ (with $\hat{\mathbf{f}}=0$ ) due to its interaction with the nearby viscous interface. By using the Peach-Koehler formulation (Ting, 1996; Lee et al., 2000; Fan and Wang, 2003), the time-dependent image force acting on the extended line dislocation can be finally derived as

$$
\begin{align*}
& F_{2}(t)=\frac{\widehat{\mathbf{b}}^{\mathrm{T}}}{4 \pi \hat{x}_{2}}\left[2 \operatorname{Re}\left\{\mathbf{N}^{-1}\right\}-\mathbf{L}_{1}-2 \sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \operatorname{Re}\left\{\mathbf{Y}_{n m k} \frac{\lambda_{k} t}{\lambda_{k} t+\mathrm{i}\left(\bar{p}_{n}-p_{m}\right) \hat{X}_{2}}\right\}\right] \widehat{\mathbf{b}}, \\
& F_{1}(t)=0 \tag{5.1}
\end{align*}
$$

where $F_{1}$ and $F_{2}$ are, respectively, the components of the image force along the $x_{1}$ and $x_{2}$ directions, and

$$
\begin{equation*}
\mathbf{Y}_{n m k}=\overline{\mathbf{Y}}_{m n k}^{\mathrm{T}}=\left(\overline{\mathbf{B}}_{1} \mathbf{I}_{n} \overline{\mathbf{B}}_{1}^{-1}\right)\left(\boldsymbol{\Psi} \mathbf{I}_{k} \mathbf{\Lambda}_{0}^{-1} \overline{\boldsymbol{\Psi}}^{\mathrm{T}}\right)\left(\mathbf{B}_{1} \mathbf{I}_{m} \mathbf{B}_{1}^{-1}\right)^{\mathrm{T}}=\left(\overline{\mathbf{B}}_{1} \mathbf{I}_{n} \overline{\mathbf{B}}_{1}^{-1}\right)\left(\frac{\mathbf{v}_{k} \overline{\mathbf{v}}_{k}^{\mathrm{T}}}{\delta_{k}}\right)\left(\mathbf{B}_{1} \mathbf{I}_{m} \mathbf{B}_{1}^{-1}\right)^{\mathrm{T}} \tag{5.2}
\end{equation*}
$$

It is observed that the term in the square brackets on the right-hand side of Eq. (5.1) is a $5 \times 5$ time-dependent real and symmetric matrix. In the following we look into the above image force expression in more detail.

### 5.1. Isotropic elastic bimaterials

First it should be stressed that Eq. (4.1) is still valid for any kind of mathematically degenerate materials such as the isotropic elastic material. For example if we assume that both half-planes are isotropic elastic, then $p_{1}=p_{2}=p_{3}=p_{4}=p_{5}=i$. Consequently we have

$$
\begin{equation*}
F_{2}(t)=\frac{\widehat{\mathbf{b}}^{\mathrm{T}}}{4 \pi \hat{x}_{2}}\left[2 \operatorname{Re}\left\{\mathbf{N}^{-1}\right\}-\mathbf{L}_{1}-2 \sum_{k=1}^{2} \frac{\lambda_{k} t}{\lambda_{k} t+2 \hat{x}_{2}} \operatorname{Re}\left(\frac{\mathbf{v}_{k} \overline{\mathbf{v}}_{k}^{\mathrm{T}}}{\delta_{k}}\right)\right] \widehat{\mathbf{b}} . \tag{5.3}
\end{equation*}
$$

Due to the fact that the Barnett-Lothe tensors $\mathbf{S}, \mathbf{H}$ and $\mathbf{L}$ for isotropic elastic materials are well known (Ting, 1996), then it is not difficult to determine the explicit value of the above expression for an elastic dislocation $\widehat{\mathbf{b}}=\left[\begin{array}{lllll}b_{1} & b_{2} & b_{3} & 0 & 0\end{array}\right]^{\mathrm{T}}$ as

$$
\begin{align*}
F_{2}(t)= & \frac{\mu_{1}}{4 \pi \hat{x}_{2}\left(1-v_{1}\right)\left(1-\beta^{2}\right)}\left[\left(\alpha+\beta^{2}\right)\left(b_{1}^{2}+b_{2}^{2}\right)-\frac{t(\alpha+1)}{t+2 t_{1}}\left(b_{1}^{2}+\beta^{2} b_{2}^{2}\right)\right] \\
& +\frac{\mu_{1} b_{3}^{2}}{4 \pi \hat{x}_{2}}\left(\frac{\mu_{2}-\mu_{1}}{\mu_{1}+\mu_{2}}-\frac{t}{t+2 t_{2}} \frac{2 \mu_{2}}{\mu_{1}+\mu_{2}}\right), \tag{5.4}
\end{align*}
$$

where $\alpha$ and $\beta$ are Dundurs constants given by

$$
\begin{equation*}
\alpha=\frac{\mu_{2}\left(1-v_{1}\right)-\mu_{1}\left(1-v_{2}\right)}{\mu_{2}\left(1-v_{1}\right)+\mu_{1}\left(1-v_{2}\right)}, \quad \beta=\frac{\mu_{2}\left(1-2 v_{1}\right)-\mu_{1}\left(1-2 v_{2}\right)}{2\left[\mu_{2}\left(1-v_{1}\right)+\mu_{1}\left(1-v_{2}\right)\right]}, \tag{5.5}
\end{equation*}
$$

with $\mu_{i}$ and $v_{i}(i=1,2)$ being the shear moduli and Poisson's ratios, and $t_{1}$ and $t_{2}$ being two relaxation times given by

$$
\begin{equation*}
t_{1}=\frac{\hat{x}_{2}}{\lambda_{1}}=\frac{2 \hat{x}_{2} \eta_{1}\left(1-v_{1}\right)\left(1-\beta^{2}\right)}{\mu_{1}(\alpha+1)}, \quad t_{2}=\frac{\hat{x}_{2}}{\lambda_{2}}=\frac{\hat{x}_{2} \eta_{3}\left(\mu_{1}+\mu_{2}\right)}{\mu_{1} \mu_{2}} \tag{5.6}
\end{equation*}
$$

If $\eta_{1}=\eta_{3}$ and $0<v_{i} \leqslant 1 / 2$ then it can be easily found that $t_{1}<t_{2}$. In addition $F_{2}(t)$ is a monotonically decreasing function of $t$. It is observed that the second term in Eq. (5.4) gives the time-dependent image force due to a screw dislocation $b_{3}$, which is in agreement with the recent result by Wang and Pan (2008). However, the first term in Eq. (5.4) gives the time-dependent image force due to an edge dislocation with Burgers vector ( $b_{1}, b_{2}$ ), which is new in the literature. At $t=0$ the first term in Eq. (5.4) is the same as that obtained by Dundurs and Sendeckyj (1965) for a perfect interface (also see Ting, 1996). It is observed that the first term in Eq. (5.4) depends not only on $b_{1}^{2}+b_{2}^{2}$ but also on $b_{1}^{2}+\beta^{2} b_{2}^{2}$, thus it varies with the direction of the vector ( $b_{1}, b_{2}$ ). This observation is different from the invariant phenomenon for a perfect interface (Ting, 1996). When $t \rightarrow \infty$ Eq. (5.4) becomes

$$
\begin{equation*}
F_{2}(\infty)=-\frac{\mu_{1}\left[b_{1}^{2}-\alpha b_{2}^{2}+\left(1-v_{1}\right) b_{3}^{2}\right]}{4 \pi \hat{x}_{2}\left(1-v_{1}\right)} \tag{5.7}
\end{equation*}
$$

which can be reduced to that derived by Chen et al. (1998, Eq. (27)) for an edge dislocation ( $b_{3}=0$ ) interacting with a freesliding interface on which the two shear stresses $\sigma_{12}$ and $\sigma_{32}$ are zero. In addition if the two isotropic elastic half-planes possess the same material properties, i.e., $\mu_{1}=\mu_{2}=\mu, v_{1}=v_{2}=v$, then Eq. (5.4) is further reduced to

$$
\begin{equation*}
F_{2}(t)=-\frac{\mu t}{4 \pi \hat{x}_{2}}\left[\frac{b_{1}^{2}}{(1-v)\left(t+2 t_{1}\right)}+\frac{b_{3}^{2}}{t+2 t_{2}}\right] \leqslant 0 \tag{5.8}
\end{equation*}
$$

where

$$
\begin{equation*}
t_{1}=\frac{2 \hat{x}_{2} \eta_{1}(1-v)}{\mu}, \quad t_{2}=\frac{2 \hat{x}_{2} \eta_{3}}{\mu} . \tag{5.9}
\end{equation*}
$$

Eq. (5.8) implies that the elastic dislocation is always attracted to the viscous interface at any non-zero time, and that the image force will always be null if the dislocation only contains the $b_{2}$ component.

### 5.2. Transversely isotropic multiferroic bimaterials

Here we consider a practical situation in which the two multiferroic half-planes are both transversely isotropic, with $\left(x_{1}, x_{2}\right)$-plane being the isotropic plane. If the extended line dislocation contains only the $b_{1}$ and $b_{2}$ components, then the
result is identical to that presented in the previous subsection due the fact that the ( $x_{1}, x_{2}$ )-plane is an isotropic plane. We thus consider the two multiferroic half-planes with a screw dislocation $\hat{\mathbf{b}}=\left[\begin{array}{lllll}0 & 0 & b_{3} & \Delta \phi & \Delta \varphi\end{array}\right]^{\mathrm{T}}$. In this case it follows from Eq. (5.1) that

$$
\begin{equation*}
F_{2}(t)=\frac{1}{4 \pi \hat{x}_{2}}\left[\mathbf{b}_{0}^{\mathrm{T}}\left(\frac{4 t_{0}}{t+2 t_{0}} \mathbf{E}^{-1}-\mathbf{C}_{1}\right) \mathbf{b}_{0}+\frac{2 t}{t+2 t_{0}} \widetilde{\mathbf{b}}_{0}^{\mathrm{T}} \mathbf{M} \widetilde{\mathbf{b}}_{0}\right] \tag{5.10}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{C}=\left[\begin{array}{ccc}
C_{44} & e_{15} & q_{15} \\
e_{15} & -\varepsilon_{11} & -\alpha_{11} \\
q_{15} & -\alpha_{11} & -\mu_{11}
\end{array}\right], \quad \mathbf{E}=\left[\begin{array}{lll}
E_{11} & E_{12} & E_{13} \\
E_{12} & E_{22} & E_{23} \\
E_{13} & E_{23} & E_{33}
\end{array}\right] \equiv \mathbf{C}_{1}^{-1}+\mathbf{C}_{2}^{-1}, \quad \mathbf{M}=\left[\begin{array}{ll}
E_{22} & E_{23} \\
E_{23} & E_{33}
\end{array}\right]^{-1},  \tag{5.11}\\
& \mathbf{b}_{0}=\left[\begin{array}{lll}
b_{3} & \Delta \phi & \Delta \varphi
\end{array}\right]^{\mathrm{T}}, \quad \widetilde{\mathbf{b}}_{0}=\left[\begin{array}{ll}
\Delta \phi & \Delta \varphi
\end{array}\right]^{\mathrm{T}}, \tag{5.12}
\end{align*}
$$

and $t_{0}$ is a relaxation time defined by

$$
\begin{equation*}
t_{0}=\frac{\hat{x}_{3} \eta_{3}|\mathbf{E}|}{E_{22} E_{33}-E_{23}^{2}}>0 \tag{5.13}
\end{equation*}
$$

### 5.3. Relaxation time

It is observed from the previous two subsections that two positive real relaxation times are inherent in the timedependent image force expression for an extended line dislocation interacting with a viscous interface between two isotropic elastic half-planes or between two transversely isotropic multiferroic half-planes. For a generally anisotropic multiferroic bimaterial with a viscous interface, if we introduce the following relaxation times:

$$
\begin{equation*}
t_{n m k}=\bar{t}_{m n k}=\frac{\mathrm{i}\left(\bar{p}_{n}-p_{m}\right)}{\left(2 \lambda_{k}\right)} \quad(n, m=1-5, \quad k=1,2) \tag{5.14}
\end{equation*}
$$

then Eq. (5.1) can be further expressed in terms of these relaxation times as

$$
\begin{equation*}
F_{2}(t)=\frac{\widehat{\mathbf{b}}^{\mathrm{T}}}{4 \pi \hat{X}_{2}}\left[2 \operatorname{Re}\left\{\mathbf{N}^{-1}\right\}-\mathbf{L}_{1}-2 \sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \operatorname{Re}\left\{\mathbf{Y}_{n m k} \frac{t}{t+2 t_{n m k}}\right\}\right] \widehat{\mathbf{b}} . \tag{5.15}
\end{equation*}
$$

It is observed from Eq. (5.14) that there exist in total as many as 50 relaxation times, among which 10 are positive real while the other 40 are complex with positive real parts and form 20 distinct conjugate pairs. For example if the two half-planes are occupied by orthotropic elastic materials, then the Stroh eigenvalues and the Hermitian matrix $\mathbf{N}$ can be obtained explicitly (Suo, 1990; Dongye and Ting, 1989). In this case there exist four relaxation times $t_{k},(k=1-4)$ given by

$$
\begin{align*}
& t_{1}=\frac{\sqrt{\xi}[(1 / 2)(s+1)]^{1 / 2}}{\lambda_{1}}, \quad t_{2,3}=\frac{\sqrt{\xi}\left([(1 / 2)(s+1)]^{1 / 2} \pm[(1 / 2)(s-1)]^{1 / 2}\right)}{\lambda_{1}} \\
& t_{4}=\frac{\sqrt{C_{55}^{(1)} / C_{44}^{(1)}}}{\lambda_{2}} \tag{5.16}
\end{align*}
$$

where

$$
\xi=\sqrt{\frac{\rho_{11}}{\rho_{22}}}
$$

and

$$
s=\frac{1+\rho_{11} \rho_{22}-\left(1+\rho_{12}\right)^{2}}{2 \sqrt{\rho_{11} \rho_{22}}}>-1
$$

with

$$
\rho_{\gamma \beta}=\frac{C_{\gamma \beta}^{(1)}}{C_{66}^{(1)}} ;
$$

and the two eigenvalues $\lambda_{1}, \lambda_{2}$ are determined by

$$
\begin{equation*}
\lambda_{1}=\frac{N_{22}}{\eta_{1}\left(N_{11} N_{22}-N_{12} \bar{N}_{12}\right)}, \quad \lambda_{2}=\frac{1}{\eta_{3} N_{33}} \tag{5.17}
\end{equation*}
$$

It is observed from Eq. (5.16) that $t_{1}$ and $t_{4}$ are always real; while $t_{2}$ and $t_{3}$ are real when $s \geqslant 1$, and complex conjugate when $-1<s<1$.

### 5.4. Image force for a perfect interface at the initial moment

At the initial moment $t=0$ the image force can be expressed as

$$
\begin{equation*}
F_{2}(0)=\frac{\widehat{\mathbf{b}}^{\mathrm{T}}}{4 \pi \hat{x}_{2}}\left[2 \operatorname{Re}\left\{\mathbf{N}^{-1}\right\}-\mathbf{L}_{1}\right] \widehat{\mathbf{b}}, \tag{5.18}
\end{equation*}
$$

which is consistent with the result obtained by Ting and Barnett (1993) for a perfect interface between two anisotropic elastic half-planes.

### 5.5. Image force for a free-sliding interface when time approaches infinity

When time $t \rightarrow \infty$ the viscous interface will evolve into a free-sliding one on which the shear stresses $\sigma_{12}$ and $\sigma_{32}$ are zero. It follows from Eq. (5.1) that

$$
\begin{equation*}
F_{2}(\infty)=\frac{\widehat{\mathbf{b}}^{\mathrm{T}}}{4 \pi \hat{x}_{2}}\left[2 \operatorname{Re}\left\{\mathbf{N}^{-1}-\sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \mathbf{Y}_{n m k}\right\}-\mathbf{L}_{1}\right] \widehat{\mathbf{b}} . \tag{5.19}
\end{equation*}
$$

If we employ the following identity:

$$
\begin{align*}
\mathbf{N}^{-1}-\sum_{n=1}^{5} \sum_{m=1}^{5} \sum_{k=1}^{2} \mathbf{Y}_{n m k} & =\mathbf{N}^{-1}-\boldsymbol{\Psi}\left(\mathbf{I}_{1}+\mathbf{I}_{2}\right) \boldsymbol{\Lambda}_{0}^{-1} \overline{\boldsymbol{\Psi}}^{\mathrm{T}}=\sum_{k=3}^{5} \frac{\mathbf{v}_{k} \overline{\mathbf{v}}_{k}^{\mathrm{T}}}{\delta_{k}} \\
& =\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & k_{22} & 0 & k_{24} & k_{25} \\
0 & 0 & 0 & 0 & 0 \\
0 & \bar{k}_{24} & 0 & k_{44} & k_{45} \\
0 & \bar{k}_{25} & 0 & \bar{k}_{45} & k_{55}
\end{array}\right] \tag{5.20}
\end{align*}
$$

where $k_{i j}$ can be concisely given by

$$
\left[\begin{array}{lll}
k_{22} & k_{24} & k_{25}  \tag{5.21}\\
\bar{k}_{24} & k_{44} & k_{45} \\
\bar{k}_{25} & \bar{k}_{45} & k_{55}
\end{array}\right]=\left[\begin{array}{lll}
N_{22} & N_{24} & N_{25} \\
\bar{N}_{24} & N_{44} & N_{45} \\
\bar{N}_{25} & \bar{N}_{45} & N_{55}
\end{array}\right]^{-1}
$$

then Eq. (5.19) can be further simplified as

$$
\begin{equation*}
F_{2}(\infty)=\frac{1}{4 \pi \hat{x}_{2}}\left[2 \widetilde{\mathbf{b}}^{\mathrm{T}} \operatorname{Re}\left\{\widetilde{\mathbf{N}}^{-1}\right\} \widetilde{\mathbf{b}}-\widehat{\mathbf{b}}^{\mathrm{T}} \mathbf{L}_{1} \widehat{\mathbf{b}}\right], \tag{5.22}
\end{equation*}
$$

where

$$
\begin{align*}
& \widetilde{\mathbf{b}}=\left[\begin{array}{lll}
b_{2} & \Delta \phi & \Delta \varphi
\end{array}\right]^{\mathrm{T}},  \tag{5.23}\\
& \widetilde{\mathbf{N}}=\left[\begin{array}{lll}
N_{22} & N_{24} & N_{25} \\
\bar{N}_{24} & N_{44} & N_{45} \\
\bar{N}_{25} & \bar{N}_{45} & N_{55}
\end{array}\right] . \tag{5.24}
\end{align*}
$$

Expression (5.22) indicates that: (1) all the rest components in $\mathbf{N}$ except for those appearing in $\widetilde{\mathbf{N}}$ have no influence on $F_{2}(\infty)$; (2) the material properties in the lower multiferroic half-plane have no influence on $F_{2}(\infty)$ for an elastic dislocation $\widetilde{\mathbf{b}}=\left[\begin{array}{lllll}b_{1} & 0 & b_{3} & 0 & 0\end{array}\right]^{\mathrm{T}}$ (i.e, $\widetilde{\mathbf{b}}=\mathbf{0}$ ); (3) $F_{2}(\infty)$ varies with rotations about the $x_{3}$-axis due to the fact that the term $\widetilde{\mathbf{b}}^{\mathrm{T}} \operatorname{Re}\left\{\widetilde{\mathbf{N}}^{-1}\right\} \widetilde{\mathbf{b}}$ changes with rotations about the $x_{3}$-axis, which is quite different to the invariance property of the image force with the orientation of the perfect interface (Ting and Barnett, 1993; Ting, 1996). Furthermore Eq. (5.22) for the image force on an extended dislocation interacting with a free-sliding interface is strikingly simple! It is of interest to notice that Wang et al. (2001) have derived an expression for the image force on a line dislocation interacting with a free-sliding interface between two piezoelectric half-planes. In the context of multiferroic bimaterial, that expression can be slightly modified as

$$
F_{2}(\infty)=\frac{\widehat{\mathbf{b}}^{\mathrm{T}}}{4 \pi \hat{x}_{2}}\left[2 \operatorname{Re}\left\{\left[\begin{array}{c}
\mathbf{J}_{1} \mathbf{N}  \tag{5.25}\\
\mathbf{J}_{2}
\end{array}\right]^{-1}\left[\begin{array}{c}
\mathbf{J}_{1} \\
\mathbf{0}_{2 \times 5}
\end{array}\right]\right\}-\mathbf{L}_{1}\right] \widehat{\mathbf{b}},
$$

where

$$
\mathbf{J}_{1}=\left[\begin{array}{lllll}
0 & 1 & 0 & 0 & 0  \tag{5.26}\\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right], \quad \mathbf{J}_{2}=\left[\begin{array}{lllll}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{array}\right]
$$

Even though it can be proved that Eqs. (5.22) and (5.25) are equivalent, Eq. (5.25) is still not as explicit and concise as Eq. (5.22).

### 5.6. A comparison of image force $F_{2}(0)$ with $F_{2}(\infty)$

Here it is of interest to compare the value of the image force $F_{2}(0)$ for a perfect interface with that of $F_{2}(\infty)$ for a freesliding interface. It follows from Eqs. (5.18) and (5.19) that

$$
\begin{equation*}
F_{2}(0)-F_{2}(\infty)=\frac{1}{2 \pi \hat{x}_{2}}\left[\frac{\left(\widehat{\mathbf{b}}^{\mathrm{T}} \mathbf{v}_{1}\right)\left(\overline{\mathbf{v}}_{1}^{\mathrm{T}} \widehat{\mathbf{b}}\right)}{\delta_{1}}+\frac{\left(\widehat{\mathbf{b}}^{\mathrm{T}} \mathbf{v}_{2}\right)\left(\overline{\mathbf{v}}_{2}^{\mathrm{T}} \widehat{\mathbf{b}}\right)}{\delta_{2}}\right] \geqslant 0 \tag{5.27}
\end{equation*}
$$

due to the fact that both $\delta_{1}$ and $\delta_{2}$ are positive. Eq. (5.27) states that the image force on an extended line dislocation interacting with a perfect interface is always equal to or larger than that on the same dislocation interacting with a freesliding interface. In other words a free-sliding interface is more attractive to the extended line dislocation than a perfect interface. If the two multiferroic half-planes are exactly the same, then $F_{2}(\infty) \leqslant F_{2}(0)=0$, which means that the dislocation is always attracted to a free-sliding interface between two identical multiferroic half-planes. If we can find a real extended Burgers vector $\hat{\mathbf{b}}$ which is orthogonal to both $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$, i.e., $\mathbf{v}_{1}^{T} \hat{\mathbf{b}}=\mathbf{v}_{2}^{\mathrm{T}} \hat{\mathbf{b}}=0$, then $F_{2}(0) \equiv F_{2}(\infty)$. Let $\mathbf{v}^{\prime}, \mathbf{v}^{\prime \prime}{ }_{i}$ be, respectively, the real and imaginary parts of $\mathbf{v}_{i}$. then the condition $\widehat{\mathbf{b}}^{\mathrm{T}} \mathbf{v}_{1}=\widehat{\mathbf{b}}^{\mathrm{T}} \mathbf{v}_{2}=0$ is equivalent to the following set of four independent linear algebraic equations for the unknown $\hat{\mathbf{b}}$

$$
\left[\begin{array}{llll}
\mathbf{v}^{\prime}{ }_{1} & \mathbf{v}^{\prime \prime}{ }_{1} & \mathbf{v}^{\prime}{ }_{2} & \mathbf{v}^{\prime \prime}{ }_{2} \tag{5.28}
\end{array}\right]^{\mathrm{T}} \hat{\mathbf{b}}=\mathbf{0}_{4 \times 1},
$$

through which we can find at least a non-zero solution for the five-dimension real vector $\hat{\mathbf{b}}$. In other words we can always find an extended line dislocation $\hat{\mathbf{b}}$ such that the image force on the dislocation interacting with a perfect interface is just equal to that on the same dislocation interacting with a free-sliding interface.

## 6. Conclusions

We have derived the elegant and exact closed-form Green's functions Eqs. (3.29) and (3.30) for a generally anisotropic multiferroic bimaterial with a viscous interface subjected to an extended line force and an extended line dislocation in the upper half-plane. The obtained Green's function solutions are then interpreted physically in terms of the image singularities in the form of an extended line force and an extended line dislocation. It is found that 25 static image singularities and 50 moving image singularities in the form of an extended line force and an extended line dislocation for the upper or lower half-plane are needed to satisfy exactly the interfacial conditions on a viscous interface. For a perfect interface, on the other hand, only 25 static image singularities are needed [in the context of pure elasticity, the number is further reduced to nine (Ting, 1992)]. The image force on an extended line dislocation due to its interaction with the viscous interface is further derived, given explicitly by Eqs. (5.1) and (5.2). In order to better understand the influence of the viscous interface on the mobility of the extended line dislocation, we then look into the image force expression for six different cases: (i) isotropic elastic bimaterials; (ii) transversely isotropic multiferroic bimaterials with the symmetry axes along the $x_{3}$-axis; (iii) relaxation time; (iv) image force at the initial moment for a perfect interface; (v) image force at infinite time for a free-sliding interface; (vi) a comparison of the image force at the initial moment to that at infinite time. The results show that: (1) the derived image force expression is valid for any kind of mathematical degenerate materials; (2) as many as 50 relaxation times are needed to describe the time-dependent image force (see Eq. (5.15)) for a viscous interface between two generally anisotropic multiferroic half-planes; (3) a free-sliding interface is more attractive to the line dislocation than a perfect interface.

It is expected that the Green's functions presented in this paper could be directly applied to the study of dislocation induced mobility in novel multiferroic bimaterials. For more complicated boundary value problems, the derived exact closed-form Green's functions can be utilized as the kernel functions in any boundary integral formulation. Since only the boundary of the problem needs to be discretized, field concentration and singularity can be more efficiently handled than the domain-based discretization method.

## Acknowledgments

This work was supported in part by AFOSR FA9550-06-1-0317. The authors would also like to thank the reviewers and the editor for their constructive comments.

## Appendix A. Proof that $\boldsymbol{p}$ cannot be real in Eq. (2.3)

If we choose $u_{k}=a_{k} f\left(z_{p}\right),(k=1-3), \phi=a_{4} f\left(z_{p}\right)$ and $\varphi=a_{5} f\left(z_{p}\right)$ with $z_{p}=x_{1}+p x_{2}$, then differentiation of $u_{k}, \phi$ and $\varphi$ leads to

$$
\begin{equation*}
u_{k, l}=\left(\delta_{l 1}+p \delta_{l 2}\right) a_{k} f^{\prime}\left(z_{p}\right), \phi_{, l}=\left(\delta_{l 1}+p \delta_{l 2}\right) a_{4} f^{\prime}\left(z_{p}\right), \quad \varphi_{l}=\left(\delta_{l 1}+p \delta_{l 2}\right) a_{5} f^{\prime}\left(z_{p}\right) \tag{A.1}
\end{equation*}
$$

where $\delta_{l i}$ is the Kronecker delta.
Consequently satisfaction of $\sigma_{i j, j}=0, \quad D_{i, i}=0, \quad B_{i, i}=0$ yields (assuming constant material properties in the solid)

$$
\begin{align*}
& \left(C_{i j k s} a_{k}+e_{s i j} a_{4}+q_{s i j} a_{5}\right)\left(\delta_{j 1}+p \delta_{j 2}\right)\left(\delta_{s 1}+p \delta_{s 2}\right)=0, \\
& \left(e_{i k s} a_{k}-\varepsilon_{i s} a_{4}-\alpha_{i s} a_{5}\right)\left(\delta_{i 1}+p \delta_{i 2}\right)\left(\delta_{s 1}+p \delta_{s 2}\right)=0, \\
& \left(q_{i k s} a_{k}-\alpha_{i s} a_{4}-\mu_{i s} a_{5}\right)\left(\delta_{i 1}+p \delta_{i 2}\right)\left(\delta_{s 1}+p \delta_{s 2}\right)=0, \tag{A.2}
\end{align*}
$$

If $p$ were real, multiplication of Eq. (A.2) 1-3 by $a_{i}, a_{4}$ and $a_{5}$, respectively, and subtraction of the results leads to

$$
\begin{align*}
& C_{i j k s}\left[a_{i}\left(\delta_{j 1}+p \delta_{j 2}\right)\right]\left[a_{k}\left(\delta_{s 1}+p \delta_{s 2}\right)\right]+\varepsilon_{i S}\left[a_{4}\left(\delta_{i 1}+p \delta_{i 2}\right)\right]\left[a_{4}\left(\delta_{s 1}+p \delta_{s 2}\right)\right] \\
& \quad+2 \alpha_{i s}\left[a_{4}\left(\delta_{i 1}+p \delta_{i 2}\right)\right]\left[a_{5}\left(\delta_{s 1}+p \delta_{s 2}\right)\right]+\mu_{i s}\left[a_{5}\left(\delta_{i 1}+p \delta_{i 2}\right)\right]\left[a_{5}\left(\delta_{s 1}+p \delta_{s 2}\right)\right]=0, \tag{A.3}
\end{align*}
$$

which violates the positive definite energy (elastic and electromagnetic energies) condition that

$$
\begin{equation*}
C_{i j k s} u_{i, j} u_{k, s}>0, \varepsilon_{i s} E_{i} E_{s}+2 \alpha_{i s} E_{i} H_{s}+\mu_{i s} H_{i} H_{s}>0, \tag{A.4}
\end{equation*}
$$

where $E_{i}=-\phi_{i}, H_{i}=-\varphi_{, i}$ are the electric field and magnetic field, respectively.
It then follows that the 10 eigenvalues of Eq. (2.3) should form five conjugate pairs since all the three matrices $\mathbf{Q} \mathbf{R}$, and $\mathbf{T}$ in Eq. (2.3) are real.

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