New Journal of Physics
The open-access journal for physics

# Role of material property gradient and anisotropy in thermoelectric materials

**X Wang**[1,2], **E Pan**[1,2,4] **and J D Albrecht**[3]

[1] Department of Civil Engineering, University of Akron, Akron, OH 44325-3905, USA

[2] Department of Applied Mathematics, University of Akron, Akron, OH 44325-3905, USA

[3] Air Force Research Laboratory, Wright-Patterson Air Force Base, Dayton, OH 45433, USA

E-mail: pan2@uakron.edu

*New Journal of Physics* **10** (2008) 083019 (15pp)
Received 30 May 2008
Published 15 August 2008
Online at http://www.njp.org/
doi:10.1088/1367-2630/10/8/083019

**Abstract.** It was recently discovered that inclusions, fatigue damage and other types of material imperfections and defects in metals can be nondestructively detected by noncontacting magnetic measurements that sense the thermoelectric currents produced by directional heating and cooling. Since detection of small defects in thermoelectric materials is ultimately limited by intrinsic thermoelectric anisotropy and inhomogeneity of the material to be inspected, a thorough study is required on their impact on the nondestructive capability. Therefore, in this investigation the induced electric current densities and thermal fluxes are first derived for a steady line heat source in an inhomogeneous and anisotropic thermoelectric material. The exact closed-form solutions are obtained by converting the original problem into two inhomogeneous Helmholtz equations via eigenvalue/eigenvector separation. The material properties are assumed to vary exponentially in the same manner in an arbitrary direction. For the corresponding homogeneous but anisotropic material case, we also present an elegant formulation based on the complex variable method. It is shown that the induced magnetic fields can be expressed in a concise and exact closed form for a line heat source in an infinite homogeneous anisotropic material and in one of the two bonded anisotropic half-planes. Our numerical results demonstrate clearly that both property anisotropy and gradient in thermoelectric materials can significantly influence the induced thermoelectric currents and magnetic fields.

[4] Author to whom any correspondence should be addressed.

## Contents



## 1. Introduction

Thermoelectric materials, which possess the inherent property of coupled transport of heat and electricity, are becoming increasingly important in the field of energy production, conversion, conservation and nondestructive testing (NDT) [1]–[6]. Recently nanostructural thermoelectric materials have shown great promise for thermoelectric applications because these nanostructural materials can reduce the thermal conductivity more than the electrical conductivity by interface scattering. In other words, with these novel materials, one can achieve an increase in the power factor, with the consequence of increasing the thermoelectric figure of merit $ZT$ [7]–[10].

It has been discovered that the noncontacting thermoelectric technique can be used to detect various imperfections in conducting metals exhibiting thermoelectric effects, when the specimen to be tested is subjected to directional heating and cooling [1, 11, 12]. The physics of the process is relatively simple: an external heating or cooling is applied to the specimen to produce a modest temperature gradient in the test material domain. Since the host material and the inclusions (more accurately called inhomogeneities) within it have different material properties, different temperature fields and thus different thermoelectric potentials will be induced along the interfaces between the host material and inclusion. These potential differences will drive opposite thermoelectric currents (and thus two local current loops) inside and outside the inclusion, which can be finally detected by scanning the specimen with a sensitive magnetometer. It should be pointed out that the capability of detecting these imperfections depends on the thermoelectric background signal produced by the intrinsic anisotropy and inhomogeneity (or grading composition) of the material. In other words, a clear understanding of the effect of both anisotropy and inhomogeneity on the material response is required. Nayfeh *et al* [2] presented an analytic model to calculate the magnetic field produced by thermoelectric currents in homogeneous but anisotropic materials under two-dimensional (2D) heating and cooling. Carreon *et al* [3], on the other hand, presented another analytic model to calculate the normal and tangential magnetic fields produced by thermoelectric currents in an isotropic but linearly gradient slender rectangular bar under axial heating and cooling. However, since most materials are both inhomogeneous and anisotropic (even though weakly), a study is needed on the coupling influence of the material anisotropy and inhomogeneity on the material response, which motivates this investigation. For simplicity, here we consider an exponentially graded anisotropic material under 2D heating and cooling, and we derive the explicit solutions by converting the original problem into two inhomogeneous Helmholtz equations via eigenvalue/eigenvector separation. The corresponding 2D problem in homogeneous anisotropic materials is also discussed by means of the elegant complex variable method. These include the explicit solutions for a line heat source in an infinite homogeneous anisotropic material and in one of the two bonded anisotropic half-planes. The correctness of the methods is verified by reducing our results to the existing simple solutions obtained by Nayfeh *et al* [2], and by comparing the results from our eigenvalue/eigenvector separation approach and complex-variable method. The effect of the material gradient and anisotropy on the induced fields is clearly demonstrated and its impact on the nondestructive evaluation of materials is also discussed.

## 2. A line heat source in an infinite exponentially gradient anisotropic thermoelectric material

In this section, we present an analytic model to predict the magnetic background signal caused in inhomogeneous anisotropic media.

For anisotropic materials, the electrical current density $\mathbf{j}$ and thermal heat flux $\mathbf{h}$ vectors are related to both electric potential $\Phi$ and temperature $T$ through [2, 4]

$$\begin{aligned} j_i &= -\sigma_{ij}\Phi_{,j} - \in_{ij} T_{,j}, \\ h_i &= -\bar{\in}_{ij}\Phi_{,j} - \kappa_{ij} T_{,j}, \end{aligned} \tag{1}$$

where the subscript ‘$,j$’ to $\Phi$ and $T$ denotes the derivative with respect to the $j$th coordinate $x_j$ ($x_1 = x,\ x_2 = y,\ x_3 = z$), $\sigma_{ij}$ denotes the electrical conductivity measured at a uniform temperature, $\kappa_{ij}$ is the thermal conductivity at zero electrical field, $\in_{ij}$ and $\bar{\in}_{ij}$ are thermoelectric coupling coefficients. These coupling coefficients can be expressed in terms of the absolute thermoelectric power $S$ and the electrical conductivity $\sigma_{ij}$ of the material as $\in_{ij} = S\sigma_{ij}$ and $\bar{\in}_{ij} = ST\sigma_{ij}$. Furthermore, for gradient materials as studied in this section, $\sigma_{ij}$, $\kappa_{ij}$, $\in_{ij}$ and $\bar{\in}_{ij}$ are all functions of the coordinates $x_1 = x$, $x_2 = y$ and $x_3 = z$.

The electrical current density and thermal heat flux vectors should satisfy the following equations:

$$\begin{aligned} \nabla \cdot \mathbf{j} &= 0, \\ \nabla \cdot \mathbf{h} &= q_{\text{gen}}, \end{aligned} \tag{2}$$

where $q_{\text{gen}}$ is the power generated per unit volume. It is obvious that the induced electrical current density (thus the magnetic field) is proportional to the strength of the heat power emanated from the source. Thus the magnitude of the induced magnetic field can be controlled by the applied temperature source so that the associated magneto thermal transport effect can be neglected.

Let us consider an infinite line heat source in an infinite inhomogeneous anisotropic material. We first assume that $\sigma_{ij}$, $\kappa_{ij}$, $\in_{ij}$ and $\bar{\in}_{ij}$ are exponentially varied in the ($x$–$y$)-plane in the same manner along the $x$- and $y$-directions as follows:

$$\begin{bmatrix} \sigma_{ij} & \in_{ij} & \bar{\in}_{ij} & \kappa_{ij} \end{bmatrix} = \exp(2\beta_1 x + 2\beta_2 y) \begin{bmatrix} \sigma^0_{ij} & \in^0_{ij} & \bar{\in}^0_{ij} & \kappa^0_{ij} \end{bmatrix}, \tag{3}$$

where $\sigma^0_{ij}$, $\in^0_{ij}$, $\bar{\in}^0_{ij}$, $\kappa^0_{ij}$, $\beta_1$ and $\beta_2$ are all material constants.

We point out that functionally graded materials (FGMs) can be fabricated for special applications (see e.g. [13, 14]). They can be varied in many different ways and in different orientations in space. The exponential variation is a special case that has been investigated in the FGM community for over 20 years (see e.g. [15, 16]). For such a special variation, it is often possible to derive either an analytical solution or even an exact-closed form solution (such as the one presented in this paper), serving as benchmarks for future numerical simulation. However, other complicated spatial variation can be piecewise approximated when solving the real problem via a boundary integral equation formulation. In particular, when the material gradients are not too large, the exponential variation in material properties studied here will approximate a linear variation in material properties. We also point out that closed-form solutions can still be obtained for some non-exponential variations of the material properties [17, 18].

Under the above assumption the principal directions of the electrical conductivity, thermal conductivity and thermoelectric coupling tensors do not change at different locations. Furthermore, for the convenience of analysis, we assume that the electrical conductivity, thermal conductivity and thermoelectric coupling tensors all exhibit the same principal directions, though their degree of anisotropy might be very different [2]. Without loss of generality, in the following we assume that the three Cartesian coordinates $x$, $y$ and $z$ are established along the principal directions. The infinite line heat source is parallel to the $z$-axis and passes through $x = x_0$ and $y = y_0$. Therefore, we can write $q_{\text{gen}} = Q\delta(x - x_0)\delta(y - y_0)$, where $Q$ denotes the total heat power emanated from a unit length of the line source and $\delta(\,)$ is the Dirac delta function. Consequently the problem becomes 2D, with the electric potential and temperature being only functions of the coordinates $x$ and $y$. By keeping the above assumptions in mind and by substituting equation (1) into (2), we finally obtain the following set of coupled inhomogeneous partial differential equations:

$$
\begin{bmatrix} \sigma_1^0 & \in_1^0 \\ \in_1^0 & \dfrac{\kappa \in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2\Phi}{\partial x^2} \\ \dfrac{\partial^2 T}{\partial x^2} \end{bmatrix} + \begin{bmatrix} \sigma_2^0 & \in_2^0 \\ \in_2^0 & \dfrac{\in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2\Phi}{\partial y^2} \\ \dfrac{\partial^2 T}{\partial y^2} \end{bmatrix}
$$

$$
+2\beta_1 \begin{bmatrix} \sigma_1^0 & \in_1^0 \\ \in_1^0 & \dfrac{\kappa \in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \begin{bmatrix} \dfrac{\partial\Phi}{\partial x} \\ \dfrac{\partial T}{\partial x} \end{bmatrix} + 2\beta_2 \begin{bmatrix} \sigma_2^0 & \in_2^0 \\ \in_2^0 & \dfrac{\in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \begin{bmatrix} \dfrac{\partial\Phi}{\partial y} \\ \dfrac{\partial T}{\partial y} \end{bmatrix}
$$

$$
= -\frac{\exp(-2\beta_1 x_0 - 2\beta_2 y_0) \in_2^{02} Q}{\eta\sigma_2^0\kappa_2^0}\delta(x - x_0)\delta(y - y_0) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{4}
$$

where $\eta = \in_2^0 \bar{\in}_2^0/(\sigma_2^0\kappa_2^0)$ is the dimensionless thermoelectric coupling parameter. We mention that during the derivation, we have utilized the identity $\in_1^0 / \in_2^0 = \bar{\in}_1^0/\bar{\in}_2^0$. It is also worth pointing out that, for typical metals the coupling parameter $\eta$ is relatively small, somewhere between $10^{-3}$ and $10^{-2}$ [1]–[3].

In order to solve equation (4), we next consider the following eigenvalue problem:

$$\begin{bmatrix} \sigma_1^0 & \in_1^0 \\ \in_1^0 & \dfrac{\kappa \in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \mathbf{v} = \lambda \begin{bmatrix} \sigma_2^0 & \in_2^0 \\ \in_2^0 & \dfrac{\in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \mathbf{v}. \tag{5}$$

The two eigenvalues of the above equation can be easily determined as

$$\begin{aligned} \lambda_1 &= \frac{\sigma + \kappa - 2 \in \eta + \sqrt{(\sigma-\kappa)^2 - 4\eta(\sigma - \in)(\in - \kappa)}}{2(1-\eta)} > 0, \\ \lambda_2 &= \frac{\sigma + \kappa - 2 \in \eta - \sqrt{(\sigma-\kappa)^2 - 4\eta(\sigma - \in)(\in - \kappa)}}{2(1-\eta)} > 0, \end{aligned} \tag{6}$$

where $\sigma = \sigma_1^0/\sigma_2^0$, $\in = \in_1^0 / \in_2^0$ and $\kappa = \kappa_1^0/\kappa_2^0$ are ratios of the material properties. The two eigenvectors associated with the eigenvalues are

$$\mathbf{v}_1 = \begin{bmatrix} \in_2^0 (\lambda_1 - \in) \\ \sigma_2^0(\sigma - \lambda_1) \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \in_2^0 (\lambda_2 - \in) \\ \sigma_2^0(\sigma - \lambda_2) \end{bmatrix}. \tag{7}$$

Apparently due to the fact that both

$$\begin{bmatrix} \sigma_1^0 & \in_1^0 \\ \in_1^0 & \dfrac{\kappa \in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sigma_2^0 & \in_2^0 \\ \in_2^0 & \dfrac{\in_2^{02}}{\eta\sigma_2^0} \end{bmatrix}$$

are real and symmetric, then we have the following orthogonal relationships:

$$\begin{aligned} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \sigma_2^0 & \in_2^0 \\ \in_2^0 & \dfrac{\in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} &= \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \\ \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \sigma_1^0 & \in_1^0 \\ \in_1^0 & \dfrac{\kappa \in_2^{02}}{\eta\sigma_2^0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} &= \begin{bmatrix} \lambda_1\delta_1 & 0 \\ 0 & \lambda_2\delta_2 \end{bmatrix}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \delta_1 &= \sigma_2^0 \in_2^{02} \left[ (\lambda_1 - \in)(2\sigma - \in - \lambda_1) + \frac{(\sigma - \lambda_1)^2}{\eta} \right] > 0, \\ \delta_2 &= \sigma_2^0 \in_2^{02} \left[ (\lambda_2 - \in)(2\sigma - \in - \lambda_2) + \frac{(\sigma - \lambda_2)^2}{\eta} \right] > 0. \end{aligned} \tag{9}$$

Now we introduce two new functions $F$ and $G$, which are related to electric potential and temperature through

$$\begin{bmatrix} \Phi \\ T \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}. \tag{10}$$

In view of the orthogonal relationships in equations (8), equation (4) can now be decoupled into

$$\begin{aligned}
&\frac{\partial^2 F}{\partial x^2}+\frac{1}{\lambda_1}\frac{\partial^2 F}{\partial y^2}+2\beta_1\frac{\partial F}{\partial x}+\frac{2\beta_2}{\lambda_1}\frac{\partial F}{\partial y}=-\frac{2\pi\exp(-\beta_1 x_0-\beta_2 y_0)P_1}{\sqrt{\lambda_1}}\delta(x-x_0)\delta(y-y_0),\\
&\frac{\partial^2 G}{\partial x^2}+\frac{1}{\lambda_2}\frac{\partial^2 G}{\partial y^2}+2\beta_1\frac{\partial G}{\partial x}+\frac{2\beta_2}{\lambda_2}\frac{\partial G}{\partial y}=-\frac{2\pi\exp(-\beta_1 x_0-\beta_2 y_0)P_2}{\sqrt{\lambda_2}}\delta(x-x_0)\delta(y-y_0),
\end{aligned}\tag{11}$$

where

$$\begin{aligned}
P_1&=\frac{\exp(-\beta_1 x_0-\beta_2 y_0)(\sigma-\lambda_1)\in_2^{02}Q}{2\pi\eta\kappa_2^0\sqrt{\lambda_1}\delta_1},\\
P_2&=\frac{\exp(-\beta_1 x_0-\beta_2 y_0)(\sigma-\lambda_2)\in_2^{02}Q}{2\pi\eta\kappa_2^0\sqrt{\lambda_2}\delta_2}.
\end{aligned}\tag{12}$$

Equations (11) can be equivalently expressed by

$$\begin{aligned}
&\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial\left(\sqrt{\lambda_1}y\right)^2}+2\beta_1\frac{\partial F}{\partial x}+\frac{2\beta_2}{\sqrt{\lambda_1}}\frac{\partial F}{\partial\left(\sqrt{\lambda_1}y\right)}\\
&\qquad=-2\pi\exp(-\beta_1 x_0-\beta_2 y_0)P_1\delta(x-x_0)\delta\left(\sqrt{\lambda_1}y-\sqrt{\lambda_1}y_0\right),\\
&\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial\left(\sqrt{\lambda_2}y\right)^2}+2\beta_1\frac{\partial G}{\partial x}+\frac{2\beta_2}{\sqrt{\lambda_2}}\frac{\partial G}{\partial\left(\sqrt{\lambda_2}y\right)}\\
&\qquad=-2\pi\exp(-\beta_1 x_0-\beta_2 y_0)P_2\delta(x-x_0)\delta\left(\sqrt{\lambda_2}y-\sqrt{\lambda_2}y_0\right).
\end{aligned}\tag{13}$$

In order to solve equations (13), we further introduce two new functions such that

$$F=\exp(-\beta_1 x-\beta_2 y)f,\qquad G=\exp(-\beta_1 x-\beta_2 y)g.\tag{14}$$

As a result, equations (13) can be expressed into two inhomogeneous Helmholtz equations for $f$ and $g$ as

$$\begin{aligned}
&\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial(\sqrt{\lambda_1}y)^2}-\rho_1^2 f=-2\pi P_1\delta(x-x_0)\delta\left(\sqrt{\lambda_1}y-\sqrt{\lambda_1}y_0\right),\\
&\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial\left(\sqrt{\lambda_2}y\right)^2}-\rho_2^2 g=-2\pi P_2\delta(x-x_0)\delta\left(\sqrt{\lambda_2}y-\sqrt{\lambda_2}y_0\right),
\end{aligned}\tag{15}$$

where

$$\rho_1=\sqrt{\beta_1^2+\frac{\beta_2^2}{\lambda_1}},\qquad \rho_2=\sqrt{\beta_1^2+\frac{\beta_2^2}{\lambda_2}}.\tag{16}$$

The solutions to equations (15) can now be conveniently given by

$$\begin{aligned} f &= P_1 K_0\left[\rho_1\sqrt{(x-x_0)^2+\lambda_1(y-y_0)^2}\right], \\ g &= P_2 K_0\left[\rho_2\sqrt{(x-x_0)^2+\lambda_2(y-y_0)^2}\right], \end{aligned} \tag{17}$$

where $K_n$ is the $n$th-order modified Bessel function of the second kind.

In view of equations (10), (14) and (17), the electric potential and the temperature induced by the line heat source can be determined as

$$\begin{aligned} \frac{\Phi}{Q} &= \frac{\in_2^{03}(\sigma-\lambda_1)(\lambda_1-\in)}{2\pi\eta\kappa_2^0\sqrt{\lambda_1}\delta_1}\frac{K_0\left[\rho_1\sqrt{(x-x_0)^2+\lambda_1(y-y_0)^2}\right]}{\exp\left[\beta_1(x+x_0)+\beta_2(y+y_0)\right]} \\ &\quad+\frac{\in_2^{03}(\sigma-\lambda_2)(\lambda_2-\in)}{2\pi\eta\kappa_2^0\sqrt{\lambda_2}\delta_2}\frac{K_0\left[\rho_2\sqrt{(x-x_0)^2+\lambda_2(y-y_0)^2}\right]}{\exp\left[\beta_1(x+x_0)+\beta_2(y+y_0)\right]}, \\ \frac{T}{Q} &= \frac{\sigma_2^0\in_2^{02}(\sigma-\lambda_1)^2}{2\pi\eta\kappa_2^0\sqrt{\lambda_1}\delta_1}\frac{K_0\left[\rho_1\sqrt{(x-x_0)^2+\lambda_1(y-y_0)^2}\right]}{\exp\left[\beta_1(x+x_0)+\beta_2(y+y_0)\right]} \\ &\quad+\frac{\sigma_2^0\in_2^{02}(\sigma-\lambda_2)^2}{2\pi\eta\kappa_2^0\sqrt{\lambda_2}\delta_2}\frac{K_0\left[\rho_2\sqrt{(x-x_0)^2+\lambda_2(y-y_0)^2}\right]}{\exp\left[\beta_1(x+x_0)+\beta_2(y+y_0)\right]}. \end{aligned} \tag{18}$$

When $\beta_1=\beta_2=0$ for the homogeneous material, the present solutions are reduced to those in Nayfeh *et al* [2] by noticing the following asymptotic behavior for $K_0(x)$

$$K_0(x)\to-\ln(x/2)-\gamma,\quad \text{when}\quad x\to 0^+ \tag{19}$$

with $\gamma=0.577\,21$ being the Euler constant.

Once we obtain the expressions for the electric potential and temperature, it is not difficult to derive the electrical current density and heat flux vectors as follows:

$$\begin{aligned} j_1 &= \frac{Q\sigma_2^0\in_2^{03}(\sigma-\lambda_1)(\sigma-\in)}{2\pi\eta\kappa_2^0\sqrt{\lambda_1}\delta_1}Y_1(x,y,\lambda_1,\rho_1)+\frac{Q\sigma_2^0\in_2^{03}(\sigma-\lambda_2)(\sigma-\in)}{2\pi\eta\kappa_2^0\sqrt{\lambda_2}\delta_2}Y_1(x,y,\lambda_2,\rho_2), \\ j_2 &= \frac{Q\sigma_2^0\in_2^{03}(\sigma-\lambda_1)(\sigma-\in)}{2\pi\eta\kappa_2^0\sqrt{\lambda_1}\delta_1}Y_2(x,y,\lambda_1,\rho_1)+\frac{Q\sigma_2^0\in_2^{03}(\sigma-\lambda_2)(\sigma-\in)}{2\pi\eta\kappa_2^0\sqrt{\lambda_2}\delta_2}Y_2(x,y,\lambda_2,\rho_2), \end{aligned} \tag{20}$$

$$\begin{aligned} h_1 &= \frac{Q\sigma_2^0\in_2^{02}(\sigma-\lambda_1)\left[\sigma-\lambda_1+\eta(\lambda_1-\in)\right]}{2\pi\eta\sqrt{\lambda_1}\delta_1}Y_1(x,y,\lambda_1,\rho_1) \\ &\quad+\frac{Q\sigma_2^0\in_2^{02}(\sigma-\lambda_2)\left[\sigma-\lambda_2+\eta(\lambda_2-\in)\right]}{2\pi\eta\sqrt{\lambda_2}\delta_2}Y_1(x,y,\lambda_2,\rho_2), \\ h_2 &= \frac{Q\sigma_2^0\in_2^{02}(\sigma-\lambda_1)\left[\sigma-\lambda_1+\eta(\lambda_1-\in)\right]}{2\pi\eta\sqrt{\lambda_1}\delta_1}Y_2(x,y,\lambda_1,\rho_1) \\ &\quad+\frac{Q\sigma_2^0\in_2^{02}(\sigma-\lambda_2)\left[\sigma-\lambda_2+\eta(\lambda_2-\in)\right]}{2\pi\eta\sqrt{\lambda_2}\delta_2}Y_2(x,y,\lambda_2,\rho_2), \end{aligned} \tag{21}$$

where the functions $Y_1$ and $Y_2$ are defined as

$$
\begin{aligned}
Y_1(x, y, \lambda, \rho) &= \frac{\beta_1 \lambda K_0\left[\rho\sqrt{(x-x_0)^2+\lambda(y-y_0)^2}\right]}{\exp\left[-\beta_1(x-x_0)-\beta_2(y-y_0)\right]} \\
&\quad + \frac{\rho\lambda(x-x_0)K_1\left[\rho\sqrt{(x-x_0)^2+\lambda(y-y_0)^2}\right]}{\exp\left[-\beta_1(x-x_0)-\beta_2(y-y_0)\right]\sqrt{(x-x_0)^2+\lambda(y-y_0)^2}}, \\
Y_2(x, y, \lambda, \rho) &= \frac{\beta_2 K_0\left[\rho\sqrt{(x-x_0)^2+\lambda_1(y-y_0)^2}\right]}{\exp\left[-\beta_1(x-x_0)-\beta_2(y-y_0)\right]} \\
&\quad + \frac{\rho\lambda(y-y_0)K_1\left[\rho\sqrt{(x-x_0)^2+\lambda(y-y_0)^2}\right]}{\exp\left[-\beta_1(x-x_0)-\beta_2(y-y_0)\right]\sqrt{(x-x_0)^2+\lambda(y-y_0)^2}}.
\end{aligned}
\tag{22}
$$

The magnetic field can be obtained from Maxwell's equation $\nabla \times \mathbf{H} = \mathbf{j}$ by integration. For the current 2D problem, we have

$$
H_z = \int j_1 \mathrm{d}y \quad \text{or} \quad H_z = -\int j_2 \mathrm{d}x, \tag{23}
$$

which can be obtained by simple numerical quadrature. It is interesting to point out that if the anisotropic effect is ignored by letting $\sigma, \in, \kappa \to 1$, the inhomogeneity of the material as described by equation (3) will not produce any nonvanishing thermoelectric current distribution and the associated nonvanishing thermoelectric magnetic field. This fact can be easily observed from the previous theoretical development.

This analytical solution can be utilized to study the effect of the grading composition on the material behavior/response under different loadings, with the results being applied as guidelines in noncontacting and nondestructive evaluation of materials. However, in order to verify the correctness of the derived solution, we now present the complex variable method for the corresponding anisotropic but homogeneous ($\beta_1 = \beta_2 = 0$) material case. It is shown that the solution based on the complex variable method is elegant. Furthermore, the corresponding bimaterial case can be also obtained in a very concise form. Our solution contains many previous results as special cases, and in section 4, numerical results will be presented for both grading and homogeneous material cases based on the two different solution approaches. The effect of the material grading composition and anisotropy is further discussed, showing clearly the importance of them to the material response.

## 3. Complex variable formulation for homogeneous anisotropic thermoelectric materials and its applications

In the following discussions, we will first present the basic complex variable formulation for 2D problems in homogeneous anisotropic thermoelectric materials. Then we derive the explicit solutions for a line heat source in a homogeneous anisotropic thermoelectric material and in one of the two bonded anisotropic thermoelectric half-planes, demonstrating the proposed complex variable method.

### 3.1. Complex variable formulation

For homogeneous materials ($\beta_1 = \beta_2 = 0$) in the absence of a line source, it is found from equation (4) that

$$\begin{bmatrix} \sigma_1 & \epsilon_1 \\ \epsilon_1 & \dfrac{\kappa \epsilon_2^2}{\eta\sigma_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 \Phi}{\partial x^2} \\ \dfrac{\partial^2 T}{\partial x^2} \end{bmatrix} + \begin{bmatrix} \sigma_2 & \epsilon_2 \\ \epsilon_2 & \dfrac{\epsilon_2^2}{\eta\sigma_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 \Phi}{\partial y^2} \\ \dfrac{\partial^2 T}{\partial y^2} \end{bmatrix} = 0. \tag{24}$$

If we further introduce two functions $F$ and $G$ defined in equation (10), it is found that

$$\frac{\partial^2 F}{\partial x^2} + \frac{1}{\lambda_1}\frac{\partial^2 F}{\partial y^2} = 0, \qquad \frac{\partial^2 G}{\partial x^2} + \frac{1}{\lambda_2}\frac{\partial^2 G}{\partial y^2} = 0, \tag{25}$$

whose general solutions can be conveniently given by

$$F = \mathrm{Im}\{f_1(z_1)\}, \qquad G = \mathrm{Im}\{f_2(z_2)\}, \tag{26}$$

where $z_1 = x + \mathrm{i}\sqrt{\lambda_1}y$ and $z_2 = x + \mathrm{i}\sqrt{\lambda_2}y$. Consequently the electric potential and temperature can be expressed as

$$\mathbf{u} = \begin{bmatrix} \Phi \\ T \end{bmatrix} = \mathbf{A}\,\mathrm{Im}\,\{\mathbf{f}(z)\}\,, \tag{27}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \\ \mathbf{f}(z) &= \begin{bmatrix} f_1(z_1) & f_2(z_2) \end{bmatrix}^T. \end{aligned} \tag{28}$$

On the other hand, in the absence of source, we can introduce two functions $\phi_1$ and $\phi_2$ such that

$$j_1 = -\frac{\partial \phi_1}{\partial y}, \quad j_2 = \frac{\partial \phi_1}{\partial x}, \quad h_1 = -\frac{\eta\sigma_2\kappa_2}{\epsilon_2^2}\frac{\partial \phi_2}{\partial y}, \quad h_2 = \frac{\eta\sigma_2\kappa_2}{\epsilon_2^2}\frac{\partial \phi_2}{\partial x}. \tag{29}$$

As a result, it follows that the two functions $\phi_1$ and $\phi_2$ can also be concisely expressed in terms of the analytic function vector $\mathbf{f}(z)$ as

$$\varphi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \mathbf{B}\,\mathrm{Re}\{\mathbf{f}(z)\}, \tag{30}$$

where

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} \sigma_2 & \epsilon_2 \\ \epsilon_2 & \dfrac{\epsilon_2^2}{\eta\sigma_2} \end{bmatrix} \mathbf{A} \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{\lambda_1}\sigma_2 \epsilon_2 (\sigma - \epsilon) & \sqrt{\lambda_2}\sigma_2 \epsilon_2 (\sigma - \epsilon) \\ \sqrt{\lambda_1}\, \epsilon_2^2 \left(\lambda_1 - \epsilon + \dfrac{\sigma - \lambda_1}{\eta}\right) & \sqrt{\lambda_2}\, \epsilon_2^2 \left(\lambda_2 - \epsilon + \dfrac{\sigma - \lambda_2}{\eta}\right) \end{bmatrix}. \end{aligned} \tag{31}$$

The electrical current density and heat flux vectors can be obtained from equations (29) and (30) as

$$\begin{bmatrix} j_1 \\ h_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\eta\sigma_2\kappa_2}{\epsilon_2^2} \end{bmatrix} \mathbf{B} \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \mathrm{Im}\{\mathbf{f}'(z)\}, \qquad \begin{bmatrix} j_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\eta\sigma_2\kappa_2}{\epsilon_2^2} \end{bmatrix} \mathbf{B}\,\mathrm{Re}\{\mathbf{f}'(z)\}. \tag{32}$$

Due to the fact that $\nabla \times \mathbf{H} = \mathbf{j}$, or equivalently

$$\frac{\partial H_z}{\partial y} = j_1, \qquad \frac{\partial H_z}{\partial x} = -j_2, \tag{33}$$

then the nonzero magnetic field component $H_z$ can also be concisely expressed in terms of $\mathbf{f}(z)$ as

$$H_z = -\phi_1 = -\begin{bmatrix}1 & 0\end{bmatrix} \mathbf{B}\,\mathrm{Re}\{\mathbf{f}(z)\} = \sigma_2 \in_2 (\in - \sigma) \begin{bmatrix}\sqrt{\lambda_1} & \sqrt{\lambda_2}\end{bmatrix} \mathrm{Re}\{\mathbf{f}(z)\}. \tag{34}$$

In addition it is found that the matrix $\mathbf{AB}^{-1}$ is symmetric, real and positive definite given by

$$\mathbf{AB}^{-1} = (\mathbf{AB}^{-1})^T = \sum_{k=1}^{2} \left( \begin{bmatrix} \dfrac{\in_2^2 (\lambda_k - \in)^2}{\delta_k \sqrt{\lambda_k}} & \dfrac{\sigma_2 \in_2 (\lambda_k - \in)(\sigma - \lambda_k)}{\delta_k \sqrt{\lambda_k}} \\ \dfrac{\sigma_2 \in_2 (\lambda_k - \in)(\sigma - \lambda_k)}{\delta_k \sqrt{\lambda_k}} & \dfrac{\sigma_2^2 (\sigma - \lambda_k)^2}{\delta_k \sqrt{\lambda_k}} \end{bmatrix} \right). \tag{35}$$

### 3.2. Applications

In the following, we apply the derived complex variable formulation to two interesting problems in order to demonstrate its versatility.

*3.2.1. A steady line heat source in a homogeneous thermoelectric material.* We first consider a steady line heat source of strength $Q$ located at the origin in a homogeneous material. It follows from equation (30) that

$$\mathbf{f}(z) = \frac{\mathrm{i} \in_2^2 Q}{2\pi \eta \sigma_2 \kappa_2} \langle \ln(z_\alpha) \rangle \mathbf{B}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{36}$$

where $\langle \ln(z_\alpha) \rangle = \mathrm{diag}[\ln(z_1) \quad \ln(z_2)]$.

Consequently, the electric potential and temperature can be obtained from equation (27), whereas the electrical current density and heat flux vectors can be obtained from equations (32). Particularly it follows from equation (34) that

$$H_z = \frac{Q \in_2 (\in - \sigma)}{2\pi \kappa_2 (\lambda_1 - \lambda_2)(1 - \eta)} \left[ \tan^{-1}\left( \frac{\sqrt{\lambda_1} y}{x} \right) - \tan^{-1}\left( \frac{\sqrt{\lambda_2} y}{x} \right) \right], \tag{37}$$

which is just the result obtained by Nayfeh *et al* [2]. Thus the correctness of the complex variable method is verified. In addition, it is observed that the solution procedure presented here is very simple as compared to previous methods in this field.

*3.2.2. A steady line heat source in anisotropic thermoelectric bimaterials.* Next, we consider a steady line heat source of strength $Q$ located at $(0, d)$, $(d > 0)$ in the upper one of the two bonded different anisotropic half-planes $y > 0$ (#1) and $y < 0$ (#2). It is assumed that the principal directions for both half-planes are parallel to the $x$- and $y$-axes. In addition, the superscripts (1) and (2) are attached to scalars in the upper and lower half-planes, respectively; whereas the subscripts 1 and 2 are attached to matrices/vectors in the upper and lower half-planes, respectively. In the following derivations, we will first replace the complex variables $z_1$ and $z_2$ by the common complex variable $z = x + \mathrm{i}y$ due to the fact that $z_1 = z_2 = z$ on the real axis [19].

When the analysis is finished, the complex variable $z = x + \mathrm{i}y$ shall be changed back to the corresponding complex variables $z_1$ and $z_2$.

We assume that the interface between the two half-planes is perfect, i.e. [1]

$$\Phi^{(1)} = \Phi^{(2)}, \quad T^{(1)} = T^{(2)}, \quad j_2^{(1)} = j_2^{(2)}, \quad h_2^{(1)} = h_2^{(2)}, \quad y = 0. \tag{38}$$

The above continuity conditions on the interface $y = 0$ can also be equivalently expressed as

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\varphi}_1 = \boldsymbol{\Lambda}\boldsymbol{\varphi}_2, \quad y = 0, \tag{39}$$

where $\boldsymbol{\Lambda}$ is a $2 \times 2$ diagonal matrix defined by

$$\boldsymbol{\Lambda} = \mathrm{diag}\begin{bmatrix} 1 & \dfrac{\eta^{(2)}\sigma_2^{(2)}\kappa_2^{(2)}\ \epsilon_2^{(1)2}}{\eta^{(1)}\sigma_2^{(1)}\kappa_2^{(1)}\ \epsilon_2^{(2)2}} \end{bmatrix}. \tag{40}$$

In view of equations (27) and (30), the interfacial continuity conditions in equations (39) can also be expressed in terms of the two analytic function vectors $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ as

$$\left.\begin{aligned} \mathbf{A}_1\mathbf{f}_1^+(x) - \mathbf{A}_1\bar{\mathbf{f}}_1^-(x) &= \mathbf{A}_2\mathbf{f}_2^-(x) - \mathbf{A}_2\bar{\mathbf{f}}_2^+(x), \\ \mathbf{B}_1\mathbf{f}_1^+(x) + \mathbf{B}_1\bar{\mathbf{f}}_1^-(x) &= \boldsymbol{\Lambda}\mathbf{B}_2\mathbf{f}_2^-(x) + \boldsymbol{\Lambda}\mathbf{B}_2\bar{\mathbf{f}}_2^+(x), \end{aligned}\right. \quad \text{on } y = 0. \tag{41}$$

It follows from equation $(41)_1$ that

$$\begin{aligned} \mathbf{f}_1(z) &= -\mathbf{A}_1^{-1}\mathbf{A}_2\bar{\mathbf{f}}_2(z) + \mathbf{f}_0(z) + \bar{\mathbf{f}}_0(z), \\ \bar{\mathbf{f}}_1(z) &= -\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{f}_2(z) + \mathbf{f}_0(z) + \bar{\mathbf{f}}_0(z), \end{aligned} \tag{42}$$

where $\mathbf{f}_0(z)$ is the analytic function vector for a line heat source located at $(0, d)$, $(d > 0)$ in a homogeneous infinite plane given by

$$\mathbf{f}_0(z) = \frac{\mathrm{i}\,\epsilon_2^{(1)2}\,Q}{2\pi\,\eta^{(1)}\sigma_2^{(1)}\kappa_2^{(1)}}\left\langle \ln\left(z - \mathrm{i}\sqrt{\lambda_\alpha}d\right)\right\rangle \mathbf{B}_1^{-1}\begin{bmatrix}0\\1\end{bmatrix}. \tag{43}$$

Substituting equations (42) into equations $(41)_2$, we finally obtain

$$\mathbf{f}_2(z) = 2\mathbf{B}_2^{-1}(\mathbf{A}_1\mathbf{B}_1^{-1}\boldsymbol{\Lambda} + \mathbf{A}_2\mathbf{B}_2^{-1})^{-1}\mathbf{A}_1\mathbf{f}_0(z). \tag{44}$$

Consequently, we can derive the expression of $\mathbf{f}_1(z)$ as

$$\mathbf{f}_1(z) = \mathbf{f}_0(z) + \mathbf{A}_1^{-1}(\mathbf{A}_1\mathbf{B}_1^{-1}\boldsymbol{\Lambda} - \mathbf{A}_2\mathbf{B}_2^{-1})(\mathbf{A}_1\mathbf{B}_1^{-1}\boldsymbol{\Lambda} + \mathbf{A}_2\mathbf{B}_2^{-1})^{-1}\mathbf{A}_1\bar{\mathbf{f}}_0(z). \tag{45}$$

Therefore, the full-field expressions of $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ are obtained as follows:

$$\mathbf{f}_1(z) = \frac{\mathrm{i}\,\epsilon_2^{(1)2}\,Q}{2\pi\,\eta^{(1)}\sigma_2^{(1)}\kappa_2^{(1)}}\left[\left\langle \ln\left(z_\alpha - \mathrm{i}\sqrt{\lambda_\alpha}d\right)\right\rangle + \sum_{k=1}^{2}\left\langle \ln\left(z_\alpha + \mathrm{i}\sqrt{\lambda_k}d\right)\right\rangle \mathbf{M}\mathbf{I}_k\right]\mathbf{B}_1^{-1}\begin{bmatrix}0\\1\end{bmatrix}, \tag{46}$$

$$\mathbf{f}_2(z) = \frac{\mathrm{i}\,\epsilon_2^{(1)2}\,Q}{2\pi\,\eta^{(1)}\sigma_2^{(1)}\kappa_2^{(1)}}\sum_{k=1}^{2}\left\langle \ln\left(z_\alpha^* - \mathrm{i}\sqrt{\lambda_k}d\right)\right\rangle \mathbf{N}\mathbf{I}_k\mathbf{B}_1^{-1}\begin{bmatrix}0\\1\end{bmatrix}, \tag{47}$$

where the superscript '*' is utilized to distinguish the eigenvalues associated with the lower half-plane from those associated with the upper half-plane, and

$$\mathbf{I}_1 = \mathrm{diag}\begin{bmatrix}1 & 0\end{bmatrix}, \quad \mathbf{I}_2 = \mathrm{diag}\begin{bmatrix}0 & 1\end{bmatrix}, \tag{48}$$

$$\begin{aligned}\mathbf{M} &= \mathbf{A}_1^{-1}(\mathbf{A}_2\mathbf{B}_2^{-1} - \mathbf{A}_1\mathbf{B}_1^{-1}\mathbf{\Lambda})(\mathbf{A}_2\mathbf{B}_2^{-1} + \mathbf{A}_1\mathbf{B}_1^{-1}\mathbf{\Lambda})^{-1}\mathbf{A}_1,\\ \mathbf{N} &= 2\mathbf{B}_2^{-1}(\mathbf{A}_2\mathbf{B}_2^{-1} + \mathbf{A}_1\mathbf{B}_1^{-1}\mathbf{\Lambda})^{-1}\mathbf{A}_1.\end{aligned} \tag{49}$$

It follows from equation (34) that the induced magnetic field in the upper half-plane $y > 0$ is given by

$$\begin{aligned}H_z^{(1)} = \frac{\in_2^{(1)2} Q}{2\pi\eta^{(1)}\sigma_2^{(1)}\kappa_2^{(1)}}\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{B}_1\Bigg[\left\langle \tan^{-1}\left(\frac{\sqrt{\lambda_\alpha}(y-d)}{x}\right)\right\rangle\\ + \sum_{k=1}^{2}\left\langle \tan^{-1}\left(\frac{\sqrt{\lambda_\alpha}y + \sqrt{\lambda_k}d}{x}\right)\right\rangle \mathbf{M}\mathbf{I}_k\Bigg]\mathbf{B}_1^{-1}\begin{bmatrix}0\\1\end{bmatrix},\end{aligned} \tag{50}$$

whereas that in the lower half-plane $y < 0$ is given by

$$H_z^{(2)} = \frac{\in_2^{(1)2} Q}{2\pi\eta^{(1)}\sigma_2^{(1)}\kappa_2^{(1)}}\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{B}_2\sum_{k=1}^{2}\left\langle \tan^{-1}\left(\frac{\sqrt{\lambda_\alpha^*}y - \sqrt{\lambda_k}d}{x}\right)\right\rangle \mathbf{N}\mathbf{I}_k\mathbf{B}_1^{-1}\begin{bmatrix}0\\1\end{bmatrix}. \tag{51}$$

## 4. Numerical examples

In this section, we first consider the following material properties:

$$\begin{aligned}&\sigma_2^0 = 5.7\times 10^5\,\mathrm{A\,mV^{-1}}, && \kappa_2^0 = 7.3\,\mathrm{W\,(m\,^\circ C)^{-1}}, && \in_2^0 = -2.793\,\mathrm{A\,(m\,^\circ C)^{-1}},\\ &\sigma = 1.023, && \kappa = 1.01, && \in = 1.055,\end{aligned}$$

which are typical values of homogeneous Ti–6Al–4V, the popular aerospace titanium alloy. In addition, we take the thermoelectric coupling parameter $\eta = 10^{-2}$. Figure 1 demonstrates the distributions of the electric current density component $j_1$ along the $x$-axis ($y = 0$) induced by a line heat source of unit strength ($Q = 1$) located at the origin for different values of the gradient parameter $\beta_1 = -5, -1, 0, 1, 5\ (\mathrm{m^{-1}})$ with $\beta_2 = 0$ (in this way the material is gradient along the $x$-direction). It is observed from this figure that the induced $j_1$ for nonzero $\beta_1$ is no longer anti-symmetric with respect to the origin. The magnitude of $j_1$ at $x (> 0)$ is greater (or lower) than that at $-x$ for $\beta_1 > 0$ (or $\beta_1 < 0$). In addition the magnitude difference becomes larger when the absolute value of $\beta_1$ increases. Therefore, the material property gradient in an anisotropic material can exert a significant influence on the induced thermoelectric currents, and consequently on the magnetic field. Also shown in figure 1 is the electric current density component $j_1$ for the homogeneous but anisotropic material using equations (32) and (36) based on the complex-variable method (in open circles). It is clear that for this case, both solutions (eigenvalue/eigenvector separation based and complex-variable based) predict exactly the same results, which partially and mutually verify the correctness of the derived solutions.

Next, we present in figure 2 the distribution of the electric current density component $j_1$ along the $x$-axis ($y = 0$) induced by a line heat source of unit strength ($Q = 1$) located at the origin for different combinations of the material property ratios $\sigma$, $\kappa$ and $\in$ with $\beta_1 = 1\,\mathrm{m^{-1}}$ and $\beta_2 = 0$. The values of $\sigma_2^0$, $\kappa_2^0$, $\in_2^0$ and $\eta$ are also the same as before. It is clearly observed that the magnitude of $j_1$ decreases as the material anisotropic effect becomes weak (i.e. the ratios $\sigma$, $\kappa$ and $\in$ are close to 1) and that there is no induced electric current density when the material is isotropic ($\sigma = \kappa = \in = 1$) even if it is inhomogeneous.

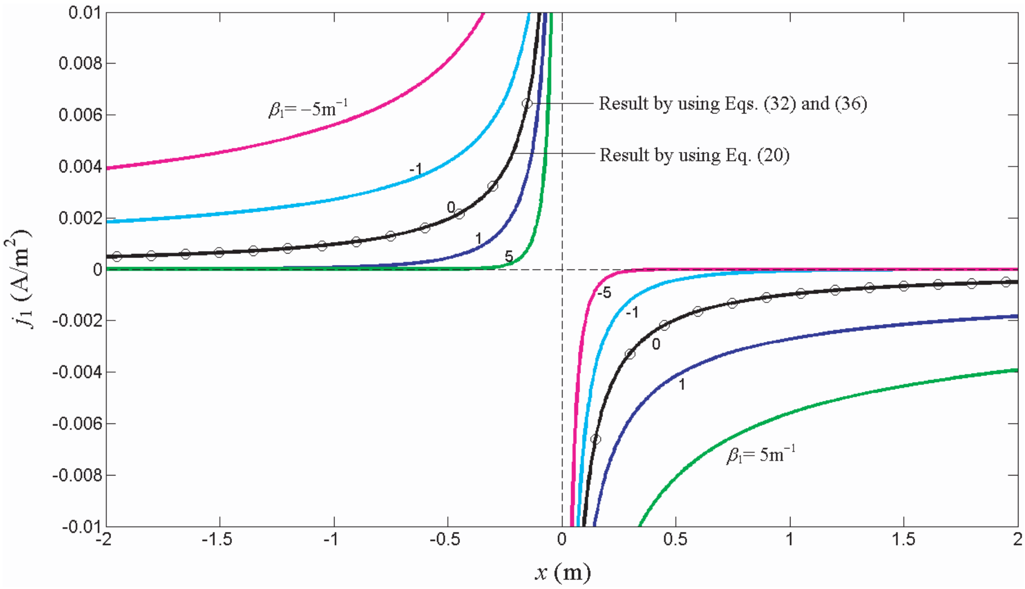


**Figure 1.** Distributions of the thermoelectric current density component $j_1$ along the $x$-axis ($y = 0$) induced by a line heat source of unit strength ($Q = 1$) located at the origin for different values of the gradient parameter $\beta_1 = -5, -1, 0, 1$ and $5$ (m$^{-1}$) with $\beta_2 = 0$. The result for the homogeneous material case ($\beta_1 = \beta_2 = 0$) is the same as that based on the complex variable method (equations (32) and (36)) shown with open circles.

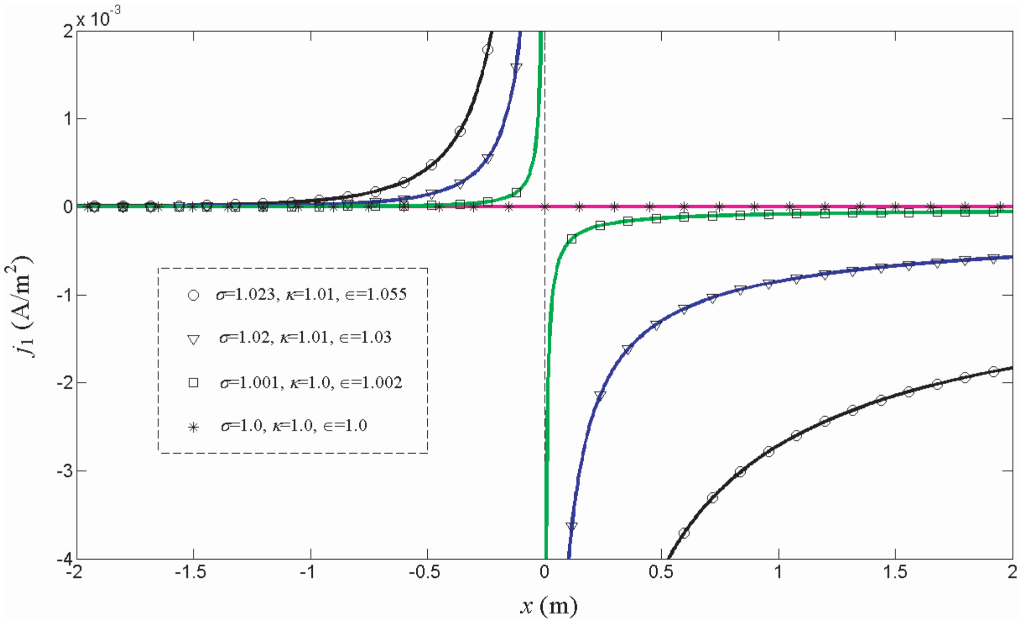


**Figure 2.** Distribution of the thermoelectric current density component $j_1$ along the $x$-axis ($y = 0$) induced by a line heat source of unit strength ($Q = 1$) located at the origin for different combinations of the material property ratios $\sigma$, $\kappa$ and $\in$ with $\beta_1 = 1\,\text{m}^{-1}$ and $\beta_2 = 0$.

## 5. Conclusion

In this research, we presented analytical expressions of the electric potential, temperature, electric current densities and thermal fluxes due to a steady line heat source in an exponentially gradient and anisotropic thermoelectric material by introducing the eigenvalue/eigenvector separation approach. We also developed an elegant complex variable formulation to study 2D problems in the corresponding anisotropic but homogeneous thermoelectric material, which was also utilized to verify the eigenvalue-based solutions for the special case (i.e. when the material is homogeneous). The correctness of the developed solutions was further verified by reducing to the existing solutions for some special cases. Our numerical results clearly indicate the effect of both material anisotropy and gradient on the induced thermoelectric current density and thus the magnetic fields to be detected by the magnetometer. Recent theoretical and experimental studies have shown that the material anisotropy is required in this special noncontacting nondestructive evaluation approach. However, since most materials are both anisotropic and inhomogeneous, the spurious signal from material grading as well as anisotropy has to be clearly separated from the true signal due to the material flaw. The solutions presented can be directly used to calculate the background signal for given material anisotropy and grading (in terms of exponential variation), and therefore, comparing this signal with the detected signal by the magnetometer will help to identify potential material flaws in the specimen.

Besides their direct application to nondestructive evaluation, it is expected that the solutions developed in this research can also be applied to other practical 2D problems (for example, a crack on the interface between two anisotropic half-planes, or an elliptical anisotropic cylinder embedded in another anisotropic matrix). Results of these problems, which are pertinent to noncontacting thermoelectric NDT, will be reported later.

## Acknowledgments

This work was supported in part by AFOSR FA9550-06-1-0317. We thank the editor and the reviewers for their constructive comments.

## References

[1] Nagy P B and Nayfeh A H 2000 *J. Appl. Phys.* **87** 7481
[2] Nayfeh A H, Carreon H and Nagy P B 2002 *J. Appl. Phys.* **91** 225
[3] Carreon H, Lakshminarayan B, Faidi W I, Nayfeh A H and Nagy P B 2003 *NDT&E Int.* **36** 339
[4] Bies W E, Radtke R J and Ehrenreich H 2002 *Phys. Rev.* B **65** 085208
[5] Hsu K F, Loo S, Guo F, Chen W, Dyck J S, Uher C, Hogan T, Polychroniadis E K and Kanatzidis M G 2004 *Science* **303** 818
[6] Quarez E, Hsu K F, Pcionek R, Frangis N, Polychroniadis E K and Kanatzidis M G 2005 *J. Am. Chem. Soc.* **127** 9177
[7] Dresselhaus M S, Chen G, Tang M Y, Yang R G, Lee H, Wang D Z, Ren Z F, Fleurial J P and Gognan P 2007 *Adv. Mater.* **19** 1043
[8] Boukai A I, Bunimovich Y, Tahir-Kheli J, Yu J K, Goddard W A and Heath J R 2008 *Nature* **451** 168
[9] Hochbaum A I, Chen R K, Delgado R D, Liang W J, Garnett E C, Najarian M, Majumdar A and Yang P D 2008 *Nature* **451** 163
[10] Snyder G J and Toberer E S 2008 *Nat. Mater.* **7** 105

[11] Hinken J H and Tavrin Y 2000 *Review of Progress in Quantitative Nondestructive Evaluation* vol 19 (Melville, NY: AIP) pp 2085–92

[12] Maslov K and Kinra V K 2001 *Mater. Eval.* **59** 1081

[13] Suresh S 2001 *Science* **292** 2447

[14] Dong L, Gu G Q and Yu K W 2003 *Phys. Rev.* B **67** 224205

[15] Chan Y S, Gray L J, Kaplan T and Paulino G H 2004 *Proc. R. Soc. Lond.* A **460** 1689

[16] Pan E and Han F 2005 *Int. J. Solids Struct.* **42** 3207

[17] Sutradhar A and Paulino G H 2004 *Int. J. Numer. Methods Eng.* **60** 2203

[18] Collet B, Destrade M and Maugin G A 2006 *Eur. J. Mech.* A **25** 695

[19] Wang X, Pan E and Feng W J 2007 *Eur. J. Mech.* A **26** 901