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# On Partially Debonded Circular Inclusions in Finite Plane Elastostatics of Harmonic Materials 


#### Abstract

We investigate a partially debonded circular elastic inclusion embedded in a particular class of harmonic materials by using the complex variable method under finite planestrain deformations. A complete (or full-field) solution is derived. It is observed that the stresses in general exhibit oscillatory singularities near the two tips of the arc shaped interface crack. Particularly the traditional inverse square root singularity for stresses is observed when the asymptotic behavior of the harmonic materials obeys a constitutive restriction proposed by Knowles and Sternberg (1975, "On the Singularity Induced by Certain Mixed Boundary Conditions in Linearized and Nonlinear Elastostatics," Int. J. Solids Struct., 11, pp. 1173-1201). It is also found that the number of admissible states under finite plane deformations for given external loads can be two, one, or even zero. [DOI: 10.1115/1.3000023]


Keywords: finite deformation, circular inclusion, arc shaped crack

## 1 Introduction

The problem of a circular arc shaped crack lying along the interface of an elastic inclusion is a classical one and has received considerable amount of attention [1-4]. In these studies, the mixed-boundary value problem was formulated on the basis of the complex variable approach and was finally reduced to an inhomogeneous Riemann-Hilbert problem whose exact solution can be easily derived. These studies show that the stresses near the tips of an interface arc crack still exhibit the same oscillatory singularities as those obtained for a straight crack between dissimilar media $[5,6]$. In previous investigations [1,4], the problem of an arc shaped interfacial crack was solved within the framework of linear elastostatics. In contrast, an exact solution to the analogous problem in finite elasticity is still absent.

An elegant complex variable formulation of a class of problems involving the finite plane-strain deformations of a set of compressible hyperelastic materials of harmonic type was recently developed [7]. The complex variable formulation [7] has also been applied to (i) get a complete solution for a planar interface crack between two half-planes occupied by two dissimilar harmonic materials [7], (ii) obtain a complete solution for an elliptical inclusion with uniform interior stress field perfectly bonded to a matrix of harmonic materials under any uniform remote stress distribution [8], (iii) identify the harmonic shapes for harmonic materials [9], and (iv) analyze the surface instability of a harmonic solid attracted by a rigid body through the influence of van der Waals forces [10].

The objective of the present work is to investigate in detail a two-dimensional crack along the interface of a circular elastic inclusion embedded in an unbounded matrix of harmonic materials loaded by remote uniform Piola stresses. By using the complex variable method, the original mixed-boundary value problem is finally reduced to an inhomogeneous Riemann-Hilbert problem. It is found that the Piola stresses near the tips of the arc interface

[^0]crack still exhibit the oscillatory singularities as those obtained for a straight crack [7]. Particularly the conventional inverse square root singularity for stresses near the tips of the interface arc crack is still observed when the asymptotic behavior of the harmonic materials obeys a constitutive restriction proposed by Knowles and Sternberg [11].

## 2 Basic Formulations

Let the complex variable $z=x_{1}+i x_{2}$ be the initial coordinates of a material particle in the undeformed configuration and $w(z)$ $=y_{1}(z)+i y_{2}(z)$ be the corresponding spatial coordinates in the deformed configuration. The deformation gradient tensor is defined as

$$
\begin{equation*}
F_{i j}=\frac{\partial y_{i}}{\partial x_{j}} \tag{1}
\end{equation*}
$$

For a particular class of harmonic materials, the strain energy density $W$ defined with respect to the undeformed unit area can be expressed by

$$
\begin{equation*}
W=2 \mu[F(I)-J], \quad F^{\prime}(I)=\frac{1}{4 \alpha}\left[I+\sqrt{I^{2}-16 \alpha \beta}\right] \tag{2}
\end{equation*}
$$

Here $I$ and $J$ are the scalar invariants of $F F^{T}$ given by

$$
\begin{equation*}
I=\lambda_{1}+\lambda_{2}=\sqrt{F_{i j} F_{i j}+2 J}, \quad J=\lambda_{1} \lambda_{2}=\operatorname{det}[F] \tag{3}
\end{equation*}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are the principal stretches, $\mu$ is the shear modulus, and $1 / 2 \leq \alpha<1, \beta>0$ are two material constants. This special class of harmonic materials has attracted considerable attention [12,13].

According to the formulation developed by Ru [7], the deformation $w(z)$ can be written in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ as

$$
\begin{equation*}
i w(z)=\alpha \varphi(z)+i \overline{\psi(z)}+\frac{\beta z}{\varphi^{\prime}(z)} \tag{4}
\end{equation*}
$$

and the complex Piola stress function $\chi(z)$ is given by


Fig. 1 An arc crack along the interface between a circular inclusion and the matrix

$$
\begin{equation*}
\chi(z)=2 i \mu\left[(\alpha-1) \varphi(z)+i \overline{\psi(z)}+\frac{\beta z}{\overline{\varphi^{\prime}(z)}}\right] \tag{5}
\end{equation*}
$$

In addition, the Piola stress components can be written in terms of the Piola stress function $\chi$ as

$$
\begin{equation*}
-\sigma_{21}+i \sigma_{11}=\chi_{, 2}, \quad \sigma_{22}-i \sigma_{12}=\chi_{, 1} \tag{6}
\end{equation*}
$$

## 3 Complete Solution for a Partially Debonded Circular Inclusion

As shown in Fig. 1, we consider a circular inclusion of radius $R$ partially bonded to an infinite matrix. We take the origin at the center of the inclusion and assume that an interfacial arc crack, whose surface is traction-free, is made along the $\operatorname{arc} L_{c}$ of the interface while along the remaining arc $L_{b}$ the inclusion is still perfectly bonded to the matrix. Furthermore, let the center of the $\operatorname{arc} L_{c}$ lie on the positive $x_{1}$-axis and the central angle subtended by the $\operatorname{arc} L_{c}$ is $2 \theta_{0} . a=R e^{i \theta_{0}}$ and $b=R e^{-i \theta_{0}}$ are two crack tips. The elastic materials occupying the inclusion and the matrix belong to the special class of harmonic materials characterized by Eq. (2) with the associated elastic constants $\mu_{1}, \alpha_{1}$, and $\beta_{1}$ and $\mu_{2}, \alpha_{2}$, and $\beta_{2}$, respectively. The composite system is assumed to be under the remote uniform Piola stresses $\sigma_{11}^{\infty}, \sigma_{22}^{\infty}, \sigma_{12}^{\infty}$, and $\sigma_{21}^{\infty}$. Throughout this paper, all physical quantities associated with the circular inclusion and unbounded matrix are identified by the subscripts 1 and 2, respectively.

The continuity condition of tractions across the total interface $|z|=R$ can be expressed as

$$
\begin{align*}
& \Gamma\left(\alpha_{1}-1\right) \varphi_{1}^{+}(z)+i \Gamma \bar{\psi}_{1}^{-}\left(R^{2} / z\right)+\frac{\Gamma \beta_{1} z}{\bar{\varphi}_{1}^{\prime-}\left(R^{2} / z\right)} \\
& \quad=\left(\alpha_{2}-1\right) \varphi_{2}^{-}(z)+i \bar{\psi}_{2}^{+}\left(R^{2} / z\right)+\frac{\beta_{2} z}{\bar{\varphi}_{2}^{\prime+}\left(R^{2} / z\right)}, \quad(|z|=R) \tag{7}
\end{align*}
$$

where $\Gamma=\mu_{1} / \mu_{2}$, and the superscripts " + " and " - " denote the limit values from the inner and outer sides of the interface $|z|=R$, respectively.

It follows from Eq. (7) that

$$
\begin{align*}
& \Gamma\left(\alpha_{1}-1\right) \varphi_{1}^{+}(z)-i \bar{\psi}_{2}^{+}\left(R^{2} / z\right)-\frac{\beta_{2} z}{\bar{\varphi}_{2}^{\prime+}\left(R^{2} / z\right)} \\
& \quad=\left(\alpha_{2}-1\right) \varphi_{2}^{-}(z)-i \Gamma \bar{\psi}_{1}^{-}\left(R^{2} / z\right)-\frac{\Gamma \beta_{1} z}{\bar{\varphi}_{1}^{\prime-}\left(R^{2} / z\right)}, \quad(|z|=R) \tag{8}
\end{align*}
$$

At infinity, it is assumed that the remote Piola stresses are uniform. Then $\varphi_{2}(z)$ and $\psi_{2}(z)$ exhibit the following asymptotic behavior:

$$
\begin{equation*}
\varphi_{2}(z)=A z+o(1), \quad \psi_{2}(z)=B z+o(1), \quad|z| \rightarrow \infty \tag{9}
\end{equation*}
$$

where the two complex constants $A$ and $B$ are related to the remote Piola stresses $\sigma_{11}^{\infty}, \sigma_{22}^{\infty}, \sigma_{12}^{\infty}$, and $\sigma_{21}^{\infty}$ through the following relations:

$$
\begin{gather*}
-\sigma_{21}^{\infty}+i \sigma_{11}^{\infty}=2 \mu_{2}\left[\left(1-\alpha_{2}\right) A+i \bar{B}-\frac{\beta_{2}}{\bar{A}}\right] \\
i \sigma_{22}^{\infty}+\sigma_{12}^{\infty}=2 \mu_{2}\left[\left(1-\alpha_{2}\right) A-i \bar{B}-\frac{\beta_{2}}{\bar{A}}\right] \tag{10}
\end{gather*}
$$

In view of Eqs. (8) and (9), we now define the following new function $\Omega(z)$ as

$$
\Omega(z)= \begin{cases}\Gamma\left(\alpha_{1}-1\right) \varphi_{1}(z)-i \bar{\psi}_{2}\left(R^{2} / z\right)-\frac{\beta_{2} z}{\bar{\varphi}_{2}^{\prime}\left(R^{2} / z\right)}+\left[\frac{\Gamma \beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-\left(\alpha_{2}-1\right) A\right] z+i \bar{B} R^{2} z^{-1}, & |z|<R  \tag{11}\\ \left(\alpha_{2}-1\right) \varphi_{2}(z)-i \Gamma \bar{\psi}_{1}\left(R^{2} / z\right)-\frac{\Gamma \beta_{1} z}{\bar{\varphi}_{1}^{\prime}\left(R^{2} / z\right)}+\left[\frac{\Gamma \beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-\left(\alpha_{2}-1\right) A\right] z+i \bar{B} R^{2} z^{-1}, & |z|>R\end{cases}
$$

It is apparent that $\Omega(z)$ is continuous across the interface $|z|$ $=R$ and then analytic in the whole plane including the points at zero and at infinity. Consequently, $\Omega(z)=0$. As a result we arrive at the following relationships:

$$
\begin{align*}
i \bar{\psi}_{2}\left(R^{2} / z\right)+\frac{\beta_{2} z}{\bar{\varphi}_{2}^{\prime}\left(R^{2} / z\right)}= & \Gamma\left(\alpha_{1}-1\right) \varphi_{1}(z) \\
& +\left[\frac{\Gamma \beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-\left(\alpha_{2}-1\right) A\right] z+i \bar{B} R^{2} z^{-1}  \tag{12a}\\
i \bar{\psi}_{1}\left(R^{2} / z\right)+\frac{\beta_{1} z}{\bar{\varphi}_{1}^{\prime}\left(R^{2} / z\right)}= & \Gamma^{-1}\left(\alpha_{2}-1\right) \varphi_{2}(z) \\
& +\left[\frac{\beta_{1}}{\varphi_{1}^{\prime}(0)}-\Gamma^{-1}\left(\alpha_{2}-1\right) A\right] z+i \Gamma^{-1} \bar{B} R^{2} z^{-1} \tag{12b}
\end{align*}
$$

The traction-free condition of the cracked part $L_{c}$ of the interface can be expressed as

$$
\begin{equation*}
\left(\alpha_{1}-1\right) \varphi_{1}^{+}(z)+i \bar{\psi}_{1}^{-}\left(R^{2} / z\right)+\frac{\beta_{1} z}{\bar{\varphi}_{1}^{\prime}\left(R^{2} / z\right)}=0, \quad z \in L_{c} \tag{13}
\end{equation*}
$$

Substituting the result of Eq. (12b) into Eq. (13) yields

$$
\begin{align*}
& \left(\alpha_{1}-1\right) \varphi_{1}^{+}(z)+\Gamma^{-1}\left(\alpha_{2}-1\right) \varphi_{2}^{-}(z)+\left[\frac{\beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-\Gamma^{-1}\left(\alpha_{2}-1\right) A\right] z \\
& \quad+i \Gamma^{-1} \bar{B} R^{2} z^{-1}=0, \quad z \in L_{c} \tag{14}
\end{align*}
$$

In view of Eq. (14), we introduce an auxiliary function $h(z)$ defined by
$h(z)$

$$
= \begin{cases}\left(\alpha_{1}-1\right) \varphi_{1}(z)+\xlongequal[\beta_{1}^{\prime}(0)]{\beta_{1}} z, & |z|<R  \tag{15}\\ \Gamma^{-1}\left(1-\alpha_{2}\right) \varphi_{2}(z)+\Gamma^{-1}\left(\alpha_{2}-1\right) A z-i \Gamma^{-1} \bar{B} R^{2} z^{-1}, & |z|>R\end{cases}
$$

Apparently $h(z)$ is holomorphic in $|z|<R$ and $|z|>R$, respectively. $h(z)=o(1)$ as $|z| \rightarrow \infty$. Furthermore,

$$
\begin{equation*}
h^{+}(z)-h^{-}(z)=0, \quad z \in L_{c} \tag{16}
\end{equation*}
$$

The continuity condition of displacements across the bonded part $L_{b}$ of the interface can be expressed as

$$
\begin{align*}
\alpha_{1} \varphi_{1}^{+}(z)+i \bar{\psi}_{1}^{-}\left(R^{2} / z\right)+\frac{\beta_{1} z}{\bar{\varphi}_{1}^{\prime}\left(R^{2} / z\right)}= & \alpha_{2} \varphi_{2}^{-}(z)+i \bar{\psi}_{2}^{+}\left(R^{2} / z\right) \\
& +\frac{\beta_{2} z}{\bar{\varphi}_{2}^{\prime+}\left(R^{2} / z\right)}, \quad z \in L_{b} \tag{17}
\end{align*}
$$

Utilizing Eqs. (12) and (15), the above expression can be equivalently expressed in terms of $h(z)$ as

$$
\begin{align*}
h^{+}(z)+k h^{-}(z)= & \frac{\alpha_{1}-1}{\alpha_{1}-\Gamma\left(\alpha_{1}-1\right)}\left[\frac{\beta_{1}}{\left(\alpha_{1}-1\right) \overline{\varphi_{1}^{\prime}(0)}}+A\right] z \\
& -\frac{i \bar{B} R^{2}\left(\alpha_{1}-1\right)}{\left(\alpha_{2}-1\right)\left[\alpha_{1}-\Gamma\left(\alpha_{1}-1\right)\right]^{2}} z^{-1} \quad z \in L_{b} \tag{18}
\end{align*}
$$

where

$$
\begin{equation*}
k=\frac{\left(1-\alpha_{1}\right)\left[\Gamma \alpha_{2}+\left(1-\alpha_{2}\right)\right]}{\left(1-\alpha_{2}\right)\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right]}>0 \tag{19}
\end{equation*}
$$

Therefore, it is observed that the stresses near the two crack tips $a=R e^{i \theta_{0}}$ and $b=R e^{-i \theta_{0}}$ exhibit oscillatory singularities, a phenomenon in agreement with that observed for a planar interface crack [7].

If we choose $\alpha_{1}=\alpha_{2}=1 / 2$ for the situation in which $F^{\prime}(I) / I$ approaches unity as $I$ tends to infinity [7,11], then Eq. (18) simplifies to

$$
\begin{equation*}
h^{+}(z)+h^{-}(z)=\frac{1}{1+\Gamma}\left(\frac{2 \beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-A\right) z-\frac{2 i \bar{B} R^{2}}{1+\Gamma} z^{-1}, \quad z \in L_{b} \tag{20}
\end{equation*}
$$

The solution to the inhomogeneous Riemann-Hilbert problem (i.e., Eqs. (16) and (20)) can be expediently given by

$$
\begin{equation*}
h(z)=\frac{1}{1+\Gamma}\left(\frac{\beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-\frac{A}{2}\right)[z-X(z)]-\frac{i \bar{B} R^{2}}{1+\Gamma}\left[z^{-1}-z^{-1} X^{-1}(0) X(z)\right] \tag{21}
\end{equation*}
$$

where the multivalued function $X(z)=\sqrt{(z-a)(z-b)}$ is discontinuous across the bonded part $L_{b}$ of the interface, and $X(z)=z+o(1)$ as $|z| \rightarrow \infty$. It follows from the above result and Eq. (15) that

$$
\begin{array}{ll}
\varphi_{1}(z)=-\frac{2}{1+\Gamma}\left(\frac{\beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-\frac{A}{2}\right)[z-X(z)]+\frac{2 i \bar{B} R^{2}}{1+\Gamma}\left[z^{-1}-z^{-1} X^{-1}(0) X(z)\right]+\frac{2 \beta_{1}}{\varphi_{1}^{\prime}(0)} z, & (|z|<R)  \tag{22}\\
\varphi_{2}(z)=\frac{2 \Gamma}{1+\Gamma}\left(\frac{\beta_{1}}{\overline{\varphi_{1}^{\prime}(0)}}-\frac{A}{2}\right)[z-X(z)]-\frac{2 i \Gamma \bar{B} R^{2}}{1+\Gamma}\left[z^{-1}-z^{-1} X^{-1}(0) X(z)\right]+2 i \bar{B} R^{2} z^{-1}+A z, & (|z|>R)
\end{array}
$$

Once $\varphi_{1}(z)$ and $\varphi_{2}(z)$ are determined, the other two analytic functions $\psi_{1}(z)$ and $\psi_{2}(z)$ can be determined from Eq. (12) as

$$
\begin{array}{ll}
\psi_{1}(z)=i \Gamma^{-1}\left(\alpha_{2}-1\right) \bar{\varphi}_{2}\left(R^{2} / z\right)-\frac{i \beta_{1} R^{2}}{z \varphi_{1}^{\prime}(z)}+i\left[\frac{\beta_{1}}{\varphi_{1}^{\prime}(0)}-\Gamma^{-1}\left(\alpha_{2}-1\right) \bar{A}\right] R^{2} z^{-1}+\Gamma^{-1} B z, & (|z|<R) \\
\psi_{2}(z)=i \Gamma\left(\alpha_{1}-1\right) \bar{\varphi}_{1}\left(R^{2} / z\right)-\frac{i \beta_{2} R^{2}}{z \varphi_{2}^{\prime}(z)}+i\left[\frac{\Gamma \beta_{1}}{\varphi_{1}^{\prime}(0)}-\left(\alpha_{2}-1\right) \bar{A}\right] R^{2} z^{-1}+B z, & (|z|>R) \tag{23}
\end{array}
$$

Now the unknown $\varphi_{1}^{\prime}(0)$ has to be determined by the following equation:

$$
\begin{equation*}
\varphi_{1}^{\prime}(0)-\frac{2 \beta_{1}\left(X^{\prime}(0)+\Gamma\right)}{1+\Gamma} \xlongequal[\overline{\varphi_{1}^{\prime}(0)}]{1}=\frac{A\left[1-X^{\prime}(0)\right]}{1+\Gamma}-\frac{i \bar{B} R^{2}}{1+\Gamma} \frac{X^{\prime \prime}(0)}{X(0)} \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
X(0)=R, \quad X^{\prime}(0)=-\cos \theta_{0}, \quad X^{\prime \prime}(0)=\frac{\sin ^{2} \theta_{0}}{R} \tag{25}
\end{equation*}
$$

Next we discuss the roots to Eq. (24) according to the three cases: $\Gamma>\cos \theta_{0}, \Gamma=\cos \theta_{0}$, and $\Gamma<\cos \theta_{0}$.

- $X^{\prime}(0)+\Gamma>0\left(\right.$ or $\left.\Gamma>\cos \theta_{0}\right)$

In this case there exist two distinct roots for $\varphi_{1}^{\prime}(0)$.
$\qquad$

$$
\begin{align*}
\left|\varphi_{1}^{\prime}(0)\right|= & \frac{\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|+\sqrt{\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|^{2}+8 \beta_{1}(1+\Gamma)\left(\Gamma-\cos \theta_{0}\right)}}{2(1+\Gamma)} \\
& \arg \left\{\varphi_{1}^{\prime}(0)\right\}=\arg \left\{A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right\} \tag{26a}
\end{align*}
$$

or

$$
\begin{align*}
\left|\varphi_{1}^{\prime}(0)\right|= & \frac{-\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|+\sqrt{\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|^{2}+8 \beta_{1}(1+\Gamma)\left(\Gamma-\cos \theta_{0}\right)}}{2(1+\Gamma)} \\
& \arg \left\{\varphi_{1}^{\prime}(0)\right\}=\arg \left\{A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right\}-\pi \tag{26b}
\end{align*}
$$

- $X^{\prime}(0)+\Gamma=0\left(\right.$ or $\left.\Gamma=\cos \theta_{0}\right)$

In this case there exists only one single root for $\varphi_{1}^{\prime}(0)$.

$$
\begin{equation*}
\varphi_{1}^{\prime}(0)=A-\frac{i \bar{B} \sin ^{2} \theta_{0}}{1+\cos \theta_{0}} \tag{27}
\end{equation*}
$$

- $X^{\prime}(0)+\Gamma<0$ (or $\Gamma<\cos \theta_{0}$ )

This case includes three subcases.

- If $\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|>2 \sqrt{2 \beta_{1}(1+\Gamma)\left(\cos \theta_{0}-\Gamma\right)}$, then there also exist two roots for $\varphi_{1}^{\prime}(0)$.

$$
\begin{align*}
\left|\varphi_{1}^{\prime}(0)\right|= & \frac{\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right| \pm \sqrt{\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|^{2}+8 \beta_{1}(1+\Gamma)\left(\Gamma-\cos \theta_{0}\right)}}{2(1+\Gamma)} \\
& \arg \left\{\varphi_{1}^{\prime}(0)\right\}=\arg \left\{A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right\} \tag{28}
\end{align*}
$$

- If $\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|=2 \sqrt{2 \beta_{1}(1+\Gamma)\left(\cos \theta_{0}-\Gamma\right)}$, then there exists only one root for $\varphi_{1}^{\prime}(0)$.

$$
\begin{equation*}
\varphi_{1}^{\prime}(0)=\frac{A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}}{2(1+\Gamma)} \tag{29}
\end{equation*}
$$

- If $\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|<2 \sqrt{2 \beta_{1}(1+\Gamma)\left(\cos \theta_{0}-\Gamma\right)}$, then there is no possible root for $\varphi_{1}^{\prime}(0)$.

Due to the fact that the mean Piola stress within the circular inclusion is given by [9]

$$
\begin{equation*}
\sigma_{11}+\sigma_{22}=4 \mu_{1} \operatorname{Im}\left\{\left(1-\alpha_{1}\right) \varphi_{1}^{\prime}(z)+\frac{\beta_{1}}{\varphi_{1}^{\prime}(z)}\right\}, \quad(|z|<R) \tag{30}
\end{equation*}
$$

Then the average mean Piola stress within the circular inclusion is

$$
\begin{equation*}
\left[\sigma_{11}+\sigma_{22}\right]_{\text {average }}=4 \mu_{1} \operatorname{Im}\left\{\left(1-\alpha_{1}\right) \varphi_{1}^{\prime}(0)+\frac{\beta_{1}}{\varphi_{1}^{\prime}(0)}\right\} \tag{31}
\end{equation*}
$$

It is observed from the above expression that the average mean Piola stress within the circular inclusion is closely related with $\varphi_{1}^{\prime}(0)$.

## 4 Complete Solution for a Perfectly Bonded Circular Inclusion

If the circular inclusion is perfectly bonded to the matrix, then the two pairs of analytic functions $\varphi_{1}(z), \psi_{1}(z)$ and $\varphi_{2}(z), \psi_{2}(z)$ can be easily determined to be

$$
\begin{equation*}
\varphi_{1}(z)=\frac{A-\frac{(1-\Gamma) \beta_{1}}{\varphi_{1}^{\prime}(0)}}{\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)} z, \quad \psi_{1}(z)=\frac{B}{\Gamma \alpha_{2}+\left(1-\alpha_{2}\right)} z, \quad(|z|<R) \tag{32}
\end{equation*}
$$

$$
\begin{align*}
\varphi_{2}(z)= & A z-\frac{i(\Gamma-1) \bar{B} R^{2}}{\Gamma \alpha_{2}+\left(1-\alpha_{2}\right)} z^{-1}, \quad \psi_{2}(z)=B z+i\left[\left(\frac{\alpha_{1}}{\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)}\right.\right. \\
& \left.\left.-\alpha_{2}\right) \bar{A}+\frac{\Gamma \beta_{1}}{\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)} \frac{1}{\varphi_{1}^{\prime}(0)}\right] R^{2} z^{-1} \\
& -\frac{i \beta_{2} R^{2} z}{A z^{2}+\frac{i(\Gamma-1) \bar{B} R^{2}}{\Gamma \alpha_{2}+\left(1-\alpha_{2}\right)}}, \quad(|z|>R) \tag{33}
\end{align*}
$$

It then follows from Eq. (32) that stresses are uniform within the perfectly bonded circular inclusion. The uniformity of stresses within a more general elliptical inclusion has been observed by Ru et al. [8]. The unknown $\varphi_{1}^{\prime}(0)$ in Eqs. (32) and (33) can be determined by the following equation.

$$
\begin{equation*}
\varphi_{1}^{\prime}(0)+\frac{(1-\Gamma) \beta_{1}}{\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)} \frac{1}{\overline{\varphi_{1}^{\prime}(0)}}=\frac{A}{\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)} \tag{34}
\end{equation*}
$$

In the following, we discuss the roots to Eq. (34) for the three cases: $\Gamma>1$ (the inclusion is stiffer than the matrix), $\Gamma=1$ (the inclusion and the matrix have the same shear modulus), and $\Gamma$ $<1$ (the inclusion is softer than the matrix).

- $\Gamma>1$ (the inclusion is stiffer than the matrix) In this case there are two roots for $\varphi_{1}^{\prime}(0)$.

$$
\begin{align*}
\left|\varphi_{1}^{\prime}(0)\right|= & \frac{|A|+\sqrt{|A|^{2}+4\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right](\Gamma-1) \beta_{1}}}{2\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right]} \\
& \arg \left\{\varphi_{1}^{\prime}(0)\right\}=\arg \{A\} \tag{35a}
\end{align*}
$$

or

$$
\begin{align*}
\left|\varphi_{1}^{\prime}(0)\right|= & \frac{-|A|+\sqrt{|A|^{2}+4\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right](\Gamma-1) \beta_{1}}}{2\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right]} \\
& \arg \left\{\varphi_{1}^{\prime}(0)\right\}=\arg \{A\}-\pi \tag{35b}
\end{align*}
$$

- $\Gamma=1$ (the inclusion and matrix have the same shear modulus)

In this case there is only one root for $\varphi_{1}^{\prime}(0)$.

$$
\begin{equation*}
\varphi_{1}^{\prime}(0)=A \tag{36}
\end{equation*}
$$

- $\Gamma<1$ (the inclusion is softer than the matrix)

This case includes three subcases.

- If $|A|>2 \sqrt{\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right](1-\Gamma) \beta_{1}}$, then there also exist two roots for $\varphi_{1}^{\prime}(0)$.

$$
\begin{align*}
\left|\varphi_{1}^{\prime}(0)\right|= & \frac{|A| \pm \sqrt{|A|^{2}-4\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right](1-\Gamma) \beta_{1}}}{2\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right]} \\
& \arg \left\{\varphi_{1}^{\prime}(0)\right\}=\arg \{A\} \tag{37}
\end{align*}
$$

- If $|A|=2 \sqrt{\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right](1-\Gamma) \beta_{1}}$, then there exists only one root for $\varphi_{1}^{\prime}(0)$.

$$
\begin{equation*}
\varphi_{1}^{\prime}(0)=\frac{A}{2\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right]} \tag{38}
\end{equation*}
$$

- If $|A|<2 \sqrt{\left[\alpha_{1}+\Gamma\left(1-\alpha_{1}\right)\right](1-\Gamma) \beta_{1}}$, then there is no possible root for $\varphi_{1}^{\prime}(0)$.

Finally if we let $\theta_{0}=0$ (there is no crack on the interface)
in Eq. (24), and let $\alpha_{1}=\alpha_{2}=1 / 2$ in Eq. (34), then Eqs. (24) and (34) will both reduce to the following equation.

$$
\begin{equation*}
\varphi_{1}^{\prime}(0)+\frac{2 \beta_{1}(1-\Gamma)}{1+\Gamma} \xlongequal[\varphi_{1}^{\prime}(0)]{1}=\frac{2 A}{1+\Gamma} \tag{39}
\end{equation*}
$$

which partially verifies the correctness of both Eqs. (24) and (34).

## 5 Conclusions

We have investigated the finite plane-strain deformation of a circular elastic inclusion bonded partially to an unbounded matrix. The elastic materials occupying the inclusion and the matrix belong to the class of harmonic materials. A complete solution to the arc interface crack problem is derived by means of the complex variable method. During the derivation, we focus on the case $\alpha_{1}$ $=\alpha_{2}=1 / 2$ in which the oscillatory singularity will disappear. We also present the complete solution for a circular inclusion perfectly bonded to the matrix. The results show that:

- When one of the two conditions (i) $\Gamma>\cos \theta_{0}$ or (ii) $\Gamma<\cos \theta_{0}$ and $\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|$ $>2 \sqrt{2 \beta_{1}(1+\Gamma)\left(\cos \theta_{0}-\Gamma\right)}$ is satisfied, it is possible to find two different states under finite plane deformations for the given remote uniform Piola stresses.
- When one of the two conditions (i) $\Gamma=\cos \theta_{0}$ or (ii) $\Gamma<\cos \theta_{0}$ and $\left|A\left(1+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0}\right|$ $=2 \sqrt{2 \beta_{1}(1+\Gamma)\left(\cos \theta_{0}-\Gamma\right)}$ is satisfied, there exists only one possible state under finite plane deformations for the given remote uniform Piola stresses.
- Otherwise when the condition $\Gamma<\cos \theta_{0}$ and $\mid A(1$ $\left.+\cos \theta_{0}\right)-i \bar{B} \sin ^{2} \theta_{0} \mid<2 \sqrt{2 \beta_{1}(1+\Gamma)\left(\cos \theta_{0}-\Gamma\right)}$ is met, there exists no possible state under finite plane deformations for the given remote uniform Piola stresses.


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