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A second-order theory for magnetoelectroelastic materials with transverse isotropy

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Abstract
In this paper, we concentrate on the basic governing equations of three-dimensional problems in transversely isotropic and nonlinear magnetoelectroelastic materials (i.e., $6\underline{m}\,\underline{m}$ magnetic crystals). We place emphasis on developing the nonlinear and fully coupled constitutive relations between extended traction (including elastic stress, polarization and magnetization) and extended strain (including elastic strain, electric field and magnetic induction). Simplified results are also presented for the corresponding small deformation problems in the case of both strong magnetic and electric fields and in the case of both weak magnetic and electric fields. The derived concise equations are important in investigating the nonlinear magnetoelectric effects of novel magnetoelectroelastic materials.

1. Introduction

Magnetoelectroelastic materials refers to novel materials which exhibit full coupling between magnetic, electric and mechanical fields. Because of their remarkable magnetoelectric coupling effect, piezoelectric–piezomagnetic composites are potential candidates for use as magnetoelectric memory elements, smart sensors and transducers.

Van Suchtelen (1972) first reported that piezoelectric–piezomagnetic composites may exhibit a new material property—the magnetoelectric coupling effect. Later on, Van den Boomgaard *et al* (1974) and Van Run *et al* (1974) investigated the magnetoelectric effect of $BaTiO_3$–$CoFe_2O_4$ composites. In the past several decades, many achievements have been made in the micromechanical modeling of magnetoelectroelastic materials and consequently on the determination of their effective properties, especially the magnetoelectric coupling effect (Harshe *et al* 1993, Avellaneda and Harshe 1994, Li and Dunn 1998, Aboudi 2001, Srinivasan *et al* 2002, Benveniste and Milton 2003, Zheng *et al* 2004, Fiebig 2005, Nan *et al* 2005, Lin *et al* 2005).

The development of magnetoelectroelastic materials has recently stimulated the studies of some fundamental problems. These investigations include: the existence of surface waves (Alshits *et al* 1992, Chen *et al* 2007, Liu *et al* 2007), wave scattering (Du *et al* 2004, Feng *et al* 2006), uniqueness and reciprocity theorems (Li 2003), various Green's function solutions (Chung and Ting 1995, Kirchner and Alshits 1996, Li 2002, Pan 2002, Wang and Shen 2002, Soh *et al* 2003, Lee and Ma 2007), deformation of multilayered magnetoelectroelastic plates (Pan 2001, Wang *et al* 2003) and circular tubes or bars (Wang and Zhong 2003), free vibration of simply supported and multilayered magnetoelectroelastic plates (Pan and Heyliger 2002), static fracture problems (Gao *et al* 2003a, 2003b, Song and Sih 2003, Spyropoulos *et al* 2003, Hu *et al* 2006, Wang and Mai 2007), and impact problems (Feng *et al* 2005, Zhou *et al* 2005, Feng and Su 2006). In addition, Soh and Liu (2005) derived eight types of constitutive equation for magnetoelectroelastic solids in which different independent variables were considered. The theoretical investigations mentioned above were all carried out under the assumption of linear constitutive relations.

Nonlinearity has always been an important issue in piezoelectric and magnetoelectroelastic materials/structures. While some simple nonlinear piezoelectric models were proposed to study shock waves in piezoelectric semiconductors (Lysne 1972, Chen *et al* 1976), these models are essentially semi-coupled and consequently the modeling results are

different from the experimental ones (Davison and Graham 1979). Recently, various formulations for coupled nonlinear electroelastic and magnetoelastic materials were proposed (i.e., Dorfmann and Ogden 2004, 2005; Steigmann 2004, 2008; Kankanala and Triantafyllidis 2004; Suo *et al* 2008). A review of nonlinear electroelasticity under static deformation was given by Bustamante *et al* (2008) in which different expressions for the nonlinear constitutive relations and governing equations were compared and discussed. While certain magnetoelectroelastic composites were shown to be highly nonlinear (Bichurin *et al* 2005, Nan *et al* 2008), a fully coupled nonlinear constitutive relation of them has not been developed yet.

Thus, in this paper, we derive the basic constitutive equations for three-dimensional nonlinear magnetoelectroelastic materials. The polynomial constitutive relations for a transversely isotropic (i.e., $6\underline{m}\,\underline{m}$) magnetic crystal are derived in a concise form using an invariant integrity basis. Furthermore the constitutive equations are fully coupled and can be specialized to the simple case of small deformation and strong magnetic and electric fields, and to the simple case of small deformation and relatively weak magnetic and electric fields. The present work should be useful to researchers in the investigation of the mechanics and physics of magnetoelectroelastic solids undergoing nonlinear deformations.

2. Equations for a nonlinear magnetoelectroelastic material

Let the coordinates of a material particle with respect to a rectangular Cartesian coordinate system be X_K in the reference (undeformed) configuration and its spatial coordinates in the current (deformed) configuration be x_k. For a transversely isotropic magnetoelectroelastic material within the hexagonal system with class symmetry $6\underline{m}\,\underline{m}$, the basic equations for this type of nonlinear magnetoelectroelastic material can be written as follows.

2.1. Governing equations

In the absence of body force, free charge density and free current density with the nonlinear magnetoelectroelastic material at rest, the balance laws and the quasistatic approximation to the Maxwell equation in the deformed configuration (lowercase subscripts) can be written as (Pao 1978, Kiral and Eringen 1990)

$$T_{kl,k} - (P_{k,k}E_l + M_{k,k}B_l) = \rho\ddot{u}_l, \tag{1}$$

$$(\varepsilon_0 E_k + P_k)_{,k} = 0, \tag{2}$$

$$B_{k,k} = 0, \tag{3}$$

where $T_{kl} = \sigma_{kl} + P_k E_l + M_k B_l$ (Pao 1978) is a symmetric stress tensor called the elastic stress, σ_{kl} the Cauchy stress, ρ the mass density, u_l the displacement vector, P_k the electric polarization, M_k the magnetization, E_k the electric field, B_k the magnetic induction, ε_0 the permittivity of free space, and a dot above a quantity signifies its material time derivative.

As shown in section 2.2, the constitutive equations are generally derived in the material (undeformed) configuration (uppercase subscripts). For the sake of application, the spatial quantities are expressed by the corresponding material ones as (Kiral and Eringen 1990, Jordan and Eringen 1964)

$$T_{kl} = J^{-1}T_{KL}x_{k,K}x_{l,L}, \tag{4}$$

$$P_k = J^{-1}\Pi_K x_{k,K}, \tag{5}$$

$$M_k = J^{-1}M_K x_{k,K}, \tag{6}$$

$$E_k = X_{K,k}E_K, \tag{7}$$

$$B_k = X_{K,k}B_K, \tag{8}$$

where

$$\Pi_K = P_K/\rho, \tag{9}$$

$$J = \det(x_{k,K}) = \rho_0/\rho, \tag{10}$$

where ρ_0 is the mass density in the material configuration; again, an uppercase subscript corresponds to the undeformed configuration, and a lowercase subscript to the deformed configuration.

2.2. Constitutive equations

Modern texts in physics (Feynman *et al* 1964, Pao 1978) tend to regard the electric field (**E**) and magnetic induction (**B**) as the basic variables. The other four electromagnetic field variables, i.e. the electric displacement, polarization, magnetic field and magnetization vectors (**D**, **P**, **H**, **M**), can be related by the following two vector equations (Pao 1978):

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{11}$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \tag{12}$$

where μ_0 is the permeability of free space, which is related to ε_0 by $\mu_0\varepsilon_0 = c^{-2}$, with c being the speed of light *in vacuo*.

For an isothermal or adiabatic process, in the material configuration, the free energy function in the absence of heat conduction or heat sources can be defined as (Pao 1978)

$$\Sigma = \Sigma(\Gamma_{KL}, E_K, B_K), \tag{13}$$

where Γ_{KL} is the Lagrangian strain (see equation (17) below).

The constitutive equations can be derived from equation (13) as (Pao 1978, Kiral and Eringen 1990)

$$T_{KL} = \frac{\partial\Sigma}{\partial\Gamma_{KL}}, \tag{14}$$

$$\Pi_K = -\frac{\partial\Sigma}{\partial E_K}, \tag{15}$$

$$M_K = -\frac{\partial\Sigma}{\partial B_K}. \tag{16}$$

We point out that the free energy expression (13) for the general and fully coupled nonlinear magnetoelectroelastic deformation can be reduced to the simple magnetoelastic and electroelastic cases previously studied. For example, in the nonlinear magnetoelastic deformation analysis, Dorfmann

and Ogden (2004) expressed the free energy in terms of the deformation gradient tensor and the magnetic field (or magnetic induction); for the nonlinear electroelastic case, the pair of the Lagrangian strain with polarization, electric field, or electric displacement could be the independent variables in the free energy expression (Bustamante *et al* 2008). It is also well known that the free energy (13) can be expressed by other triads but the work conjugates among the electric and magnetic quantities need to be carefully checked (see, e.g., Suo *et al* 2008).

2.3. Extended geometry equations (i.e., relations of strain–displacement, electric field–electric potential and magnetic field–magnetic potential)

In the material configuration, the Lagrangian strain, electric field, and magnetic field can be expressed as

$$\Gamma_{KL} = \tfrac{1}{2}(x_{k,K}x_{k,L} - \delta_{KL}) = \tfrac{1}{2}(u_{K,L} + u_{L,K} + u_{M,K}u_{M,L}), \tag{17}$$

$$E_K = -\frac{\partial \phi}{\partial X_K} = -\phi_{,K}, \tag{18}$$

$$H_K = -\frac{\partial \psi}{\partial X_K} = -\psi_{,K}. \tag{19}$$

Similar to equation (7), we have

$$H_k = -\frac{\partial \psi}{\partial x_k} = -\frac{\partial \psi}{\partial X_K}\frac{\partial X_K}{\partial x_k} = X_{K,k}H_K. \tag{20}$$

2.4. Boundary and initial conditions

(a) Mechanical boundary conditions

$$u_i = \bar{u}_i(x_1, x_2, x_3) \in S_1, \tag{21}$$

$$\sigma_{ij}n_j = \bar{p}_i(x_1, x_2, x_3) \in S_2, \tag{22}$$

where $S_1 + S_2 = S$ is the total boundary of the problem domain, and $\bar{u}_i$ and $\bar{p}_i$ are, respectively, the given displacement and traction on the boundary.

(b) Magnetic boundary conditions

$$\psi = \bar{\psi}(x_1, x_2, x_3) \in S_3, \tag{23}$$

$$B_i n_i = \bar{B}_n(x_1, x_2, x_3) \in S_4, \tag{24}$$

where $S_3 + S_4 = S$, $\bar{\psi}$ and $\bar{B}_n$ are, respectively, the given magnetic potential and normal magnetic induction on the boundary.

(c) Electric boundary conditions

$$\phi = \bar{\phi}(x_1, x_2, x_3) \in S_5, \tag{25}$$

$$D_i n_i = \bar{D}_n(x_1, x_2, x_3) \in S_6, \tag{26}$$

where $S_5 + S_6 = S$, and $\bar{\phi}$ and $\bar{D}_n$ are, respectively, the given electric potential and normal electric displacement on the boundary.

(d) Initial conditions

$$\begin{aligned} u_i(x_1, x_2, x_3, 0) &= u_i^0(x_1, x_2, x_3), \\ \dot{u}_i(x_1, x_2, x_3, 0) &= \dot{u}_i^0(x_1, x_2, x_3), \end{aligned} \tag{27}$$

where u_i^0 is the known initial displacement and $\dot{u}_i^0$ the known initial velocity within the problem domain.

3. Second-order nonlinear constitutive equations of three-dimensional problems

As shown in section 2, for the nonlinear problem of concern in this paper, the nonlinear constitutive equations are the key issues. To derive these equations, we assume that the constitutive equations for stress, electric polarization and magnetization are polynomial functions of the strain, electric field and magnetic induction.

3.1. Polynomial integrity basis

Two basic requirements of invariance that must be imposed upon the constitutive equations are the spatial invariance and material invariance (Jordan and Eringen 1964). For a $6\underline{m}\,\underline{m}$ magnetic crystal with both the elastic symmetric axis and the poling direction as the X_3-axis, the polynomial integrity basis, the degree of which is less than three, can be given in the following concise form (Kiral and Eringen 1990).

- Elements in Γ_{IJ} only:

$$\text{Degree 1:}\quad \Gamma_{33}, \qquad \Gamma_{11} + \Gamma_{22}, \tag{28}$$

$$\text{Degree 2:}\quad \Gamma_{11}\Gamma_{22} - \Gamma_{12}^2, \qquad \Gamma_{13}^2 + \Gamma_{23}^2, \tag{29}$$

$$\text{Degree 3:}\quad \Gamma_{11}\left(\Gamma_{11}^2 + 6\Gamma_{11}\Gamma_{22} - 12\Gamma_{12}^2 + 9\Gamma_{22}^2\right), \quad \Gamma_{11}\Gamma_{23}^2 + \Gamma_{22}\Gamma_{13}^2 - \Gamma_{13}\Gamma_{23}\Gamma_{12}. \tag{30}$$

- Elements in E_I only:

$$\text{Degree 1:}\quad E_3, \tag{31}$$

$$\text{Degree 2:}\quad E_1^2 + E_2^2. \tag{32}$$

- Elements in B_I only:

$$\text{Degree 1:}\quad B_3, \tag{33}$$

$$\text{Degree 2:}\quad B_1^2 + B_2^2. \tag{34}$$

- Elements in Γ_{IJ} and E_I only:

$$\text{Degree 2:}\quad \Gamma_{31}E_1 + \Gamma_{23}E_2, \tag{35}$$

$$\text{Degree 3:}\quad (E_1\Gamma_{23} + E_2\Gamma_{31})\Gamma_{12} - E_1\Gamma_{22}\Gamma_{31} - E_2\Gamma_{11}\Gamma_{23}, \tag{36a}$$

$$\Gamma_{11}E_2^2 + \Gamma_{22}E_1^2 - 2E_1E_2\Gamma_{12}. \tag{36b}$$

- Elements in Γ_{IJ} and B_I only:

$$\text{Degree 2:}\quad \Gamma_{31}B_1 + \Gamma_{23}B_2, \tag{37}$$

$$\text{Degree 3:}\quad (B_1\Gamma_{23} + B_2\Gamma_{31})\Gamma_{12} - B_1\Gamma_{22}\Gamma_{31} - B_2\Gamma_{11}\Gamma_{23}, \tag{38a}$$

$$\Gamma_{11}B_2^2 + \Gamma_{22}B_1^2 - 2B_1B_2\Gamma_{12}. \tag{38b}$$

- Elements in E_I and B_I only:

$$\text{Degree 2:}\quad E_1B_1 + E_2B_2. \tag{39}$$

- Elements in Γ_{IJ}, E_I and B_I only:

$$\text{Degree 3:}\quad \mathrm{Im}\{(B_1 + \mathrm{i}B_2)(E_1 + \mathrm{i}E_2) \times [2\Gamma_{12} + \mathrm{i}(\Gamma_{11} - \Gamma_{22})]\}. \tag{40}$$

3.2. Polynomial free energy function

In order to derive the second-order nonlinear constitutive equations, the free energy Σ should be formed as a third-order polynomial function of Γ_{IJ}, E_I and B_I (Yang and Betra 1995). From equations (28) to (40), after complicated symbolic mathematical manipulations using the *Mathematica* software package, the free energy Σ can be finally expressed in the following form:

$$\Sigma = \Sigma_a + \Sigma_b + \Sigma_c + \Sigma_d + \Sigma_s + \Sigma_f + \Sigma_g, \tag{41}$$

where

$$\begin{aligned}\Sigma_a = {} & a_1\Gamma_{33} + a_2(\Gamma_{11} + \Gamma_{22}) + a_3(\Gamma_{11}\Gamma_{22} - \Gamma_{12}^2) \\ & + a_4(\Gamma_{23}^2 + \Gamma_{31}^2) + a_5\Gamma_{33}^2 + a_6(\Gamma_{11} + \Gamma_{22})\Gamma_{33} \\ & + a_7(\Gamma_{11} + \Gamma_{22})^2 + a_8\Gamma_{11}(\Gamma_{11}^2 - 12\Gamma_{12}^2 + 6\Gamma_{11}\Gamma_{22} \\ & + 9\Gamma_{22}^2) + a_9(\Gamma_{11}\Gamma_{23}^2 + \Gamma_{31}(\Gamma_{22}\Gamma_{31} - \Gamma_{12}\Gamma_{23})) \\ & + a_{10}(\Gamma_{11}\Gamma_{22} - \Gamma_{12}^2)\Gamma_{33} + a_{11}(\Gamma_{23}^2 + \Gamma_{31}^2)\Gamma_{33} \\ & + a_{12}(\Gamma_{11} + \Gamma_{22})(\Gamma_{11}\Gamma_{22} - \Gamma_{12}^2) \\ & + a_{13}(\Gamma_{11} + \Gamma_{22})(\Gamma_{23}^2 + \Gamma_{31}^2) + a_{14}\Gamma_{33}^3 \\ & + a_{15}(\Gamma_{11} + \Gamma_{22})\Gamma_{33}^2 + a_{16}(\Gamma_{11} + \Gamma_{22})^3 \\ & + a_{17}(\Gamma_{11} + \Gamma_{22})^2\Gamma_{33},\end{aligned} \tag{42a}$$

$$\begin{aligned}\Sigma_b = {} & b_1E_3 + b_2(E_1^2 + E_2^2) + b_3E_3^2 + b_4(E_1^2 + E_2^2)E_3 \\ & + b_5E_3^3,\end{aligned} \tag{42b}$$

$$\begin{aligned}\Sigma_c = {} & c_1B_3 + c_2(B_1^2 + B_2^2) + c_3B_3^2 + c_4(B_1^2 + B_2^2)B_3 \\ & + c_5B_3^3,\end{aligned} \tag{42c}$$

$$\begin{aligned}\Sigma_d = {} & d_1(\Gamma_{31}E_1 + \Gamma_{23}E_2) + d_2\Gamma_{33}E_3 + d_3(\Gamma_{11} + \Gamma_{22})E_3 \\ & + d_4(\Gamma_{12}\Gamma_{23}E_1 - \Gamma_{22}\Gamma_{31}E_1 - \Gamma_{11}\Gamma_{23}E_2 + \Gamma_{12}\Gamma_{31}E_2) \\ & + d_5\Gamma_{33}(\Gamma_{31}E_1 + \Gamma_{23}E_2) + d_6(\Gamma_{11} + \Gamma_{22})(\Gamma_{31}E_1 \\ & + \Gamma_{23}E_2) + d_7(-\Gamma_{12}^2 + \Gamma_{11}\Gamma_{22})E_3 + d_8(\Gamma_{23}^2 + \Gamma_{31}^2)E_3 \\ & + d_9\Gamma_{33}^2E_3 + d_{10}(\Gamma_{11} + \Gamma_{22})^2E_3 + d_{11}(\Gamma_{11} + \Gamma_{22})\Gamma_{33}E_3 \\ & + d_{12}(\Gamma_{22}E_1^2 + E_2(-2\Gamma_{12}E_1 + \Gamma_{11}E_2)) + d_{13}\Gamma_{33}(E_1^2 \\ & + E_2^2) + d_{14}(\Gamma_{11} + \Gamma_{22})(E_1^2 + E_2^2) \\ & + d_{15}(\Gamma_{31}E_1 + \Gamma_{23}E_2)E_3 + d_{16}\Gamma_{33}E_3^2 \\ & + d_{17}(\Gamma_{11} + \Gamma_{22})E_3^2,\end{aligned} \tag{42d}$$

$$\begin{aligned}\Sigma_s = {} & s_1(\Gamma_{31}B_1 + \Gamma_{23}B_2) + s_2\Gamma_{33}B_3 + s_3(\Gamma_{11} + \Gamma_{22})B_3 \\ & + s_4(\Gamma_{12}\Gamma_{23}B_1 - \Gamma_{22}\Gamma_{31}B_1 - \Gamma_{11}\Gamma_{23}B_2 + \Gamma_{12}\Gamma_{31}B_2) \\ & + s_5\Gamma_{33}(\Gamma_{31}B_1 + \Gamma_{23}B_2) + s_6(\Gamma_{11} + \Gamma_{22})(\Gamma_{31}B_1 \\ & + \Gamma_{23}B_2) + s_7(-\Gamma_{12}^2 + \Gamma_{11}\Gamma_{22})B_3 + s_8(\Gamma_{23}^2 + \Gamma_{31}^2)B_3 \\ & + s_9\Gamma_{33}^2B_3 + s_{10}(\Gamma_{11} + \Gamma_{22})^2B_3 + s_{11}(\Gamma_{11} + \Gamma_{22})\Gamma_{33}B_3 \\ & + s_{12}(\Gamma_{22}B_1^2 + B_2(-2\Gamma_{12}B_1 + \Gamma_{11}B_2)) + s_{13}\Gamma_{33}(B_1^2 \\ & + B_2^2) + s_{14}(\Gamma_{11} + \Gamma_{22})(B_1^2 + B_2^2) \\ & + s_{15}(\Gamma_{31}B_1 + \Gamma_{23}B_2)B_3 + s_{16}\Gamma_{33}B_3^2 \\ & + s_{17}(\Gamma_{11} + \Gamma_{22})B_3^2,\end{aligned} \tag{42e}$$

$$\begin{aligned}\Sigma_f = {} & f_1(B_1E_1 + B_2E_2) + f_2B_3E_3 + f_3(B_1^2 + B_2^2)E_3 \\ & + f_4B_3(B_1E_1 + B_2E_2) + f_5B_3^2E_3 \\ & + f_6(B_1E_1 + B_2E_2)E_3 + f_7B_3(E_1^2 + E_2^2) \\ & + f_8B_3E_3^2,\end{aligned} \tag{42f}$$

$$\begin{aligned}\Sigma_g = {} & g_1(2\Gamma_{12}(B_2E_1 + B_1E_2) + (\Gamma_{11} - \Gamma_{22})(B_1E_1 \\ & - B_2E_2)) + g_2\Gamma_{33}(B_1E_1 + B_2E_2) + g_3(\Gamma_{11} + \Gamma_{22}) \\ & \times (B_1E_1 + B_2E_2) + g_4(\Gamma_{31}B_1 + \Gamma_{23}B_2)E_3 \\ & + g_5B_3(\Gamma_{31}E_1 + \Gamma_{23}E_2) + g_6\Gamma_{33}B_3E_3 \\ & + g_7(\Gamma_{11} + \Gamma_{22})B_3E_3.\end{aligned} \tag{42g}$$

Equation (42) indicates that the free energy consists of seven parts with Σ_a, Σ_b, Σ_c, Σ_d, Σ_s, Σ_f and Σ_g corresponding, respectively, to the purely mechanical properties, purely electric properties, purely magnetic properties, electromechanical coupling properties, magnetomechanical coupling properties, magnetoelectric coupling properties and magnetoelectromechanical coupling properties. It is also observed that there are a total of 76 independent material constants for a $6\underline{m}\,\underline{m}$ magnetic crystal and that there are 21 independent constants for the corresponding linearized material.

3.3. Second-order constitutive theory

Substituting equations (41) and (42) into equations (14)–(16), we can express the elastic stresses, polarization and magnetization as follows:

$$T_{IJ} = T_{IJa} + T_{IJd} + T_{IJs} + T_{IJg}, \tag{43}$$

$$\Pi_I = \Pi_{Ib} + \Pi_{Id} + \Pi_{If} + \Pi_{Ig}, \tag{44}$$

$$M_I = M_{Ic} + M_{Is} + M_{If} + M_{Ig}, \tag{45}$$

where

$$\begin{aligned}T_{11a} = {} & a_2 + a_3\Gamma_{22} + a_6\Gamma_{33} + 2a_7(\Gamma_{11} + \Gamma_{22}) \\ & + 3a_8(\Gamma_{11}^2 - 4\Gamma_{12}^2 + 4\Gamma_{11}\Gamma_{22} + 3\Gamma_{22}^2) \\ & + a_9\Gamma_{23}^2 + a_{10}\Gamma_{22}\Gamma_{33} + a_{12}(-\Gamma_{12}^2 + 2\Gamma_{11}\Gamma_{22} + \Gamma_{22}^2) \\ & + a_{13}(\Gamma_{23}^2 + \Gamma_{31}^2) + a_{15}\Gamma_{33}^2 - 3a_{16}(\Gamma_{11} + \Gamma_{22})^2 \\ & + 2a_{17}(\Gamma_{11} + \Gamma_{22})\Gamma_{33},\end{aligned} \tag{46a}$$

$$\begin{aligned}T_{11d} = {} & d_3E_3 - d_4\Gamma_{23}E_2 + d_6(\Gamma_{31}E_1 + \Gamma_{23}E_2) + d_7\Gamma_{22}E_3 \\ & + 2d_{10}(\Gamma_{11} + \Gamma_{22})E_3 + d_{11}\Gamma_{33}E_3 + d_{12}E_2^2 \\ & + d_{14}(E_1^2 + E_2^2) + d_{17}E_3^2,\end{aligned} \tag{46b}$$

$$\begin{aligned}T_{11s} = {} & s_3B_3 - s_4\Gamma_{23}B_2 + s_6(\Gamma_{31}B_1 + \Gamma_{23}B_2) + s_7\Gamma_{22}B_3 \\ & + 2s_{10}(\Gamma_{11} + \Gamma_{22})B_3 + s_{11}\Gamma_{33}B_3 + s_{12}B_2^2 \\ & + s_{14}(B_1^2 + B_2^2) + s_{17}B_3^2,\end{aligned} \tag{46c}$$

$$\begin{aligned}T_{11g} = {} & g_1(B_1E_1 - B_2E_2) + g_3(B_1E_1 + B_2E_2) \\ & + g_7B_3E_3,\end{aligned} \tag{46d}$$

$$\begin{aligned}T_{22a} = {} & a_2 + a_3\Gamma_{11} + a_6\Gamma_{33} \\ & + 2a_7(\Gamma_{11} + \Gamma_{22}) + 6a_8\Gamma_{11}(\Gamma_{11} + 3\Gamma_{22}) \\ & + a_9\Gamma_{31}^2 + a_{10}\Gamma_{11}\Gamma_{33} + a_{12}(-\Gamma_{12}^2 + 2\Gamma_{11}\Gamma_{22} + \Gamma_{11}^2) \\ & + a_{13}(\Gamma_{23}^2 + \Gamma_{31}^2) + a_{15}\Gamma_{33}^2 + 3a_{16}(\Gamma_{11} + \Gamma_{22})^2 \\ & + a_{17}(\Gamma_{11} + \Gamma_{22})\Gamma_{33},\end{aligned} \tag{47a}$$

$$\begin{aligned}T_{22d} = {} & d_3E_3 - d_4\Gamma_{31}E_1 + d_6(\Gamma_{31}E_1 + \Gamma_{23}E_2) + d_7\Gamma_{11}E_3 \\ & + 2d_{10}(\Gamma_{11} + \Gamma_{22})E_3 + d_{11}\Gamma_{33}E_3 + d_{12}E_1^2 \\ & + d_{14}(E_1^2 + E_2^2) + d_{17}E_3^2,\end{aligned} \tag{47b}$$

$$\begin{aligned}T_{22s} = {} & s_3B_3 - s_4\Gamma_{31}B_1 + s_6(\Gamma_{31}B_1 + \Gamma_{23}B_2) + s_7\Gamma_{11}B_3 \\ & + 2s_{10}(\Gamma_{11} + \Gamma_{22})B_3 + s_{11}\Gamma_{33}B_3 + s_{12}B_1^2 \\ & + s_{14}(B_1^2 + B_2^2) + s_{17}B_3^2,\end{aligned} \tag{47c}$$

$$\begin{aligned}T_{22g} = {} & g_1(-B_1E_1 + B_2E_2) + g_3(B_1E_1 + B_2E_2) \\ & + g_7B_3E_3,\end{aligned} \tag{47d}$$

$$T_{33a} = a_1 + 2a_5\Gamma_{33} + a_6(\Gamma_{11} + \Gamma_{22}) + a_{10}(-\Gamma_{12}^2 + \Gamma_{11}\Gamma_{22}) + a_{11}(\Gamma_{23}^2 + \Gamma_{31}^2) + 3a_{14}\Gamma_{33}^2 + 2a_{15}(\Gamma_{11} + \Gamma_{22})\Gamma_{33} + a_{17}(\Gamma_{11} + \Gamma_{22})^2, \quad (48a)$$

$$T_{33d} = d_2E_3 + d_5(\Gamma_{31}E_1 + \Gamma_{23}E_2) + 2d_9\Gamma_{33}E_3 + d_{11}(\Gamma_{11} + \Gamma_{22})E_3 + d_{13}(E_1^2 + E_2^2) + d_{16}E_3^2, \quad (48b)$$

$$T_{33s} = s_2B_3 + s_5(\Gamma_{31}B_1 + \Gamma_{23}B_2) + 2s_9\Gamma_{33}B_3 + s_{11}(\Gamma_{11} + \Gamma_{22})B_3 + s_{13}(B_1^2 + B_2^2) + s_{16}B_3^2, \quad (48c)$$

$$T_{33g} = g_2(B_1E_1 + B_2E_2) + g_6B_3E_3, \quad (48d)$$

$$T_{23a} = a_4\Gamma_{23} + a_9(\Gamma_{11}\Gamma_{23} - 0.5\Gamma_{12}\Gamma_{31}) + a_{11}\Gamma_{23}\Gamma_{33} + a_{13}(\Gamma_{11} + \Gamma_{22})\Gamma_{23}, \quad (49a)$$

$$T_{23d} = 0.5d_1E_2 + 0.5d_4(\Gamma_{12}E_1 - \Gamma_{11}E_2) + 0.5d_5\Gamma_{33}E_2 + 0.5d_6(\Gamma_{11} + \Gamma_{22})E_2 + d_8\Gamma_{23}E_3 + 0.5d_{15}E_2E_3, \quad (49b)$$

$$T_{23s} = 0.5s_1B_2 + 0.5s_4(\Gamma_{12}B_1 - \Gamma_{11}B_2) + 0.5s_5\Gamma_{33}B_2 + 0.5s_6(\Gamma_{11} + \Gamma_{22})B_2 + s_8\Gamma_{23}B_3 + 0.5s_{15}B_2B_3, \quad (49c)$$

$$T_{23g} = 0.5g_4B_2E_3 + 0.5g_5B_3E_2, \quad (49d)$$

$$T_{31a} = a_4\Gamma_{31} + a_9(\Gamma_{22}\Gamma_{31} - 0.5\Gamma_{12}\Gamma_{23}) + a_{11}\Gamma_{31}\Gamma_{33} + a_{13}(\Gamma_{11} + \Gamma_{22})\Gamma_{31}, \quad (50a)$$

$$T_{31d} = 0.5d_1E_1 + 0.5d_4(\Gamma_{12}E_2 - \Gamma_{22}E_1) + 0.5d_5\Gamma_{33}E_1 + 0.5d_6(\Gamma_{11} + \Gamma_{22})E_1 + d_8\Gamma_{31}E_3 + 0.5d_{15}E_1E_3, \quad (50b)$$

$$T_{31s} = 0.5s_1B_1 + 0.5s_4(\Gamma_{12}B_2 - \Gamma_{22}B_1) + 0.5s_5\Gamma_{33}B_1 + 0.5s_6(\Gamma_{11} + \Gamma_{22})B_1 + s_8\Gamma_{31}B_3 + 0.5s_{15}B_1B_3, \quad (50c)$$

$$T_{31g} = 0.5g_4B_1E_3 + 0.5g_5B_3E_1, \quad (50d)$$

$$T_{12a} = -a_3\Gamma_{12} - 12a_8\Gamma_{11}\Gamma_{12} - 0.5a_9\Gamma_{23}\Gamma_{31} - a_{10}\Gamma_{12}\Gamma_{33} - a_{12}\Gamma_{12}(\Gamma_{11} + \Gamma_{12}), \quad (51a)$$

$$T_{12d} = 0.5d_4(\Gamma_{23}E_1 + \Gamma_{31}E_2) - d_7\Gamma_{12}E_3 - d_{12}E_1E_2, \quad (51b)$$

$$T_{12s} = 0.5s_4(\Gamma_{23}B_1 + \Gamma_{31}B_2) - s_7\Gamma_{12}B_3 - s_{12}B_1B_2, \quad (51c)$$

$$T_{12g} = g_1(B_2E_1 + B_1E_2), \quad (51d)$$

$$-\Pi_{1b} = 2b_2E_1 + 2b_4E_1E_3, \quad (52a)$$

$$-\Pi_{1d} = d_1\Gamma_{31} + d_4(\Gamma_{12}\Gamma_{23} - \Gamma_{22}\Gamma_{31}) + d_5\Gamma_{31}\Gamma_{33} + d_6(\Gamma_{11} + \Gamma_{22})\Gamma_{31} + 2d_{12}(\Gamma_{22}E_1 - \Gamma_{12}E_2) + 2d_{13}\Gamma_{33}E_1 + 2d_{14}(\Gamma_{11} + \Gamma_{22})E_1 + d_{15}\Gamma_{31}E_3, \quad (52b)$$

$$-\Pi_{1f} = f_1B_1 + f_4B_1B_3 + f_6B_1E_3 + 2f_7B_3E_1, \quad (52c)$$

$$-\Pi_{1g} = g_1((\Gamma_{11} - \Gamma_{22})B_1 + 2\Gamma_{12}B_2) + g_2\Gamma_{33}B_1 + g_3(\Gamma_{11} + \Gamma_{22})B_1 + g_5\Gamma_{31}B_3, \quad (52d)$$

$$-\Pi_{2b} = 2b_2E_2 + 2b_4E_2E_3, \quad (53a)$$

$$-\Pi_{2d} = d_1\Gamma_{23} + d_4(-\Gamma_{11}\Gamma_{23} + \Gamma_{12}\Gamma_{31}) + d_5\Gamma_{23}\Gamma_{33} + d_6(\Gamma_{11} + \Gamma_{22})\Gamma_{23} + 2d_{12}(-\Gamma_{12}E_1 + \Gamma_{11}E_2) + 2d_{13}\Gamma_{33}E_2 + 2d_{14}(\Gamma_{11} + \Gamma_{22})E_2 + d_{15}\Gamma_{23}E_3, \quad (53b)$$

$$-\Pi_{2f} = f_1B_2 + f_4B_2B_3 + f_6B_2E_3 + 2f_7B_3E_2, \quad (53c)$$

$$-\Pi_{2g} = g_1((\Gamma_{22} - \Gamma_{11})B_2 + 2\Gamma_{12}B_1) + g_2\Gamma_{33}B_2 + g_3(\Gamma_{11} + \Gamma_{22})B_2 + g_5\Gamma_{23}B_3, \quad (53d)$$

$$-\Pi_{3b} = b_1 + 2b_3E_3 + b_4(E_1^2 + E_2^2) + 3b_5E_3^2, \quad (54a)$$

$$-\Pi_{3d} = d_2\Gamma_{33} + d_3(\Gamma_{11} + \Gamma_{22}) + d_7(-\Gamma_{12}^2 + \Gamma_{11}\Gamma_{22}) + d_8(\Gamma_{23}^2 + \Gamma_{31}^2) + d_9\Gamma_{33}^2 + d_{10}(\Gamma_{11} + \Gamma_{22})^2 + d_{11}(\Gamma_{11} + \Gamma_{22})\Gamma_{33} + d_{15}(\Gamma_{31}E_1 + \Gamma_{23}E_2) + 2d_{16}\Gamma_{33}E_3 + 2d_{17}(\Gamma_{11} + \Gamma_{22})E_3, \quad (54b)$$

$$-\Pi_{3f} = f_2B_3 + f_3(B_1^2 + B_2^2) + f_5B_3^2 + f_6(B_1E_1 + B_2E_2) + 2f_8B_3E_3, \quad (54c)$$

$$-\Pi_{3g} = g_4(\Gamma_{31}B_1 + \Gamma_{23}B_2) + g_6\Gamma_{33}B_3 + g_7(\Gamma_{11} + \Gamma_{22})B_3, \quad (54d)$$

$$-M_{1c} = 2c_2B_1 + 2c_4B_1B_3, \quad (55a)$$

$$-M_{1s} = s_1\Gamma_{31} + s_4(\Gamma_{12}\Gamma_{23} - \Gamma_{22}\Gamma_{31}) + s_5\Gamma_{31}\Gamma_{33} + s_6(\Gamma_{11} + \Gamma_{22})\Gamma_{31} + 2s_{12}(\Gamma_{22}B_1 - \Gamma_{12}B_2) + 2s_{13}\Gamma_{33}B_1 + 2s_{14}(\Gamma_{11} + \Gamma_{22})B_1 + s_{15}\Gamma_{31}B_3, \quad (55b)$$

$$-M_{1f} = f_1E_1 + 2f_3B_1E_3 + f_4B_3E_1 + f_6E_1E_3, \quad (55c)$$

$$-M_{1g} = g_1((\Gamma_{11} - \Gamma_{22})E_1 + 2\Gamma_{12}E_2) + g_2\Gamma_{33}E_1 + g_3(\Gamma_{11} + \Gamma_{22})E_1 + g_4\Gamma_{31}E_3, \quad (55d)$$

$$-M_{2c} = 2c_2B_2 + 2c_4B_2B_3, \quad (56a)$$

$$-M_{2s} = s_1\Gamma_{23} + s_4(\Gamma_{12}\Gamma_{31} - \Gamma_{11}\Gamma_{23}) + s_5\Gamma_{23}\Gamma_{33} + s_6(\Gamma_{11} + \Gamma_{22})\Gamma_{23} + 2s_{12}(\Gamma_{11}B_2 - \Gamma_{12}B_1) + 2s_{13}\Gamma_{33}B_2 + 2s_{14}(\Gamma_{11} + \Gamma_{22})B_2 + s_{15}\Gamma_{23}B_3, \quad (56b)$$

$$-M_{2f} = f_1E_2 + 2f_3B_2E_3 + f_4B_3E_2 + f_6E_2E_3, \quad (56c)$$

$$-M_{2g} = g_1((\Gamma_{22} - \Gamma_{11})E_2 + 2\Gamma_{12}E_1) + g_2\Gamma_{33}E_2 + g_3(\Gamma_{11} + \Gamma_{22})E_2 + g_4\Gamma_{23}E_3, \quad (56d)$$

$$-M_{3c} = c_1 + 2c_3B_3 + c_4(B_1^2 + B_2^2) + 3c_5B_3^2, \quad (57a)$$

$$-M_{3s} = s_2\Gamma_{33} + s_3(\Gamma_{11} + \Gamma_{22}) + s_7(-\Gamma_{12}^2 + \Gamma_{11}\Gamma_{22}) + s_8(\Gamma_{23}^2 + \Gamma_{31}^2) + s_9\Gamma_{33}^2 + s_{10}(\Gamma_{11} + \Gamma_{22})^2 + s_{11}(\Gamma_{11} + \Gamma_{22})\Gamma_{33} + s_{15}(\Gamma_{31}B_1 + \Gamma_{23}B_2) + 2s_{16}\Gamma_{33}B_3 + 2s_{17}(\Gamma_{11} + \Gamma_{22})B_3, \quad (57b)$$

$$-M_{3f} = f_2E_3 + f_4(B_1E_1 + B_2E_2) + 2f_5B_3E_3 + f_7(E_1^2 + E_2^2) + f_8E_3^2, \quad (57c)$$

$$-M_{3g} = g_5(\Gamma_{31}E_1 + \Gamma_{23}E_2) + g_6\Gamma_{33}E_3 + g_7(\Gamma_{11} + \Gamma_{22})E_3. \quad (57d)$$

Equations (46*a*), (47*a*), (48*a*), (54*a*) and (57*a*) imply that a_2 and a_1 correspond to the initial stresses, b_1 to the initial polarization, and c_1 to the initial magnetization. Thus, if the magnetoelectroelastic medium considered here is in a free state, there are, in fact, only 72 independent material constants for the nonlinear magnetoelectroelastic material.

Therefore, we have derived the concise expressions of the nonlinear fully coupled constitutive equations for a $6\underline{m}\,\underline{m}$ magnetic crystal. While our results should be the foundation of future numerical analysis on the nonlinear problems of magnetoelectroelastic materials, the correctness of these equations can be further verified indirectly by reducing them to different special cases. We discuss two special cases below.

4. Small deformation but strong magnetic and electric fields

Let us assume that the magnetoelectroelastic domain is under a small deformation but with strong magnetic and electric

fields. Then, there is no difference between the Lagrangian and Eulerian strains. In other words,

$$\Gamma_{KL} = \tfrac{1}{2}(u_{K,L} + u_{L,K}) \equiv \bar{\Gamma}_{KL}. \tag{58}$$

If all the quadratic terms with respect to $\Gamma_{KL}(\bar{\Gamma}_{KL})$ are dropped, we can directly obtain, from section 3, the constitutive equations corresponding to small deformation but strong magnetic and electric fields.

In this case, the Lagrangian strain Γ_{KL} in the constitutive equations should be replaced by $\bar{\Gamma}_{KL}$, with equations (46*a*), (47*a*), (48*a*), (49*a*), (50*a*), (51*a*), (52*b*), (53*b*), (54*b*), (55*b*), (56*b*) and (57*b*) being further reduced to

$$T_{11a} = a_2 + a_3\bar{\Gamma}_{22} + a_6\bar{\Gamma}_{33} + 2a_7(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}), \tag{59}$$

$$T_{22a} = a_2 + a_3\bar{\Gamma}_{11} + a_6\bar{\Gamma}_{33} + 2a_7(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}), \tag{60}$$

$$T_{33a} = a_1 + 2a_5\bar{\Gamma}_{33} + a_6(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}), \tag{61}$$

$$T_{23a} = a_4\bar{\Gamma}_{23}, \tag{62}$$

$$T_{31a} = a_4\bar{\Gamma}_{31}, \tag{63}$$

$$T_{12a} = -a_3\bar{\Gamma}_{12}, \tag{64}$$

$$\Pi_{1d} = -d_1\bar{\Gamma}_{31} - 2d_{12}(\bar{\Gamma}_{22}E_1 - \bar{\Gamma}_{12}E_2) - 2d_{13}\bar{\Gamma}_{33}E_1 - 2d_{14}(\bar{\Gamma}_{11} + \bar{\Gamma}_{22})E_1 - d_{15}\bar{\Gamma}_{31}E_3, \tag{65}$$

$$\Pi_{2d} = -d_1\bar{\Gamma}_{23} - 2d_{12}(-\bar{\Gamma}_{12}E_1 + \bar{\Gamma}_{11}E_2) - 2d_{13}\bar{\Gamma}_{33}E_2 - 2d_{14}(\bar{\Gamma}_{11} + \tilde{\Gamma}_{22})E_2 - d_{15}\bar{\Gamma}_{23}E_3, \tag{66}$$

$$\Pi_{3d} = -d_2\bar{\Gamma}_{33} - d_3(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}) - d_{15}(\bar{\Gamma}_{31}E_1 + \bar{\Gamma}_{23}E_2) - 2d_{16}\bar{\Gamma}_{33}E_3 - 2d_{17}(\bar{\Gamma}_{11} + \bar{\Gamma}_{22})E_3, \tag{67}$$

$$M_{1s} = -s_1\bar{\Gamma}_{31} - 2s_{12}(\Gamma_{22}B_1 - \Gamma_{12}B_2) - 2s_{13}\Gamma_{33}B_1 - 2s_{14}(\Gamma_{11} + \Gamma_{22})B_1 - s_{15}\Gamma_{31}B_3, \tag{68}$$

$$M_{2s} = -s_1\bar{\Gamma}_{23} - 2s_{12}(\bar{\Gamma}_{11}B_2 - \bar{\Gamma}_{12}B_1) - 2s_{13}\bar{\Gamma}_{33}B_2 - 2s_{14}(\bar{\Gamma}_{11} + \bar{\Gamma}_{22})B_2 - s_{15}\bar{\Gamma}_{23}B_3, \tag{69}$$

$$M_{3s} = -s_2\bar{\Gamma}_{33} - s_3(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}) - s_{15}(\bar{\Gamma}_{31}B_1 + \bar{\Gamma}_{23}B_2) - 2s_{16}\bar{\Gamma}_{33}B_3 - 2s_{17}(\bar{\Gamma}_{11} + \bar{\Gamma}_{22})B_3. \tag{70}$$

5. Small deformation and weak magnetic and electric fields (linear constitutive relations)

If we further assume that the magnetic and electric fields are also weak, then all the quadratic terms with respect to E_K and B_K as well as $\bar{\Gamma}_{KL}$ can also be dropped. Therefore, the corresponding constitutive relations in the case of small deformation and weak magnetic and electric fields are linear, and they can be expressed as

$$T_{11} = a_2 + 2a_7\bar{\Gamma}_{11} + (a_3 + 2a_7)\bar{\Gamma}_{22} + a_6\bar{\Gamma}_{33} + d_3E_3 + s_3B_3, \tag{71}$$

$$T_{22} = a_2 + (a_3 + 2a_7)\bar{\Gamma}_{11} + 2a_7\bar{\Gamma}_{22} + a_6\bar{\Gamma}_{33} + d_3E_3 + s_3B_3, \tag{72}$$

$$T_{33} = a_1 + 2a_5\bar{\Gamma}_{33} + a_6(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}) + d_2E_3 + s_2B_3, \tag{73}$$

$$T_{23} = a_4\bar{\Gamma}_{23} + 0.5d_1E_2 + 0.5s_1B_2, \tag{74}$$

$$T_{31} = a_4\bar{\Gamma}_{31} + 0.5d_1E_1 + 0.5s_1B_1, \tag{75}$$

$$T_{12} = -a_3\bar{\Gamma}_{12}, \tag{76}$$

$$\Pi_1 = -d_1\bar{\Gamma}_{31} - 2b_2E_1 - f_1B_1, \tag{77}$$

$$\Pi_2 = -d_1\bar{\Gamma}_{23} - 2b_2E_2 - f_1B_2, \tag{78}$$

$$\Pi_3 = -b_1 - d_2\bar{\Gamma}_{33} - d_3(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}) - 2b_3E_3 - f_2B_3, \tag{79}$$

$$M_1 = -s_1\bar{\Gamma}_{31} - f_1E_1 - 2c_2B_1, \tag{80}$$

$$M_2 = -s_1\bar{\Gamma}_{23} - f_1E_2 - 2c_2B_2, \tag{81}$$

$$M_3 = -c_1 - s_2\bar{\Gamma}_{33} - s_3(\bar{\Gamma}_{11} + \bar{\Gamma}_{22}) - f_2E_3 - 2c_3B_3. \tag{82}$$

We note that, regardless of the initial stresses, initial polarization and magnetization, there are, in fact, only 17 independent material coefficients, $a_3, a_4, a_5, a_6, a_7, b_2, b_3, c_2, c_3, d_1, d_2, d_3, s_1, s_2, s_3, f_1$ and f_2 for the corresponding linear materials.

Equations (71)–(82) can be compared to the corresponding linear constitutive relations in the more conventional form characterized by matrices (e.g., Lee and Ma 2007):

$$\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \bar{\Gamma}_{11} \\ \bar{\Gamma}_{22} \\ \bar{\Gamma}_{33} \\ 2\bar{\Gamma}_{23} \\ 2\bar{\Gamma}_{31} \\ 2\bar{\Gamma}_{12} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -E_1 \\ -E_2 \\ -E_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & q_{31} \\ 0 & 0 & q_{31} \\ 0 & 0 & q_{33} \\ 0 & q_{15} & 0 \\ q_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -B_1 \\ -B_2 \\ -B_3 \end{Bmatrix}, \tag{83}$$

$$\begin{Bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\Gamma}_{11} \\ \bar{\Gamma}_{22} \\ \bar{\Gamma}_{33} \\ 2\bar{\Gamma}_{23} \\ 2\bar{\Gamma}_{31} \\ 2\bar{\Gamma}_{12} \end{Bmatrix} - \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{11} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} -E_1 \\ -E_2 \\ -E_3 \end{Bmatrix} - \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \begin{Bmatrix} -B_1 \\ -B_2 \\ -B_3 \end{Bmatrix}, \tag{84}$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & q_{15} & 0 \\ 0 & 0 & 0 & q_{15} & 0 & 0 \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\Gamma}_{11} \\ \bar{\Gamma}_{22} \\ \bar{\Gamma}_{33} \\ 2\bar{\Gamma}_{23} \\ 2\bar{\Gamma}_{31} \\ 2\bar{\Gamma}_{12} \end{Bmatrix} - \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \begin{Bmatrix} -E_1 \\ -E_2 \\ -E_3 \end{Bmatrix} - \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix} \begin{Bmatrix} -B_1 \\ -B_2 \\ -B_3 \end{Bmatrix}. \tag{85}$$

From such a comparison, one immediately identifies

$$a_3 = c_{12} - c_{11} = -2c_{66}, \qquad a_4 = 2c_{44}, \qquad a_5 = \tfrac{1}{2}c_{33},$$
$$a_6 = c_{13}, \qquad a_7 = \tfrac{1}{2}c_{11}, \tag{86a}$$
$$b_2 = -\tfrac{1}{2}k_{11}, \qquad b_3 = -\tfrac{1}{2}k_{33}, \tag{86b}$$
$$c_2 = -\tfrac{1}{2}\mu_{11}, \qquad c_3 = -\tfrac{1}{2}\mu_{33}, \tag{86c}$$
$$d_1 = -2e_{15}, \qquad d_2 = -e_{33}, \qquad d_3 = -e_{31}, \tag{86d}$$
$$s_1 = -2q_{15}, \qquad s_2 = -q_{33}, \qquad s_3 = -q_{31}, \tag{86e}$$
$$f_1 = -\alpha_{11}, \qquad f_2 = -\alpha_{33}. \tag{86f}$$

6. Conclusions

In this paper, we have derived the three-dimensional and fully coupled constitutive equations for a second-order nonlinear transversely isotropic (i.e., $6\underline{m}\,\underline{m}$) magnetic crystal. They are expressed in terms of independent material coefficients which need to be determined experimentally or via *ab initio* calculation or through both approaches. As an application, the constitutive relations for the special case of small deformations together with different magnetic and electric fields are also presented, and the material coefficients corresponding to small deformation and weak magnetic and electric fields can be related to those in the standard linear theory (e.g., Lee and Ma 2007). The present work forms the basis for future nonlinear analysis of dynamically loaded structures composed of magnetoelectroelastic materials with the symmetry class $6\underline{m}\,\underline{m}$ based on a numerical method (i.e., finite element formation).

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