# Surface Loading of a Multilayered Viscoelastic Pavement: Semianalytical Solution 

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#### Abstract

In this paper a new method is proposed to analyze the mechanical response of a linear viscoelastic pavement. The material parameters of the asphalt concrete are characterized by the relaxation modulus and creep compliance, which are further represented by the Prony series. By virtue of the Laplace transform and the correspondence principle, the solution in the Laplace domain is first derived. The interconversion between the relaxation modulus and creep compliance is then applied to treat the complicated inverse Laplace transform. The displacement, strain, and stress fields are represented concisely in terms of the convolution integral in the time domain, which is subsequently solved analytically. Therefore, responses of the viscoelastic pavement are finally expressed analytically in the time domain and numerically in space domain, called a semianalytical approach. Since both the relaxation modulus and creep compliance are used simultaneously, instead of only one parameter in the conventional methods, the present method is also called a dual-parameter method. The present formulation is verified at both the short- and long-term time limits analytically and at the other finite time numerically, as compared to the conventional numerical methods. We clearly show that the present dual-parameter and semianalytical method can predict accurately the time-dependent responses of the viscoelastic pavement, especially at the long-term time. The present formulation could also be employed to validate the widely used collocation method.


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## Introduction

Mechanical responses of flexible pavements are of primary importance in pavement analysis and design. Conventionally, the pavement is regarded as a layered elastic structure with many studies already being carried out based on the well-known layered elastic theory. For example, Burmister presented analytical solutions for both a two-layered pavement (Burmister 1943) and a three-layered pavement (Burmister 1945a,b,c). In the last decades, the theory has been extended for an arbitrary number of layers and various computer codes have been developed, e.g., BISAR (De Jong et al. 1979), KENLAYER (Huang 1993), JULEA (Uzen 1994), with JULEA being further incorporated into the MEPDG (NCHRP 2004). Contributions to the general layered elastic structures can also be found in the geotechnical field, e.g., in Pan (1997), Yue and Yin (1998), and Pak and Guzina (2002).

As is well known, however, an asphalt concrete (AC) layer in the flexible pavement behaves viscoelastically, and, thus, viscoelastic models are required to simulate the time-dependent behavior of the pavement (Elseifi et al. 2006). In order to predict the

[^0]viscoelastic response of the pavement, most of the previous works (Chou and Larew 1969; Huang 1967, 1973; Hopman 1996) applied the integral transform methods (e.g., Fourier and Laplace transforms) to the constitutive equation of the viscoelastic material first, and then employed the correspondence principle (Lee 1955; Radok 1957). However, there are two major challenges in solving the viscoelastic pavement response by using the correspondence principle. The first challenge lies in the difficulty of obtaining the associated elastic solution for a multilayered structure, and the other one is associated with the inverse transform where the larger the number of layers is, the more difficult to invert the transform analytically. To avoid the difficulty caused by the integral transform methods, the collocation method (Schapery 1962) was proposed along with the inverse Laplace transform (Huang 1967, 1973, 1993). The viscoelastic pavement response could also be solved by directly applying the collocation method in the time domain (Elliott and Moavenzadeh 1971; Park and Kim 1998).

Due to the challenges/difficulties mentioned above, the pavement response is often treated as elastic. For example, MEPDG (NCHRP 2004) is based on the layered elastic theory, in which the input modulus is a constant in spite of the fact that the viscoelasticity of asphalt concrete has been characterized. Even when treating the viscoelastic pavement, only simplified models, such as the beam (Hardy and Cebon 1993) or plate (Kim et al. 2002) model, were investigated. Currently, the study of viscoelastic pavement in the context of general viscoelasticity theory is usually conducted through the finite-element-based methods (Sargand 2002; Xu 2004). However, as indicated in these analyses, such methods were time-consuming and had tremendous difficulty in predicting the long-term responses of the viscoelastic pavement.

In this paper, we present a semianalytical approach to solve the time-dependent response of the viscoelastic pavement. The mate-


Fig. 1. Multilayered viscoelastic pavement structure
rial parameters of the asphalt concrete are first characterized by both the relaxation modulus and creep compliance simultaneously (we name the method as the dual-parameter method), which are further represented by the Prony series. By virtue of the Laplace transform and the correspondence principle, the solution of the layered pavement in the Laplace domain is then derived. In order to treat the complicated inverse Laplace transform, the interconversion between the relaxation modulus and creep compliance is proposed. In so doing, the displacement, strain and stress fields are finally represented concisely in terms of the convolution integral in time domain, which are subsequently solved analytically. Therefore, in terms of the proposed novel approach, responses of the viscoelastic pavement are expressed analytically in time domain and numerically in space domain, called a semianalytical approach. The present method is verified at both the short- and long-term time limits analytically and at the other finite time numerically using the conventional numerical methods. We clearly show that the present dual-parameter and semianalytical method can predict accurately the time-dependent responses of the viscoelastic pavement, especially at the long-term time. The present formulation could also be employed to validate the widely used collocation method in the future.

## Model Description and Elastic Solutions

A typical multilayered pavement is depicted in Fig. 1. The pavement is composed of $p$ layers, which are horizontally infinite with layer thickness $h_{\mathrm{i}}=z_{\mathrm{i}}-z_{\mathrm{i}-1} \quad(i=1,2, \ldots, p) . \quad H=$ total thickness above the infinite half-space, which is also called the $(p+1)$ th layer. Each layer is homogeneous, isotropic, and elastic except for the first layer, which is viscoelastic. In other words, to incorporate the viscoelasticity of asphalt concrete, layer 1 is characterized by the time-dependent modulus of elasticity $E(t)$ and time-independent Poisson's ratio $v$. The layers are perfectly bonded and, thus, the tractions and displacements are continuous across the interface. The time-dependent load $q(t)$ is uniformly applied to the circle of radius $R$ on the surface $z=0$. Since the problem is axis-symmetric, the cylindrical coordinates $(r, \theta, z)$ are employed.

The elastic pavement is examined first, in which both the Young's modulus and Poisson's ratio are time-independent constants. The constitutive equation for an elastic material is given by

$$
\begin{equation*}
\sigma_{i j}=\lambda \varepsilon_{k k} \delta_{i j}+2 G \varepsilon_{i j} \tag{1}
\end{equation*}
$$

where $\sigma_{i j}$ and $\varepsilon_{i j}=$ stress and strain tensors; $\lambda$ and $G=$ elastic Lamé constants; $\varepsilon_{k k}=$ volumetric strain (with repeated indexes taking the summation from 1 to 3 ); and $\delta_{i j}=$ Kronecker delta.

In this study, a step load is applied on the surface, i.e.

$$
\sigma_{z z}(r, z=0)=\left\{\begin{array}{cc}
-q h(t) & r<R  \tag{2}\\
0 & r>R
\end{array}\right.
$$

where $h(t)=$ Heaviside step function.
According to Pan et al. (2007b), the solutions for the purely elastic pavement at the location $(r, z)$ of layer $k$ can be expressed as integrals in the space-transformed domain. For example

$$
\begin{gather*}
u_{r}(r, z)=-\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\eta U_{M}\right) J_{1}(\eta r) \eta d \eta  \tag{3}\\
u_{z}(r, z)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(U_{L}\right) J_{0}(\eta r) \eta d \eta  \tag{4}\\
\sigma_{r z}(r, z)=-\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(T_{M}\right) J_{1}(\eta r) \eta^{2} d \eta  \tag{5}\\
\sigma_{z z}(r, z)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\frac{T_{L}}{\eta}\right) J_{0}(\eta r) \eta^{2} d \eta \tag{6}
\end{gather*}
$$

where the integral kernels $U_{L}, \eta U_{M}, T_{L} / \eta$, and $T_{M}=$ components of $[H(\eta, z)]$, i.e.

$$
\begin{equation*}
[H(\eta, z)]=\left[U_{L} \eta U_{M} T_{L} / \eta T_{M}\right]^{T} \tag{7}
\end{equation*}
$$

and $J_{n}(\eta r)=$ Bessel function of order $n$.
Boundary conditions [Eq. (2)] at $t>0$ are transformed to

$$
\left\{\begin{array}{l}
T_{L}(\eta, 0 ; t)=T_{L}(\eta, 0)=-\frac{\sqrt{2 \pi} q R}{\eta} J_{1}(\eta R)  \tag{8}\\
T_{M}(\eta, 0 ; t)=T_{M}(\eta, 0)=0
\end{array}\right.
$$

By virtue of the propagator matrix method, $[H(\eta, z)]$ can then be computed from

$$
\begin{equation*}
[H(\eta, z)]_{4 \times 1}=\left[A^{E}(\eta, z)\right]_{4 \times 4}[K(\eta)]_{4 \times 1} \tag{9}
\end{equation*}
$$

where $[K(\eta)]=$ column matrix in the half-space to be determined, and the subscripts, e.g., $4 \times 1$, denote the dimension, which will be omitted for simplicity in the following sections. In Eq. (9), the superscript $E$ also denotes the elastic problem, and

$$
\begin{align*}
{\left[A^{E}(\eta, z)\right]=} & {\left[a_{k}^{E}\left(z-z_{k}\right)\right]\left[a_{k+1}^{E}\left(h_{k+1}\right)\right] \cdots\left[a_{p-1}^{E}\left(h_{p-1}\right)\right]\left[a_{p}^{E}\left(h_{p}\right)\right] } \\
& \times\left[Z_{p}^{E}(\eta, H)\right] \tag{10}
\end{align*}
$$

with $\left[a^{E}\right]$ in Eq. (10) being termed as the propagator matrix, which links $[H(\eta, z)]$ at the top to the bottom of a layer. In other words, for layer $k$, for instance

$$
\begin{equation*}
\left[H\left(\eta, z_{k-1}\right)\right]=\left[a_{k}^{E}\right]\left[H\left(\eta, z_{k}\right)\right] \tag{11}
\end{equation*}
$$

In addition, in Eq. (10), $\left[Z^{E}\right]=$ solution matrix in the homogeneous half-space evaluated at $z=H=z_{p}$

$$
\begin{equation*}
\left[H\left(\eta, z_{p}\right)\right]=\left[Z_{p}^{E}(\eta, H)\right][K(\eta)] \tag{12}
\end{equation*}
$$

The elements of $\left[a^{E}\right]$ and $\left[Z^{E}\right]$ are listed in Appendix I.

## Viscoelastic Behavior of Asphalt Concrete

The viscoelastic behavior of asphalt concrete can be modeled by either the generalized Maxwell model consisting of a spring and $M$ Maxwell elements connected in parallel, or the generalized Kelvin model consisting of a spring and $N$ Voigt elements connected in series (Gibson et al. 2003; Elseifi et al. 2006). The relaxation modulus $E(t)$ from the generalized Maxwell model and the creep compliance $D(t)$ from the generalized Kelvin model are given by (Park and Schapery 1999)

$$
\begin{gather*}
E(t)=E_{e}+\sum_{i=1}^{M} E_{i} e^{-t / \rho_{i}}  \tag{13}\\
D(t)=D_{0}+\sum_{j=1}^{N} D_{j}\left(1-e^{-t / \tau_{j}}\right) \tag{14}
\end{gather*}
$$

where $E_{e}$ and $E_{i}=$ equilibrium modulus and relaxation strength, respectively; and $\rho_{i}=$ relaxation time; $D_{0}, D_{j}$, and $\tau_{j}=$ glassy compliance, the retardation strength, and the retardation time, respectively. It is noticed that at $t=0, E_{0}=E(0)=E_{e}+\sum_{i=1}^{M} E_{i}$ and $D_{0}=D(0)$.

The expression in the form of Eq. (13) or (14) is named as the Prony series. The Prony series of the relaxation modulus and creep compliance are mathematically equivalent if the constants are chosen properly (Park and Schapery 1999; Sargand 2002). Thus, the relaxation modulus $E(t)$ can be represented by the creep compliance $D(t)$ and vice versa, which is referred to as interconversion between the relaxation modulus and creep compliance (Park and Shapery 1999; Park and Kim 1999, 2001). It is noted that for the purely elastic material, $E(t) D(t)=1$, which was also used as a crude approximation for quasi-elasticity and weak viscoelasticity (Park and Kim 1999). For a typical viscoelastic material, however, $E(t)$ and $D(t)$ are related by (Ferry 1980; Park and Kim 1999)

$$
\begin{equation*}
\int_{0}^{t} E(t-\tau) D(\tau) d \tau=t \text { or } E(t) * D(t)=t \text { for } t>0 \tag{15}
\end{equation*}
$$

where (*) means the convolution integral.
Applying the Laplace transform $\widetilde{f}(s)=\int_{0}^{\infty} f(t) e^{-s t} d t$ to Eq. (15) and making use of Eqs. (13) and (14), we then have

$$
\begin{equation*}
\tilde{\tilde{E}}(s) \tilde{\tilde{D}}(s)=1 \tag{16}
\end{equation*}
$$

where

$$
\begin{align*}
& \tilde{\tilde{E}}(s)=s \tilde{E}(s)=E_{e}+\sum_{i=1}^{M} \frac{s E_{i} \rho_{i}}{s \rho_{i}+1}  \tag{17}\\
& \tilde{\tilde{D}}(s)=s \tilde{D}(s)=D_{0}+\sum_{j=1}^{N} \frac{D_{j}}{s \tau_{j}+1} \tag{18}
\end{align*}
$$

$\tilde{\tilde{f}}(s)=s \tilde{f}(s)$ is also called the Carson transform (Park and Schapery 1999).

## Correspondence Principle

For a general viscoelastic material, the constitutive equation is expressed as (Xu 2004)

$$
\begin{equation*}
\sigma_{i j}=\int_{-\infty}^{t} \frac{d \lambda(\tau)}{d \tau} \varepsilon_{k k}(t-\tau) \delta_{i j} d \tau+2 \int_{-\infty}^{t} \frac{d G(\tau)}{d \tau} \varepsilon_{i j}(t-\tau) d \tau \tag{19}
\end{equation*}
$$

where parameters, such as $\lambda(t), G(t)$, et al., are similar to those in Eq. (1), but are time dependent. The Lamé coefficients $\lambda(t), G(t)$ can be estimated by the following:

$$
\begin{gather*}
\lambda(t)=\frac{E(t) v}{(1+\nu)(1-2 v)}  \tag{20}\\
G(t)=\frac{E(t)}{2(1+v)} \tag{21}
\end{gather*}
$$

Applying the Laplace transform to Eqs. (19)-(21) yields

$$
\begin{gather*}
\widetilde{\sigma}_{i j}=\tilde{\tilde{\lambda}}(s) \widetilde{\varepsilon}_{k k} \delta_{i j}+2 \tilde{\tilde{G}}(s) \widetilde{\varepsilon}_{i j}  \tag{22}\\
\tilde{\lambda}(s)=\frac{\tilde{E}(s) v}{(1+\nu)(1-2 v)}  \tag{23}\\
\tilde{\tilde{G}}(s)=\frac{\tilde{\tilde{E}}(s)}{2(1+v)} \tag{24}
\end{gather*}
$$

According to the correspondence principle (Lee 1955; Radok 1957), it is observed from Eqs. (1) and (22) that the viscoelastic solution in the Laplace domain can be easily obtained from the elastic solution by directly replacing the boundary condition $q$ with $\widetilde{q}(s)$ [the Laplace transform of $q(t)$ ] and the Lamé constants $\lambda$ and $G$ with $\tilde{\tilde{\lambda}}(s)$ and $\tilde{\tilde{G}}(s)$ for the viscoelastic asphalt material.

## Dual-Parameter Method

The propagator matrix $\left[\widetilde{a}_{1}(s)\right]$ for the viscoelastic AC layer can be obtained directly by replacing $\lambda$ and $G$ with $\tilde{\tilde{\lambda}}(s)$ and $\tilde{\tilde{G}}(s)$ in the propagator matrix $\left[a_{1}\right]$. It is found that $\left[\widetilde{a}_{1}(s)\right]$ is a function of $\tilde{\tilde{E}}(s)$

$$
\begin{equation*}
\left[\widetilde{a}_{1}(s)\right]=\left[\widetilde{a}_{1}(\tilde{\tilde{E}}(s), 1 / \tilde{\tilde{E}}(s))\right] \tag{25}
\end{equation*}
$$

It can be shown that $\tilde{\tilde{E}}(s)$ is in both the numerator and denominator of the propagator matrix. For $\tilde{\tilde{E}}(s)$ in the denominator, it implies poles of $s$ in $\left[\widetilde{a}_{1}(s)\right]$, which will raise tedious mathematic complexity when the solution is transformed back to time domain using the inverse Laplace transform. In order to solve the problem, the interconversion between the relaxation modulus and creep compliance shown in Eq. (16) is employed. In other words, Eq. (25) can be equivalently expressed as

$$
\begin{equation*}
\left[\widetilde{a}_{1}(s)\right]=\left[\widetilde{a}_{1}(\tilde{\tilde{E}}(s), \tilde{\tilde{D}}(s))\right] \tag{26}
\end{equation*}
$$

Substituting Eqs. (17) and (18) into Eq. (26) and making use of Appendix I yields

$$
\begin{equation*}
\left[\widetilde{a}_{1}(s)\right]=\left[a_{1}\right]+\left[b_{1}\right] \sum_{j=1}^{N} \frac{c_{1 j}}{s \tau_{j}+1}+\left[v_{1}\right] \sum_{i=1}^{M} \frac{1}{c_{2 i}\left(s \rho_{i}+1\right)} \tag{27}
\end{equation*}
$$

where $c_{1 j}=(1+\nu) D_{j} \quad(j=0,1, \ldots, N), 1 / c_{2 i}=-E_{i} /\left[2\left(1-v^{2}\right)\right] \quad(i$ $=0,1, \ldots, M)$. The elements of $\left[a_{1}\right],\left[b_{1}\right]$, and $\left[v_{1}\right]$ are listed in Appendix II.

It is observed from Eq. (27) and Appendix II that the propagator matrix $\left[\widetilde{a}_{1}\right]$ for the viscoelastic AC layer is composed of two
parts: the first part $\left[a_{1}\right]$ is the propagator matrix for an elastic material and the other one, which includes $\left[b_{1}\right]$ and $\left[v_{1}\right]$, totally $N+M$ terms, is caused by and related to viscoelasticity.

Applying the correspondence principle to Eq. (10), substituting Eq. (27) into the result and employing the inverse Laplace transform, the expression in Eq. (9) for the field point in the viscoelastic AC layer can be derived as

$$
\begin{align*}
{[H(\eta, z ; t)]=} & {[A(\eta, z)][K(\eta ; t)]+[B(\eta, z)]\left\{\varphi_{1}(t) *[K(\eta ; t)]\right\} } \\
& +[V(\eta, z)]\left\{\varphi_{2}(t) *[K(\eta ; t)]\right\} \tag{28}
\end{align*}
$$

where

$$
\begin{gather*}
\varphi_{1}(t)=\sum_{j=1}^{N} \frac{c_{1 j}}{\tau_{j}} e^{-t / \tau_{j}}  \tag{29}\\
\varphi_{2}(t)=\sum_{i=1}^{M} \frac{1}{\rho_{i} c_{2 i}} e^{-t / \rho_{i}}  \tag{30}\\
{[A(\eta, z)]=\left[a_{1}(z)\right]\left[a_{2}\right] \cdots\left[a_{p-1}\right]\left[a_{p}\right]\left[Z_{p}(\eta, H)\right]}  \tag{31}\\
{[B(\eta, z)]=\left[b_{1}(z)\right]\left[a_{2}\right] \cdots\left[a_{p-1}\right]\left[a_{p}\right]\left[Z_{p}(\eta, H)\right]}  \tag{32}\\
{[V(\eta, z)]=\left[v_{1}(z)\right]\left[a_{2}\right] \cdots\left[a_{p-1}\right]\left[a_{p}\right]\left[Z_{p}(\eta, H)\right]} \tag{33}
\end{gather*}
$$

Again in Eq. (28) [ $K(\eta ; t)]$ is a $4 \times 1$ time-dependent column matrix in the subgrade half-space and two of its components are zeros since it is required that the solution should be finite at infinity. Therefore, $[K(\eta ; t)]_{4 \times 1}=\left[[0]_{2 \times 1},[\kappa(\eta ; t)]_{2 \times 1}\right]$. Eq. (28) can then be rewritten as

$$
\begin{equation*}
[U(\eta, z ; t)]=[A(\eta, z)]_{12}[\kappa(\eta ; t)]+[B(\eta, z)]_{12}\left\{\varphi_{1}(t) *[\kappa(\eta ; t)]\right\} \tag{34}
\end{equation*}
$$

$$
\begin{equation*}
[T(\eta, z ; t)]=[A(\eta, z)]_{22}[\kappa(\eta ; t)]+[V(\eta, z)]_{22}\left\{\varphi_{2}(t) *[\kappa(\eta ; t)]\right\} \tag{35}
\end{equation*}
$$

where $[U(\eta, z ; t)]=\left[U_{L}, \eta U_{M}\right]^{T}, \quad[T(\eta, z ; t)]=\left[T_{L} / \eta, T_{M}\right]^{T}$; and $[\sim]_{12}$ and $[\sim]_{22}=$ submatrices of $[\sim]$ with dimension of $2 \times 2$, e.g.

$$
[A(\eta, z)]=\left[\begin{array}{ccc}
{[A(\eta, z)]_{11}} & \vdots & {[A(\eta, z)]_{12}}  \tag{36}\\
\cdots \cdots \cdots & . & \cdots \cdots \cdots \\
{[A(\eta, z)]_{21}} & \vdots & {[A(\eta, z)]_{22}}
\end{array}\right]
$$

Conventionally, either the relaxation modulus or creep compliance can be used to solve such a viscoelastic problem. Here in this section, we show that by using the relaxation modulus and creep compliance simultaneously, the displacement and traction in the viscoelastic AC layer can be represented concisely and analogously as in Eqs. (34) and (35). Thus, this method is called a dual-parameter method. It is noticed that the displacement in Eq. (34) is related to the creep compliance by $\varphi_{1}(t)$ and the traction in Eq. (35) is related to the relaxation modulus by $\varphi_{2}(t)$.

## Viscoelastic Solution

## Volterra System of Equations

We consider the traction boundary condition at the surface $z=0$ in Eq. (35)

$$
\begin{equation*}
[T(\eta, 0 ; t)]=[A(\eta, 0)]_{22}[\kappa(\eta ; t)]+[V(\eta, 0)]_{22}\left\{\varphi_{2}(t) *[\kappa(\eta ; t)]\right\} \tag{37}
\end{equation*}
$$

where $[T(\eta, 0 ; t)]=[T(\eta, 0)]=$ time step load expressed in Eq. (8). It can be observed that Eq. (37) involves the convolution integral of $[\kappa(\eta ; t)]$, leading to tedious computation complexity. The evaluation of convolution integrals is always a very challenging work and gains extensive attention in the field of both mathematics and engineering (Linz 1985). Conventionally, the numerical method is adopted to deal with the convolution integral. For example, Park and Kim (1999) introduced several common numerical rules and Xu (2004) employed the trapezoid rule in dealing with the convolution integral. However, the computation at an arbitrary time by using those numerical rules has two closely related problems: the first one is that it needs all records of previous steps, and the other one is that the error in the previous step will propagate to the current step. Thus, this computation is not only very time-consuming, but also unreliable, especially for long-term response prediction. We solve this problem by employing our dual-parameter and semianalytical approach.

## Matrix Equation and Solution

We first rewrite Eq. (37) as a linear Volterra system of equations of second kind (Linz 1985)

$$
\begin{equation*}
[\kappa(\eta ; t)]=[f(\eta ; t)]-[C(\eta)] \sum_{i=1}^{M} l_{i}\left[w_{i}(\eta ; t)\right] \tag{38}
\end{equation*}
$$

where

$$
\begin{gather*}
{[f(\eta ; t)]=[A(\eta, 0)]_{22}^{-1}[T(\eta, 0 ; t)]}  \tag{39}\\
{[C(\eta)]=[A(\eta, 0)]_{22}^{-1}[V(\eta, 0)]_{22}}  \tag{40}\\
{\left[w_{i}(\eta ; t)\right]=\int_{0}^{t} e^{\beta_{i}(t-\tau)}[\kappa(\eta ; \tau)] d \tau} \tag{41}
\end{gather*}
$$

$$
\begin{equation*}
l_{i}=1 / \rho_{i} c_{2 i} \tag{42}
\end{equation*}
$$

$$
\begin{equation*}
\beta_{i}=-1 / \rho_{i} \tag{43}
\end{equation*}
$$

Since the kernel in the convolution integral [Eq. (41)] is exponential, the following expression can be derived:

$$
\begin{gather*}
{\left[\dot{w}_{i}(\eta ; t)\right]=[\kappa(\eta ; t)]+\beta_{i}\left[w_{i}(\eta ; t)\right]}  \tag{44}\\
{\left[w_{i}(\eta ; 0)\right]=0} \tag{45}
\end{gather*}
$$

where $\operatorname{dot}(\cdot)$ denotes differentiation with respect to time $t$. Substituting Eq. (38) into Eq. (44) yields

$$
\begin{equation*}
\left[\dot{w}_{i}(\eta ; t)\right]+[C(\eta)] \sum_{i=1}^{M} l_{i}\left[w_{i}(\eta ; t)\right]-\beta_{i}\left[w_{i}(\eta ; t)\right]=[f(\eta ; t)] \tag{46}
\end{equation*}
$$

Expanding Eq. (46) into the matrix form, together with the initial condition [Eq. (45)] yields

$$
\begin{gather*}
{[\dot{W}(\eta ; t)]+[Q(\eta)][W(\eta ; t)]=[F(\eta ; t)]}  \tag{47}\\
{[W(\eta ; 0)]=0} \tag{48}
\end{gather*}
$$

where

$$
\begin{gather*}
{[W(\eta ; t)]=\left[\left[w_{1}(\eta ; t)\right]^{T},\left[w_{2}(\eta ; t)\right]^{T}, \ldots,\left[w_{M}(\eta ; t)\right]^{T}\right]^{T}}  \tag{49}\\
{[F(\eta ; t)]=\left[[f(\eta ; t)]^{T},[f(\eta ; t)]^{T}, \ldots,[f(\eta ; t)]^{T}\right]^{T}}  \tag{50}\\
{[Q(\eta)]=\left[\begin{array}{ccccc}
{[C(\eta)] l_{1}-\beta_{1} I_{0}} & {[C(\eta)] l_{2}} & \cdots & {[C(\eta)] l_{i}} & \cdots \\
{[C(\eta)] l_{1}} & {[C(\eta)] l_{2}-\beta_{2} I_{0}} & \cdots & {[C(\eta)] l_{i}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
{[C(\eta)] l_{M}} \\
{[C(\eta)] l_{1}} & {[C(\eta)] l_{2}} & \cdots & {[C(\eta)] l_{i}-\beta_{i} I_{0}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots C(\eta)] l_{M} \\
{[C(\eta)] l_{1}} & {[C(\eta)] l_{2}} & \cdots & {[C(\eta)] l_{i}} & \cdots \\
\vdots & {[C(\eta)] l_{M}-\beta_{M} I_{0}}
\end{array}\right]} \tag{51}
\end{gather*}
$$

and $I_{0}=2 \times 2$ identity matrix.
Now, the general solution to the linear equation system [Eq. (47)] can be expressed as

$$
\begin{equation*}
[W(\eta ; t)]=[X]\left\langle e^{\omega_{i} t}\right\rangle[\Delta]+\left\langle e^{\omega_{i} t}\right\rangle \int\left\langle e^{-\omega_{i} \tau}\right\rangle[X]^{-1}[F(\eta ; \tau)] d \tau \tag{52}
\end{equation*}
$$

where $[\Delta]=$ unknown coefficient vector; $\omega_{i}$ and $[X]=$ eigenvalues and eigenvector matrix of the $2 M \times 2 M$ matrix [Q], i.e.

$$
\begin{equation*}
[X]\left\langle\omega_{i}\right\rangle[X]^{-1}=-[Q] \tag{53}
\end{equation*}
$$

and

$$
\left\langle e^{\omega_{i} t}\right\rangle=\left[\begin{array}{llll}
e^{\omega_{1} t} & & &  \tag{54}\\
& e^{\omega_{2} t} & & \\
& & \ddots & \\
& & & e^{\omega_{2 M} t}
\end{array}\right]
$$

For the step load in this study, $[f(\eta ; t)]=[f(\eta)]$ and $[F(\eta ; t)]$ $=[F(\eta)]$ at $t>0$. Eq. (52) can then be reduced to the following expression by combining the initial condition [Eq. (48)]

$$
\begin{equation*}
[W(\eta ; t)]=[X]\left\langle\frac{e^{\omega_{i} t}-1}{\omega_{i}}\right\rangle[X]^{-1}[F(\eta)] \tag{55}
\end{equation*}
$$

## Pavement Response

From Eq. (49), it can be seen that the elements of [ $W$ ] will also follow the form of Eq. (55) and can be further written as

$$
\left[w_{i}(\eta ; t)\right]=\left[\begin{array}{l}
W_{2 i-1}(\eta ; t)  \tag{56}\\
W_{2 i}(\eta ; t)
\end{array}\right]=\left(\left[\Psi_{i}\right]+\left[\Phi_{i}\right]\right)[f(\eta)] \quad(i=1,2, \ldots, M)
$$

where

$$
\begin{align*}
& {\left[\Psi_{i}\right]=\sum_{k=1}^{M} \sum_{m=1}^{2 M}\left[\begin{array}{cc}
\xi_{2 i-1, m} \zeta_{m, 2 k-1} & \xi_{2 i-1, m} \zeta_{m, 2 k} \\
\xi_{2 i, m} \zeta_{m, 2 k-1} & \xi_{2 i, m} \zeta_{m, 2 k}
\end{array}\right] \frac{e^{\omega_{m} t}}{\omega_{m}}}  \tag{57}\\
& {\left[\Phi_{i}\right]=\sum_{k=1}^{M} \sum_{m=1}^{2 M}\left[\begin{array}{cc}
\xi_{2 i-1, m} \zeta_{m, 2 k-1} & \xi_{2 i-1, m} \zeta_{m, 2 k} \\
\xi_{2 i, m} \zeta_{m, 2 k-1} & \xi_{2 i, m} \zeta_{m, 2 k}
\end{array}\right] \frac{-1}{\omega_{m}}} \tag{58}
\end{align*}
$$

with $\xi, \zeta$ being the elements of $[X]$ and $[X]^{-1}$, respectively.
Substituting Eq. (56) into Eq. (38) gives

$$
\begin{equation*}
[\kappa(\eta ; t)]=[f(\eta ; t)]-[C(\eta)] \sum_{i=1}^{M} l_{i}\left[w_{i}(\eta ; t)\right]=\left[\chi_{0}\right]+\sum_{m=1}^{2 M}\left[\chi_{m}\right] e^{\omega_{m} t} \tag{59}
\end{equation*}
$$

where

$$
\left[\chi_{m}\right]=-\frac{1}{\omega_{m}}[C(\eta)] \sum_{i=1}^{M} \sum_{k=1}^{M} b_{i}\left[\begin{array}{cc}
\xi_{2 i-1, m} \zeta_{m, 2 k-1} & \xi_{2 i-1, m} \zeta_{m, 2 k}  \tag{60}\\
\xi_{2 i, m} \zeta_{m, 2 k-1} & \xi_{2 i, m} \zeta_{m, 2 k}
\end{array}\right][f(\eta)]
$$

$$
\begin{equation*}
\left[\chi_{0}\right]=[f(\eta)]-\sum_{m=1}^{2 M}\left[\chi_{m}\right] \tag{61}
\end{equation*}
$$

Substituting Eq. (59) into Eqs. (34) and (35) yields the following expansion coefficients for the displacement and traction vectors at a field point in the AC layer $\left(z<h_{1}\right)$ :

$$
\begin{align*}
& {[U(\eta, z ; t)]=\left[\Omega_{0}\right]+\sum_{m=1}^{2 M}\left[\Omega_{m}\right] e^{\omega_{m} t}+\sum_{j=1}^{N}\left[\Omega_{2 M+j}\right] e^{\alpha_{j} t}}  \tag{62}\\
& {[T(\eta, z ; t)]=\left[\Theta_{0}\right]+\sum_{m=1}^{2 M}\left[\Theta_{m}\right] e^{\omega_{m} t}+\sum_{i=1}^{M}\left[\Theta_{2 M+i}\right] e^{\beta_{i} t}} \tag{63}
\end{align*}
$$

where

$$
\begin{gather*}
{\left[\Omega_{0}\right]=[A(\eta, z)]_{12}\left[\chi_{0}\right]+\sum_{j=1}^{N} \frac{-r_{j}}{\alpha_{j}}[B(\eta, z)]_{12}\left[\chi_{0}\right]}  \tag{64}\\
{\left[\Omega_{m}\right]=[A(\eta, z)]_{12}\left[\chi_{m}\right]+\sum_{j=1}^{N} \frac{r_{j}}{p_{m}-\alpha_{j}}[B(\eta, z)]_{12}\left[\chi_{m}\right]}  \tag{65}\\
{\left[\Omega_{2 M+j}\right]=\frac{r_{j}}{\alpha_{j}}[B(\eta, z)]_{12}\left[\chi_{0}\right]+r_{j} \sum_{m=1}^{2 M} \frac{-1}{p_{m}-\alpha_{j}}[B(\lambda, z)]_{12}\left[\chi_{m}\right]} \tag{66}
\end{gather*}
$$

$$
\begin{equation*}
\left[\Theta_{0}\right]=\left[A\left(\eta, z_{f}\right)\right]_{22}\left[\chi_{0}\right]+\sum_{i=1}^{M} \frac{-b_{i}}{\beta_{i}}\left[V\left(\lambda, z_{f}\right)\right]_{22}\left[\chi_{0}\right] \tag{67}
\end{equation*}
$$

$$
\begin{equation*}
\left[\Theta_{m}\right]=\left[A\left(\eta, z_{f}\right)\right]_{22}\left[\chi_{m}\right]+\sum_{i=1}^{M} \frac{l_{i}}{p_{m}-\beta_{i}}[V(\eta, z)]_{22}\left[\chi_{m}\right] \tag{68}
\end{equation*}
$$

$$
\begin{equation*}
\left[\Theta_{2 M+i}\right]=\frac{l_{i}}{\beta_{i}}[V(\eta, z)]_{22}\left[\chi_{0}\right]+l_{i} \sum_{m=1}^{2 M} \frac{-1}{p_{m}-\beta_{i}}[V(\eta, z)]_{22}\left[\chi_{m}\right] \tag{69}
\end{equation*}
$$

with $r_{j}=1 / \tau_{j} c_{1 j}$; and $\alpha_{j}=-1 / \tau_{j}$.
For a field point in the elastic layer $\left(z>h_{1}\right)$, we have

$$
\begin{align*}
{[U(\eta, z ; t)]=} & {[A(\eta, z)]_{12}[\kappa(\eta ; t)]=[A(\eta, z)]_{12}\left[\chi_{0}\right] } \\
& +\sum_{m=1}^{2 M}[A(\eta, z)]_{12}\left[\chi_{m}\right] e^{\omega_{m} t}  \tag{70}\\
{[T(\eta, z ; t)]=} & {[A(\eta, z)]_{22}[\kappa(\eta ; t)]=[A(\eta, z)]_{22}\left[\chi_{0}\right] } \\
& +\sum_{m=1}^{2 M}[A(\eta, z)]_{22}\left[\chi_{m}\right] e^{\omega_{m} t} \tag{71}
\end{align*}
$$

With these expansion coefficients, the final (nonzero) solutions in the cylindrical coordinates can be derived and expressed in the form of Bessel integrals similar to Eqs. (3)-(6)

$$
\left.\begin{array}{c}
u_{r}(r, z ; t)=-\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\eta U_{M}\right) J_{1}(\eta r) \eta d \eta \\
u_{z}(r, z ; t)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(U_{L}\right) J_{0}(\eta r) \eta d \eta \\
\sigma_{r z}(r, z ; t)=-\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(T_{M}\right) J_{1}(\eta r) \eta^{2} d \eta \\
\sigma_{z z}(r, z ; t)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\frac{T_{L}}{\eta}\right) J_{0}(\eta r) \eta^{2} d \eta \\
\sigma_{r r}(r, z ; t)=\frac{v}{1-v} \sigma_{z z}+\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty} \frac{\hat{E}(t)}{1+v} \\
*\left(\eta U_{M}\right)\left[-\frac{1}{1-v} J_{0}(\eta r) \eta+\frac{J_{1}(\eta r)}{r}\right] \eta d \eta \\
\sigma_{\theta \theta}(r, z ; t)=\frac{2 v}{1-v} \sigma_{z z}-\sigma_{r r}-\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty} \frac{\hat{E}(t)}{1-v} *\left(\eta U_{M}\right) J_{0}(\eta r) \eta^{2} d \eta \\
\varepsilon_{z z}(r, z ; t)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\frac{v}{1-v^{2}}\right)\left(\eta U_{M}\right) J_{0}(\eta r) \eta^{2} d \eta \\
\varepsilon_{r z}(r, z ; t)=-\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\frac{1-v-2 v^{2}}{1-v}\right) \hat{D}(t)^{*} *\left(\frac{T_{L}}{\eta}\right) J_{0}(\eta r) \eta^{2} d \eta \\
(1+v) \hat{D}(t)^{*}\left(T_{M}\right) J_{1}(\eta r) \eta^{2} d \eta \tag{79}
\end{array} \text { (77)} \text { (77) }\right\}
$$

$$
\begin{align*}
\varepsilon_{r r}(r, z ; t)= & \frac{-1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\eta U_{M}\right) J_{0}(\eta r) \eta^{2} d \eta \\
& +\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\eta U_{M}\right) \frac{J_{1}(\eta r)}{r} \eta d \eta  \tag{80}\\
\varepsilon_{\theta \theta}\left(r, z_{f} ; t\right)= & -\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty}\left(\eta U_{M}\right) \frac{J_{1}(\eta r)}{r} \eta d \eta \tag{81}
\end{align*}
$$

where $\left(\eta U_{M}\right),\left(U_{L}\right),\left(T_{M}\right)$, and $\left(T_{L} / \eta\right)$ are the time-dependent integral kernels solved by either Eqs. (62) and (63) or Eqs. (70) and (71). The expressions involving convolution integral ( ${ }^{*}$ ) can be further expressed as the summation of the exponential terms. In the above expressions, $\hat{E}(t)$ and $\hat{D}(t)$ are also the inverse Laplace transforms of $\tilde{\tilde{E}}(s)$ and $\tilde{\tilde{D}}(s)$, namely

$$
\begin{align*}
& \hat{E}(t)=E_{0} \delta(t)-\sum_{i=1}^{M} \frac{E_{i}}{\rho_{i}} e^{-t / \rho_{i}}  \tag{82}\\
& \hat{D}(t)=D_{0} \delta(t)+\sum_{j=1}^{M} \frac{D_{j}}{\tau_{j}} e^{-t / \tau_{j}} \tag{83}
\end{align*}
$$

where $\delta(t)=$ Dirac delta function, which will disappear after convolution integral ( ${ }^{*}$ ).

In order to find the solutions in the Cartesian coordinates, one can simply apply the coordinate transformation between the cylindrical and Cartesian systems.

It can be concluded that by using the dual-parameter method, the displacement, strain, and stress fields can be expressed as the Prony series of exponential terms, which could be applied to verify the collocation method. In the collocation method, on the other hand, the number of exponential terms is assumed to be fixed for any field point. However, it is shown that the number of exponential terms varies with the position of the field point. For instance, for certain $\eta$, there are $2 M$ exponential terms for the displacement if the field point is not located in the AC layer, while the number is $2 M+N$ for displacement and $3 M$ for traction for the field point in the AC layer.

The final solutions in the Prony series form also imply that the solutions are analytical in time domain, and one needs only to carry out the infinite integration of the products of Bessel function in the space domain as shown in Eqs. (72)-(81). As such, these solutions are semianalytical. As for the integration of the products of Bessel function, various numerical methods have been developed, and in this paper, we adopt the algorithm proposed by Lucas (1995). In other words, the infinite integration is approximated by the summation of a series of partial integrations as

$$
\begin{equation*}
\int_{0}^{+\infty} f(\eta, z) J_{m}(\eta r) d \eta=\sum_{n=1}^{N} \int_{\eta_{n}}^{\eta_{n+1}} f(\eta, z) J_{m}(\eta r) d \eta \tag{84}
\end{equation*}
$$

The numerical details on the integration of Eq. (84) are similar to those in our recent elastic pavement program MultiSmart3D, a fast and accurate program developed for the multilayered elastic pavement (Pan et al. 2007a).

## Analytical Verification

Before numerical computation, two special cases, named the initial state and the steady state, will be discussed to validate the
above solution analytically. It can be derived from Eqs. (13), (14), (17), and (18), that the following relations hold:

$$
\begin{gather*}
E(0)=E_{0}=E_{e}+\sum_{i=1}^{M} E_{i}=\lim _{s \rightarrow \infty} \tilde{E}(s)  \tag{85}\\
D(0)=D_{0}=\lim _{s \rightarrow \infty} \tilde{\tilde{D}}(s) \tag{86}
\end{gather*}
$$

Therefore

$$
\begin{equation*}
E(0) D(0)=\lim _{s \rightarrow \infty} \tilde{\tilde{E}}(s) \tilde{\tilde{D}}(s)=1 \tag{87}
\end{equation*}
$$

Similarly

$$
\begin{equation*}
E(\infty) D(\infty)=\lim _{t \rightarrow \infty} E(t) D(t)=\lim _{s \rightarrow 0} \tilde{E}(s) \tilde{\tilde{D}}(s)=1 \tag{88}
\end{equation*}
$$

For the elastic material, $E(t) D(t)=1$, thus, from Eqs. (87) and (88) it can be anticipated that at the initial and steady states the viscoelastic material behaves like an elastic material and the viscoelastic solution should be the same as the elastic solution. We verify these in the next two subsections.

## Initial-State Solution

The solutions for $z_{f}<h_{1}$ at $t=0$ can be reduced from Eqs. (62) and (63) as

$$
\begin{align*}
& {[U(\eta, z ; t=0)]=\left[\Omega_{0}(\eta, z)\right]+\sum_{m=1}^{2 M}\left[\Omega_{m}(\eta, z)\right]+\sum_{j=1}^{N}\left[\Omega_{2 M+j}(\eta, z)\right]}  \tag{89}\\
& {[T(\eta, z ; t=0)]=\left[\Theta_{0}(\eta, z)\right]+\sum_{m=1}^{2 M}\left[\Theta_{m}(\eta, z)\right]+\sum_{i=1}^{M}\left[\Theta_{2 M+i}(\eta, z)\right]} \tag{90}
\end{align*}
$$

By combining Eqs. (64)-(69) with Eq. (59), Eqs. (89) and (90) can be rewritten as

$$
\begin{align*}
& {[U(\eta, z ; t=0)]=[A(\eta, z)]_{12}[\kappa(\eta ; 0)]}  \tag{91}\\
& {[T(\eta, z ; t=0)]=[A(\eta, z)]_{22}[\kappa(\eta ; 0)]} \tag{92}
\end{align*}
$$

It has been found that $\left[a_{1}\right]$, the elastic part of the propagator matrix for the viscoelastic material in Eq. (27), can be expressed in terms of the two parameters $E(0)$ and $D(0)$ as

$$
\begin{equation*}
\left[a_{1}(\eta, z)\right]=\left[a_{1}(\eta, z ; D(0), E(0))\right] \tag{93}
\end{equation*}
$$

Making use of Eq. (87) and Appendixes I and II, it is easy to prove that

$$
\begin{equation*}
\left[a_{1}(\eta, z ; D(0), E(0))\right]=\left[a_{1}^{E}(\eta, z ; E(0))\right] \tag{94}
\end{equation*}
$$

It should be stressed that Eq. (94) holds only when $E D=1$. After straightforward computation

$$
\begin{equation*}
[A(\eta, z)]=[A(\eta, z ; D(0), E(0))]=\left[A^{E}(\eta, z ; E(0))\right] \tag{95}
\end{equation*}
$$

Finally, substituting Eq. (95) into Eqs. (91) and (92) yields the elastic solution

$$
\begin{align*}
& {[U(\eta, z ; t=0)]=\left[A^{E}(\eta, z ; E(0))\right]_{12}[\kappa(\eta ; 0)]}  \tag{96}\\
& {[T(\eta, z ; t=0)]=\left[A^{E}(\eta, z ; E(0))\right]_{22}[\kappa(\eta ; 0)]} \tag{97}
\end{align*}
$$

## Steady-State Solution

From Eqs. (62) and (63), the solutions for $z_{f}<h_{1}$ at $t \rightarrow \infty$ are reduced to

$$
\begin{align*}
& {[U(\eta, z ; t \rightarrow \infty)]=\left[\Omega_{0}(\eta, z)\right]}  \tag{98}\\
& {[T(\eta, z ; t \rightarrow \infty)]=\left[\Theta_{0}(\eta, z)\right]} \tag{99}
\end{align*}
$$

Similarly, by combining Eqs. (64)-(69) with Eq. (59), Eqs. (98) and (99) are rewritten as

$$
\begin{equation*}
[U(\eta, z ; t \rightarrow \infty)]=\left([A(\eta, z)]_{12}+\sum_{j=1}^{N} \frac{-r_{i}}{\alpha_{i}}[B(\eta, z)]_{12}\right)[\kappa(\eta ; t \rightarrow \infty)] \tag{100}
\end{equation*}
$$

$[T(\eta, z ; t \rightarrow \infty)]=\left([A(\eta, z)]_{22}+\sum_{i=1}^{M} \frac{-l_{i}}{\beta_{i}}[V(\eta, z)]_{22}\right)[\kappa(\eta ; t \rightarrow \infty)]$
(101)

After straightforward mathematical manipulations, it is found that

$$
\begin{align*}
& {[A(\eta, z)]_{12}+\sum_{j=1}^{N} \frac{-r_{i}}{\alpha_{i}}[B(\eta, z)]_{12}=[A(\eta, z ; D(\infty), E(\infty))]_{12}}  \tag{102}\\
& {[A(\eta, z)]_{22}+\sum_{i=1}^{M} \frac{-l_{i}}{\beta_{i}}[V(\eta, z)]_{22}=[A(\eta, z ; D(\infty), E(\infty))]_{22}} \tag{103}
\end{align*}
$$

Following the same procedure for the initial-state problem and making use of Eq. (88), we arrive at

$$
\begin{equation*}
[A(\eta, z ; D(\infty), E(\infty))]=\left[A^{E}(\eta, z ; E(\infty))\right] \tag{104}
\end{equation*}
$$

Making use also of Eq. (104), Eqs. (100) and (101) are finally reduced to

$$
\begin{align*}
& {[U(\eta, z ; t \rightarrow \infty)]=\left[A^{E}(\eta, z ; E(\infty))\right]_{12}[\kappa(\eta ; t \rightarrow \infty)]}  \tag{105}\\
& {[T(\eta, z ; t \rightarrow \infty)]=\left[A^{E}(\eta, z ; E(\infty))\right]_{22}[\kappa(\eta ; t \rightarrow \infty)]} \tag{106}
\end{align*}
$$

It is shown clearly from Eqs. (96) and (97) and Eqs. (105) and (106) that at $t=0$ and $\mathrm{t} \rightarrow \infty$, the viscoelastic model is reduced to the elastic one, and, thus, the viscoelastic solution can be obtained by directly solving the elastic model with $q(t)$ and $E(t)$ being replaced by their values at the corresponding times.

## Numerical Examples

As proved above, the response of the viscoelastic pavement at the initial and steady states can be evaluated directly through the elastic solution. However, to evaluate the pavement performance during its service life, it is important to predict the pavement response at any chosen time. We present the numerical results below for the time-dependent response of the selected pavements.

## Material Characterization

The proposed method can be applied to compute the response of the viscoelastic pavement at any time. An example discussed in Xu (2004) will be examined. In this example, the pavement is composed of AC, granular subbase, and subgrade with parameters listed in Table 1. This pavement is under the action of a dual-tire

Table 1. Typical Three-Layer Pavement (Xu 2004)

| Layer | Thickness <br> $(\mathrm{in})$. | Young's modulus <br> $(\mathrm{psi})$ | Poisson's <br> ratio |
| :--- | :---: | :---: | :---: |
| AC | 8.0 | Viscoelastic | 0.35 |
| Subbase | 8.0 | 30,000 | 0.3 |
| Subgrade | Infinite | 5,000 | 0.3 |

single-axle load of 18 kip. The tire centers are on the surface of the pavement with coordinates $(x, y)=(0,0)$ and $(0,12 \mathrm{in}$.), and the tire pressure and tire radius are 100 psi and 3.785 in ., respectively. The AC layer is viscoelastic and the relaxation modulus $E_{i}$ and relaxation time $\rho_{i}$ presented in Eq. (13) are listed in Table 2. The corresponding creep compliance and retardation time presented in Eq. (14) are computed using Eq. (16) and listed in Table 3.

Table 2. Relaxation Modulus and Relaxation Time for the Viscoelastic AC Layer (Xu 2004)

|  | $E_{i}$ <br> $(\mathrm{psi})$ | $\rho_{i}$ <br> $(\mathrm{~s})$ |
| :--- | :---: | :---: |
| - | $12,500\left(\right.$ i.e., $\left.E_{e}\right)$ | - |
| 1 | 735,300 | 0.008441 |
| 2 | 386,200 | 0.1319 |
| 3 | 107,500 | 1.968 |
| 4 | 20,360 | 39.25 |

Table 3. Creep Compliance and Retardation Time for the Viscoelastic AC Layer

|  | $D_{j}$ <br> $\left(\mathrm{psi}^{-1}\right)$ | $\tau_{j}$ <br> $j$ |
| :--- | :---: | :---: |
| 0 | $7.92451 \times 10^{-7}$ | - |
| 1 | $8.72639 \times 10^{-7}$ | 0.0189201 |
| 2 | $3.56881 \times 10^{-6}$ | 0.452776 |
| 3 | $1.80369 \times 10^{-5}$ | 8.61009 |
| 4 | $5.67292 \times 10^{-5}$ | 117.703 |



The variation of $E(t)$ and $D(t)$ versus time is plotted in Fig. 2(a), showing that $E(t)$ decreases quickly with time, while $D(t)$ increases slowly with time. The product of $E(t)$ and $D(t)$ is plotted in Fig. 2(b). As proved analytically, the values of $E(t) D(t)$ at $t=0$ and $\mathrm{t} \rightarrow \infty$ are equal to 1 , implying an elastic response at these two time moments. It is also observed from Fig. 2(b) that $E(t) D(t)$ varies dramatically immediately after the loads are applied. The minimum value of $E(t) D(t)$ is about 0.65 around $t=0.5 \mathrm{~s}$, indicating that the viscoelasticity will affect the responses significantly in this time interval.

## Numerical Comparison with Finite-Element-Based Method

The stresses $\sigma_{z z}, \sigma_{y z}$ and strains $\varepsilon_{x x}, \varepsilon_{y y}$, along the depth of the pavement at $t=0.01 \mathrm{~s}$ and $t=1 \mathrm{~s}$ are calculated and plotted in Fig. 3. For comparison, the results of Xu (2004) based on the finite element method are also plotted. It is clearly observed from Fig. 3 that, when $t$ is small, e.g., $t=0.01 \mathrm{~s}$, the stresses from Xu's solution and the present semianalytical solution agree well with each other. The strains from both Xu's solution and the present semianalytical solution also agree well with each other for field points above the subgrade. At $t=1 \mathrm{~s}$, however, both results deviate from each other substantially. In Xu's method and other finite-element-based methods, numerical schemes were employed to treat the convolution integral numerically in the time domain. However, the convolution integral has a property of "memory," i.e., a computation at current time requires the computation at all previous times, and, thus, errors will accumulate with time. This is why Xu's results agree well with the present method at a small time ( $t=0.01$ ), but not at a larger time $(t=1 \mathrm{~s})$. The present method, on the other hand, is analytical in the time domain and, thus, circumvents this drawback.

Fig. 4 shows the time variation of the horizontal strains $\varepsilon_{x x}$ and $\varepsilon_{y y}$ at $(x, y, z)=(0,0,7.99$ in. $)$. It is observed from Fig. 4 that the strain increases monotonically to a peak value at about $t=5 \mathrm{~s}$, and then decreases to its limiting steady state with increasing time. We point out again that while most traditional numerical methods have difficulty in predicting correctly the long-term responses be-


Fig. 2. Variation of (a) the relaxation modulus $E(t)$ and creep compliance $D(t)$; (b) the product of $E(t)$ and $D(t)$ with time


Fig. 3. Comparison between present method and finite-element-based method: (a) stress $\sigma_{z z}$; (b) stress $\sigma_{y z}$; (c) strain $\varepsilon_{x x}$; and (d) strain $\varepsilon_{y y}$
cause of the required large amount of memory in computation, the present semianalytical method can easily capture the actual trend of the field response at large times since it is analytical in the time domain.

## Numerical Comparison with the Collocation Method

The only method that can predict pavement long-term responses is the collocation method proposed by Schapery (1962). The collocation method has been widely used in computing viscoelastic pavement responses (Huang 1993; Park and Kim 1998). In terms of the collocation method, the viscoelastic response is expressed by the Prony series

$$
\begin{equation*}
R(t)=\sum_{k=1}^{K} \Gamma_{k} e^{-t / T_{k}} \tag{107}
\end{equation*}
$$

where $\Gamma_{k}$ and $T_{k}(k=1,2, \ldots, K)$ are constants. While the time
constants $T_{k}$ are preassumed, e.g., $T_{k}=\{0.01,0.03,0.1,1,10$, $30, \infty\}$ as adopted in Huang (1993), the coefficients $\Gamma_{k}$ are solved from Eq. (107) by imposing the response of Eq. (107) equal to the given results at the collocation time points.

We showed analytically that the pavement viscoelastic response can be expressed in terms of the Prony series, which involves infinite terms. However, in the classical collocation method, only finite terms were employed, i.e., Eq. (107). As a result, the collocation method may lead to inaccurate results. For instance, Figs. 5 and 6 show the time-dependent vertical displacement $u_{z}$ at $(x, y, z)=(0,0,0)$ and the critical strain $\varepsilon_{x x}$ at $(x, y, z)=(0,0,7.99$ in. $)$ based on the present semianalytical method and the collocation method. In the collocation method, two collocation schemes are used: seven-point collocations with time constants $T_{k}=\{0.01,0.03,0.1,1,10,30, \infty\}$, and 18-point collocations with $T_{k}=\{0.01,0.03,0.1,1,3,10,30,60,100,200,300$, $400,500,600,700,800,2,000, \infty\}$.


Fig. 4. Variation of horizontal strains $\varepsilon_{x x}$ and $\varepsilon_{y y}$ at $(x, y, z)=(0,0,7.99$ in.) with time $t$ : (a) short-time response; (b) long-term response

It is observed from Figs. 5 and 6 that, in short-term time, the time-dependent response predicted by the collocation method, especially the displacement, agrees very well with the result from the present method. The only discrepancy lies roughly in the time range $[2,6] \mathrm{s}$. In the long-term time, on the other hand, the prediction by the seven-point collocations shows great discrepancy in the time range $[50,400] \mathrm{s}$, while the prediction is improved greatly by the 18 -point collocations. However, it should be noticed that although the 18 -point collocations predicted results closer to those from the presented method, as compared to the seven-point collocations, it failed to capture the monotonic trend of the strain $\varepsilon_{x x}$ as shown in Fig. 6(b). Therefore, while the responses of the viscoelastic pavement could be approximated by the collocation method, one may need to preselect more time collocation points. Furthermore, the collocation scheme is problem-dependent and as such it is difficult to be executed for general complicated cases. On the other hand, the proposed semianalytical solution can be applied to conduct much more comprehensive studies in the future.

## Conclusions

In this paper, a dual-parameter method is developed to solve the time-dependent response of the multilayered viscoelastic pavement under surface loadings. By virtue of the dual parameters, a semianalytical solution is derived in terms of which the pavement responses are expressed analytically in the time domain for the first time. This method avoids the numerical complexity in integral transform methods and the expensive computational cost in the traditional numerical handling of viscoelasticity. The present solution also reveals that the responses are indeed in the form of the Prony series, with infinite series terms. Thus, the present result can be further applied to verify other numerical methods in viscoelasticity, such as the widely adopted collocation method. Since the present approach can be applied to predict accurately the pavement responses at any time quickly, it can be applied to conduct a reliable study on the mechanistic analysis of the viscoelastic pavement in the near future.


Fig. 5. Comparison of deflection $u_{\mathrm{z}}$ at $(x, y, z)=(0,0,0)$ between the present and collocation methods: (a) short-term response; (b) long-term response


Fig. 6. Comparison of the critical strain $\varepsilon_{x x}$ at $(x, y, z)=(0,0,7.99 \mathrm{in}$.) between the present and collocation methods: (a) short-term response; (b) long-term response

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## Appendix I

1. The elements of the solution matrix $\left[Z^{E}(z)\right]$ of each layer in Eq. (12) can be expressed as (omitting the superscript $E$ )

$$
\begin{align*}
& Z_{11}=c_{1} e^{\eta z} \quad Z_{12}=c_{1} e^{-\eta z} \quad Z_{13}=\left(\frac{c_{2}}{\eta}+c_{1} z\right) e^{\eta z} \\
& Z_{14}=\left(\frac{-c_{2}}{\eta}+c_{1} z\right) e^{-\eta z} \quad Z_{21}=c_{1} e^{\eta z} \\
& Z_{22}=-c_{1} e^{-\eta z} Z_{23}=\left(\frac{c_{3}}{\eta}+c_{1} z\right) e^{\eta z} \\
& Z_{24}=\left(\frac{c_{3}}{\eta}-c_{1} z\right) e^{-\eta z} \\
& Z_{31}=e^{\eta z} Z_{32}=-e^{-\eta z} \\
& Z_{33}=\left(\begin{array}{ll}
\left.-\frac{1}{\eta}+z\right) e^{\eta z} & Z_{34}=\left(\begin{array}{ll}
-\frac{1}{\eta}-z
\end{array}\right) e^{-\eta z} \\
Z_{41} & =e^{\eta z} \\
Z_{42} & =e^{-\eta z} \\
Z_{43} & =z e^{\eta z} \\
Z_{44}=z e^{-\eta z}
\end{array}\right. \tag{108}
\end{align*}
$$

where $c_{\mathrm{i}}(i=1,2,3)=$ material coefficients in the layer and are related to the Young's modulus $E$ and Poisson's ratio $v$ as

$$
\begin{equation*}
c_{1}=\frac{1+v}{E} \quad c_{2}=-\frac{2(1+v)(1-v)}{E} \quad c_{3}=\frac{(1+v)(1-2 v)}{E} \tag{109}
\end{equation*}
$$

2. The elements of the propagator matrix $\left[a^{E}\right]$ of layer $k$ in Eq. (11) can be expressed as (omitting the subscript $k$ and superscript $E$ )

$$
\begin{gather*}
a_{11}=a_{33}=\cosh (\eta h)+\gamma \eta h \sinh (\eta h) \\
a_{12}=-a_{43}=(\gamma+1) \sinh (\eta h)+\gamma \eta h \cosh (\eta h) \\
a_{13}=-c_{1}(\gamma+2) \sinh (\eta h)-\gamma \eta h c_{1} \cosh (\eta h) \\
a_{14}=-a_{23}=-\gamma \eta h c_{1} \sinh (\eta h) \\
a_{21}=-a_{34}=(\gamma+1) \sinh (\eta h)-\gamma \eta h \cosh (\eta h) \\
a_{22}=a_{44}=\cosh (\eta h)-\gamma \eta h \sinh (\eta h) \\
a_{24}=c_{1}(\gamma+2) \sinh (\eta h)+c_{1} \gamma \eta h \cosh (\eta h) \\
a_{31}=\sinh (\eta h) / c_{2}-\eta h \cosh (\eta h) / c_{2} \\
a_{32}=-a_{41}=-\eta h \sinh (\eta h) / c_{2} \\
a_{42}=\sinh (\eta h) / c_{2}+\eta h \cosh (\eta h) / c_{2} \tag{110}
\end{gather*}
$$

where $\gamma=-1 / 2(1-v)$, and $h=$ thickness of the layer.

## Appendix II

1. The elements of the propagator matrix $\left[a_{1}\right]$ of layer 1 with thickness $h_{1}$ in Eq. (27) can be expressed as

$$
\begin{gathered}
a_{11}=a_{33}=\cosh \left(\eta h_{1}\right)+\gamma_{1} \eta h_{1} \sinh \left(\eta h_{1}\right) \\
a_{12}=-a_{43}=\eta h_{1} \gamma_{1} \cosh \left(\eta h_{1}\right)+\left(\gamma_{1}+1\right) \sinh \left(\eta h_{1}\right) \\
a_{13}=-c_{10} \gamma_{1} \eta h_{1} \cosh \left(\eta h_{1}\right)-c_{10}\left(\gamma_{1}+2\right) \sinh \left(\eta h_{1}\right) \\
a_{14}=-a_{23}=-c_{10} \gamma_{1} \eta h_{1} \sinh \left(\eta h_{1}\right) \\
a_{21}=-a_{34}=\left(\gamma_{1}+1\right) \sinh \left(\eta h_{1}\right)-\gamma_{1} \eta h_{1} \cosh \left(\eta h_{1}\right)
\end{gathered}
$$

$$
\begin{gather*}
a_{22}=a_{44}=\cosh \left(\eta h_{1}\right)-\gamma_{1} \eta h_{1} \sinh \left(\eta h_{1}\right) \\
a_{24}=c_{10}\left(\gamma_{1}+2\right) \sinh \left(\eta h_{1}\right)+c_{10} \gamma_{1} \eta h_{1} \cosh \left(\eta h_{1}\right) \\
a_{31}=\sinh \left(\eta h_{1}\right) / c_{20}-\eta h_{1} \cosh \left(\eta h_{1}\right) c_{20} \\
a_{32}=-a_{41}=-\eta h_{1} \sinh \left(\eta h_{1}\right) / c_{20} \\
a_{42}=\sinh \left(\eta h_{1}\right) / c_{20}+\eta h_{1} \cosh \left(\eta h_{1}\right) / c_{20} \tag{111}
\end{gather*}
$$

where $\gamma_{1}=-1 / 2\left(1-v_{1}\right) ; \quad c_{10}=\left(1+v_{1}\right) D_{0} ;$ and $1 / c_{20}=-E_{0} /$ $\left(2\left(1-v_{1}^{2}\right)\right)$.
2. The nonzero elements of matrices $\left[b_{1}\right]$ and $\left[v_{1}\right]$ of layer 1 with thickness $h_{1}$ in Eq. (27) can be expressed as

$$
\begin{gather*}
b_{13}=-b_{24}=-\gamma_{1} \eta h_{1} \cosh \left(\eta h_{1}\right)-\left(\gamma_{1}+2\right) \sinh \left(\eta h_{1}\right) \\
b_{14}=-b_{23}=-\gamma_{1} \eta h_{1} \sinh \left(\eta h_{1}\right)  \tag{112}\\
v_{31}=v_{32}=\eta h_{1} \cosh \left(\eta h_{1}\right)-\sinh \left(\eta h_{1}\right) \\
v_{41}=-\eta h_{1} \sinh \left(\eta h_{1}\right) \\
v_{42}=-\eta h_{1} \cosh \left(\eta h_{1}\right)-\sinh \left(\eta h_{1}\right) \tag{113}
\end{gather*}
$$

## References

Burmister, D. M. (1943). "The theory of stress and displacements in layered systems and applications to the design of airport runways." Highw. Res. Board, Proc. Annu. Meet., 23, 126-144.
Burmister, D. M. (1945a). "The general theory of stress and displacements in layered soil systems. I." J. Appl. Phys., 16(2), 89-94.
Burmister, D. M. (1945b). "The general theory of stress and displacements in layered soil systems. II." J. Appl. Phys., 16(3), 126-127.
Burmister, D. M. (1945c). "The general theory of stress and displacements in layered soil systems. III." J. Appl. Phys., 16(5), 296-302.
Chou, Y. T., and Larew, H. G. (1969). "Stresses and displacements in viscoelastic pavement systems under a moving load." Highway Research Record. 282, Highway Research Board, National Research Council, Washington, D.C., 24-40.
De Jong, D. L., Peutz, M. G. F., and Korswagen, A. R. (1979). "Computer program BISAR. Layered systems under normal and tangential surface load." External Rep. No. AMSR. 0006.73, Koninklijke/Shell Laboratorium, Amsterdam, The Netherlands.
Elliott, J. F., and Moavenzadeh, F. (1971). "Analysis of stresses and displacements in three-layer viscoelastic systems." Highway Research Record. 345, Highway Research Board, National Research Council, Washington, D.C., 45-57.
Elseifi, M. A., Al-Qadi, I. L., and Yoo, P. J. (2006). "Viscoelastic modeling and field validation of flexible pavement." J. Eng. Mech., 132(2), 172-178.
Ferry, J. D. (1980). Viscoelastic properties of polymers, 3rd Ed., Wiley, New York.
Gibson, N. H., Schwartz, C. W., Schapery, R. A., and Witczak, M. W. (2003). "Viscoelastic, viscoplastic, and damage modeling of asphalt concrete in unconfined compression." Transportation Research Record. 1860, Transportation Research Board, National Research Council, Washington, D.C., 3-15.
Hardy, M. S. A., and Cebon, D. (1993). "Response of continuous pavement to moving dynamic loads." J. Eng. Mech., 119(9), 1762-1780.
Hopman, P. C. (1996). "VEROAD: A viscoelastic multilayer computer
program." Transportation Research Record. 1539, Transportation Research Board, National Research Council, Washington, D.C., 72-80.
Huang, Y. H. (1967). "Stresses and displacements in viscoelastic layered systems under circular loaded area." Proc., 2nd Int. Conf. on the Structural Design of Asphalt Pavements, 225-244.
Huang, Y. H. (1973). "Stresses and strains in viscoelastic multilayer systems subjected to moving loads." Highway Research Record. 457, Highway Research Board, National Research Council, Washington, D.C., 60-71.

Huang, Y. H. (1993). Pavement analysis and design, Prentice-Hall, Englewood Cliffs, N.J.
Kim, S. M., Won, M. C., and McCullough, B. F. (2002). "Dynamic stress response of concrete pavements to moving tandem-axle loads." Transportation Research Record. 1809, Transportation Research Board, National Research Council, Washington, D.C., 72-80.
Lee, E. H. (1955). "Stress analysis in viscoelastic bodies." Q. Appl. Math., 13, 183-190.
Linz, P. (1985). Analytical and numerical methods for Voterra equations, SIAM Studies in Applied Mathematics, SIAM, Philadelphia.
Lucas, S. K. (1995). "Evaluation of infinite integrals involving products of Bessel functions of arbitrary order." J. Comput. Appl. Math., 64, 269-282.
National Cooperative Highway Research Program (NCHRP). (2004). Guide for mechanistic-empirical design of new and rehabilitated pavement structures, NCHRP, Washington, D.C.
Pak, R. Y. S., and Guzina, B. B. (2002). "Three-dimensional Green's functions for a multilayered half-space in displacement potential." J. Eng. Mech., 128(4), 449-461.

Pan, E. (1997). "Static Green's functions in multilayered half-spaces." Appl. Math. Model., 21, 509-521.
Pan, E., Alkasawneh, W., and Chen, E. (2007a). "An exploratory study on functionally graded materials with applications to multilayered pavement design." Rep. No. FHWA/OH/2007-12.
Pan, E., Bevis, M., Han, F., Zhou, H., and Zhu, R. (2007b). "Surface development due to loading of a layered elastic half-space. I: A rapid numerical kernel based on a circular loading element." Geophys. J. Int., 171(1), 11-24.
Park, S. W., and Kim, Y. R. (1998). "Analysis of layered viscoelastic system with transient temperature." J. Eng. Mech., 124(2), 223-231.
Park, S. W., and Kim, Y. R. (1999). "Interconversion between relaxation modulus and creep compliance for viscoelastic solids." J. Mater. Civ. Eng., 11(1), 76-82.
Park, S. W., and Kim, Y. R. (2001). "Fitting Prony-series viscoelastic models with power-law presmoothing." J. Mater. Civ. Eng., 13(1), 26-32.
Park, S. W., and Shapery, R. A. (1999). "Methods of interconversion between linear viscoelastic material functions. Part I: A numerical method based on Prony series." Int. J. Solids Struct., 36, 1653-1675.
Radok, J. R. M. (1957). "Viscoelastic stress analysis." Q. Appl. Math., 15, 198-202.
Sargand, S. (2002). "Three dimensional modeling of flexible pavements." Rep. No. FHWA/HWY-02/2002.
Schapery, R. A. (1962). "Approximation method of transform inversion for viscoelastic stress analysis." Proc., 4th National Congress of Applied Mechanics, 1075-1085.
Uzen, J. (1994). "Advanced backcalculation techniques." Nondestructive testing of pavements and backcalculation of moduli, ASTM STP 1198, 3-37.
Xu, Q. (2004). "Modeling and computing for layered pavement under vehicle loading." Ph.D. thesis, North Carolina State Univ., Raleigh, N.C.

Yue, Z., and Yin, J. (1998). "Backward transfer-matrix method for elastic analysis of layered solids with imperfect bonding." J. Elast., 50, 109128.


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