An Integral Equation Formulation of Three-Dimensional Inhomogeneity Problems

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Abstract A displacement integral equation formulation of three-dimensional infinite isotropic matrix with inhomogeneities of arbitrary shapes is derived based on the assumption that both the inhomogeneity and matrix have the same Poisson's ratio. Compared to the conventional boundary integral equation formulation which requires both the tractions and displacements on the interface between the inhomogeneity and matrix, the present displacement integral formulation only contains the unknown interface displacements. Therefore, its numerical implementation can easily be carried out since the handling of corners in any irregular shaped inhomogeneity is avoided. Thus, through the interface discretization using quadrilateral boundary elements, the resulting system of equations can be formulated so that the interface displacements can be obtained. Stresses at any point of interest can also be obtained by using the corresponding stress integral equation formulation which contains only the inhomogeneity-matrix interface displacements. Numerical results from the present approach are in excellent agreement with existing ones.

1 Introduction

Since Eshelby's classic work on the problem of an elastic ellipsoidal inhomogeneity embedded in an infinite elastic medium (1957, 1959), various inclusion/inhomogeneity problems have been extensively investigated (Mura, 1987; Ting, 1996). The interaction between the inhomogeneity and matrix can be analyzed using various numerical methods, e.g. the finite element method (FEM) and boundary element method (BEM).

The FEM has been used to solve various inclusion/inhomogeneity problems, e.g. Thomson and Hancock (1984), Ghosh and Mukhopadhyay (1993) and Nakamura and Suresh (1993). For the irregular shaped inhomogeneity with random space

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distribution, the finite element discretization could be difficult. Interaction among various three-dimensional (3D) inclusions/inhomogeneities was investigated using a domain integral equation method by Dong et al. (2003a) in which only the isotropic fundamental solution is required even for anisotropic materials so that the complicated fundamental solution for anisotropic medium can be avoided. However, the drawback is that the inhomogeneity needs to be discretized into finite elements. The BEM can also be used to solve various inhomogeneity problems based on the inhomogeneity-matrix interface conditions, i.e. displacement continuity and traction equilibrium (Dong et al., 2003b). However, for inhomogeneities of irregular shapes, special technique, e.g. discontinuous elements or double nodes in coupling the equations from the inhomogeneity to the matrix, is required to satisfy the displacement continuity condition and traction equilibrium at the corner. This demand is difficult for 3D inhomogeneity problems, especially for those inhomogeneities with irregular shapes.

In this paper, following the method proposed by Leite et al. (2003) for two-dimensional (2D) reinforced solid, a displacement integral equation formulation is developed to solve 3D inhomogeneity problems by assuming that the matrix and inhomogeneity have the same Poisson's ratio. The corresponding stress integral equation formulation is also derived in this paper. Compared to the conventional BEM for the inhomogeneity problem, the present integral equation formulation only contains the interface displacements (no interface tractions). Similar to the domain integral equation approach (Dong et al., 2003a), the present integral equation approach can also overcome the corner singularity problems (due to different normal directions at the corner nodes). As such, various irregular inhomogeneity problems, especially in 3D, can be easily solved using the derived displacement integral formulation. Numerical examples are presented to show the validity and efficiency of the proposed method.

2 Basic Formulation

For an infinite isotropic matrix subjected to remote stresses, the displacement and stress integral equations at point P in the matrix can be given as follows (Dong et al., 2003a)

$$u_k(P) = u_k^0(P) + \int_\Gamma U_{ki}(P,q)\, t_i(q)\, d\Gamma(q) - \int_\Gamma T_{ki}(P,q)\, u_i(q)\, d\Gamma(q) \quad (1)$$

and

$$\sigma_{kl}(P) - \sigma_{kl}^0(P) = \int_\Gamma U_{kli}(P,q)\, t_i(q)\, d\Gamma(q) - \int_\Gamma T_{kli}(P,q)\, u_i(q)\, d\Gamma(q) \quad (2)$$

where q is the field point acting at the inhomogeneity-matrix interface Γ. u_k^0 and σ_{kl}^0 are, respectively, the displacements and stresses at point P caused by the remote stresses in an infinite homogeneous isotropic elastic matrix. U_{ki}, T_{ki}, U_{kli} and T_{kli} are the fundamental solutions to the infinite isotropic elastic medium (Brebbia and Dominguez, 1992).

When the source point P approaches the boundary point p being on the inhomogeneity-matrix interface, Eq. (1) becomes

$$c_{ki} u_i(p) = u_k^0(p) + \int_\Gamma U_{ki}(p,q)\, t_i(q)\, d\Gamma(q) - \int_\Gamma^c T_{ki}(p,q)\, u_i(q)\, d\Gamma(q) \quad (3)$$

where c_{ki} depends on the boundary geometry at the source point p. The symbol $\int^c$ denotes the Cauchy principal value integral.

For the I-th isotropic inhomogeneity, the corresponding displacement boundary integral equation can be given as (Brebbia and Dominguez, 1992)

$$c_{ki}^I(p) u_i^I(p) = \int_{\Gamma_I} U_{ki}^I(p,q)\, t_i^I(q)\, d\Gamma(q) - \int_{\Gamma_I}^c T_{ki}^I(p,q)\, u_i^I(q)\, d\Gamma(q) \quad (4)$$

where Γ_I represents the I-th inhomogeneity-matrix interface.

If we assume that the Poisson's ratio is the same for both the matrix and inhomogeneity, one can then find the following relationships (Leite et al., 2003)

$$U_{ki}^I = \frac{G}{G^I} U_{ki} \quad (5a)$$

$$T_{ki}^I = T_{ki} \quad (5b)$$

$$U_{kli}^I = U_{kli} \quad (5c)$$

$$T_{kli}^I = \frac{G^I}{G} T_{kli} \quad (5d)$$

Substituting (5a) and (5b) into Eq. (4), then adding Eqs. (3) and (4), and considering the interface conditions, one can obtain the following displacement boundary integral equation

$$c_{ki}\left(1 + \frac{G^I}{G}\right) u_i(p) = u_i^0(p) - \int_{\Gamma_I}^c \left(1 - \frac{G^I}{G}\right) T_{ki}(p,q)\, u_i(q)\, d\Gamma \quad (6)$$

For the matrix with multiple inhomogeneities, Eq. (6) can be extended to the following form

$$\begin{aligned} c_{ki}\left(1 + \frac{G^I}{G}\right) u_i(p) = u_i^0(p) &- \int_{\Gamma_I}^c \left(1 - \frac{G^I}{G}\right) T_{ki}(p,q)\, u_i(q)\, d\Gamma \\ &- \sum_{J=1,\neq I}^{NI} \int_{\Gamma_J} \left(1 - \frac{G^J}{G}\right) T_{ij}(p,q)\, u_j(q)\, d\Gamma \end{aligned} \quad (7)$$

where *NI* denotes the number of the inhomogeneities. The source point p is acting on the I-th inhomogeneity-matrix interface.

Similarly, the stress integral equation for point P being in the inhomogeneity or matrix can be written as

$$\sigma_{kl}(P) = \sigma_{kl}^{0}(P) - \sum_{I=1}^{NI} \int_{\Gamma_I} \left(1 - \frac{G^I}{G}\right) T_{kli}(P, q)\, u_i(q)\, d\Gamma \tag{8}$$

From Eqs. (7) and (8), one can find that the displacement and stress integral equations contain only the inhomogeneity-matrix interface displacements. Note that the tractions on the inhomogeneity-matrix interfaces disappear in these two integral equations. Therefore, unlike the conventional BEM in which the discontinuous element has to be used near the corner of the irregular inhomogeneities (Dong et al., 2003b), arbitrary inhomogeneity shapes can be easily treated using Eqs. (7) and (8). Therefore, Eqs. (7) and (8) can be considered as an extension of the 2D formulation of Leite et al. (2003) to the corresponding 3D elasticity.

In numerical implementation, quadratic quadrilateral boundary elements are used to discretize the inhomogeneity-matrix interfaces. Thus, the resulting system of equations can be written as

$$\boldsymbol{A}\mathbf{U} = \widetilde{\mathbf{U}} \tag{9}$$

where $\boldsymbol{A}$ is the related coefficient matrix from Eq. (7). $\mathbf{U}$ is the vectors of the inhomogeneity-matrix interface nodal displacements. $\widetilde{\mathbf{U}}$ is the vector of the displacements on the inhomogeneity-matrix interface due to the remote stresses.

Once the interface displacements are available, the stresses at point P in the inhomogeneity or matrix can be calculated using Eq. (8).

3 Numerical Examples

In this section, a couple of numerical examples are presented to show that the proposed formulation is accurate and efficient in analyzing 3D inhomogeneity problems. These include some benchmark problems where analytical solutions are available and problems that can only be solved numerically.

3.1 A Spherical Inhomogeneity Embedded in an Infinite Isotropic Medium (Matrix)

A spherical inhomogeneity is embedded in an infinite isotropic elastic medium subjected to the remote unit stresses, i.e. $\sigma_{xx}^0 = \sigma_{yy}^0 = \sigma_{zz}^0 = \sigma^0 = 1$, along x-, y-, and z-axes respectively. The Poisson's ratio is taken to be 0.3 for both the matrix and inhomogeneity. The radius of the spherical inhomogeneity is assumed to be $R = 1$.

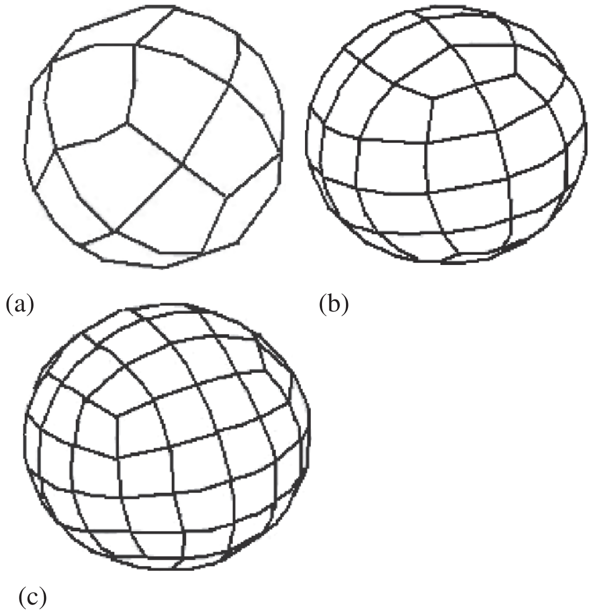

Fig. 1 Discretization of the spherical surface using quadratic quadrilateral boundary elements of 24 in (**a**), 64 in (**b**), and 96 in (**c**)

The interface between the matrix and inhomogeneity is discretized into 24, 64 and 96 quadratic quadrilateral boundary elements, respectively, as shown in Fig. 1. Assuming that the ratio of the Young's moduli between the matrix and inhomogeneity is $E_M/E_I = 1/2$ in which E_M and E_I denote the Young's moduli of the matrix and inhomogeneity, respectively. Due to the symmetry of the geometry and loadings, only the stress σ_{xx} along z-axis within the spherical inhomogeneity is displayed in Fig. 2. It can be found that with increasing elements the results from the present method are in excellent agreement with existing analytical solutions (Dong et al., 2003a).

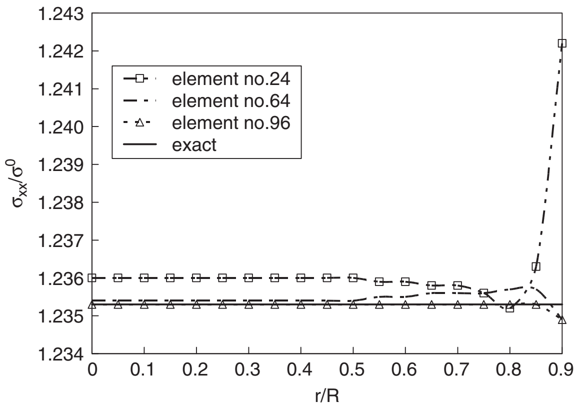

Fig. 2 Variation of the normalized stress σ_{xx}/σ^0 along z-axis within the spherical inhomogeneity (r/R is the normalized distance of the observation point to the spherical center)

3.2 A Cylindrical Inhomogeneity Embedded in an Infinite Isotropic Elastic Medium

A cylindrical inhomogeneity with radius $r = 1$ and height $h = 0.5$ is embedded in an infinite isotropic elastic medium subjected to the remote loading $\sigma_{xx}^0 = \sigma_{yy}^0 = \sigma_{zz}^0 = \sigma^0 = 1$. The ratio of the Young's moduli between the matrix and inhomogeneity is taken to be $E_M/E_I = 1/2$, and the corresponding Poisson's ratios are chosen as $\nu_M = \nu_I = 0.3$. It is noted again that the symbols with subscripts M and I denote the values related to the matrix and inhomogeneity, respectively.

The surface of the cylindrical inhomogeneity is discretized with 52 quadratic quadrilateral boundary elements as shown in Fig. 3. For comparison, the FEM is also employed to solve the same problem. The infinite medium is approximated by a cylindrical body with radius $R = 20$ and height $h = 20$. The corresponding FEM solutions are obtained by using 46337 tetrahedron 4-node elements as shown in Fig. 4 (symmetric property has been considered). The results at selected points A (0, 0, 0), B (0, 0, −0.25), C (0, 0, −0.5) and D (0, 0.081, 0.0077) are given in Table 1. It is observed clearly from Table 1 that the results from the present method are in good agreement with those from FEM.

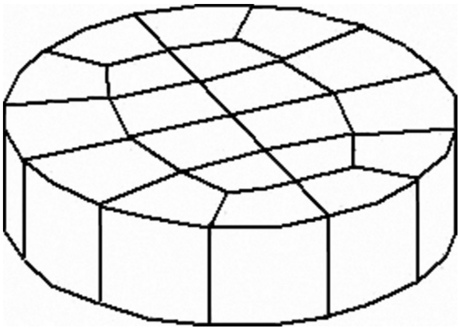

Fig. 3 Boundary element meshes of the cylindrical inhomogeneity

3.3 Two Spherical Inhomogeneities Embedded in an Infinite Isotropic Medium

Two spherical inhomogeneities with radii R_1 and R_2 and material parameters E_1, ν_1 and E_2, ν_2 are embedded in an infinite isotropic elastic matrix with material parameters E_M, ν_M (Fig. 5). The matrix is subjected to the remote loading $\sigma_{xx}^0 = \sigma_{yy}^0 = \sigma_{zz}^0 = \sigma^0 = 1$. The Poisson's ratio of the inhomogeneity and matrix are again taken to be 0.3. The distance between the centers of the two spheres is assumed to be d.

In the numerical calculation, 96 quadratic quadrilateral boundary elements are used to discretize the surface of each inhomogeneity as shown in Fig. 1(c). For

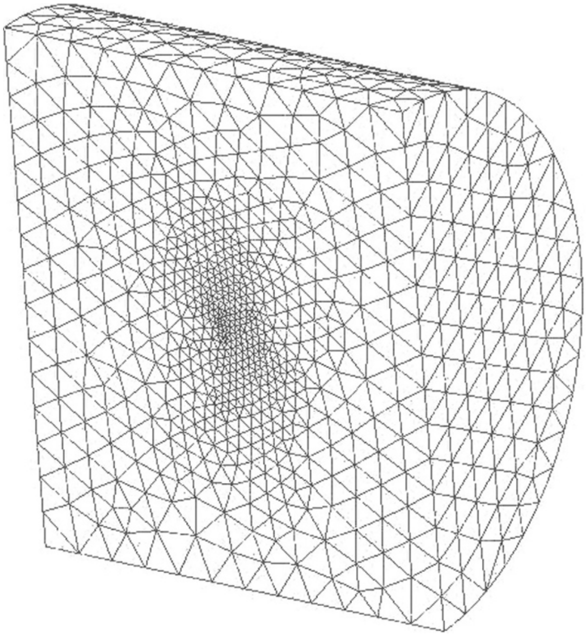

Fig. 4 Finite element meshes of one half of the cylindrical inhomogeneity

Table 1 Comparison of the results from BEM and FEM

	Point A		Point B		Point C		Point D	
	BEM	FEM	BEM	FEM	BEM	FEM	BEM	FEM
σ_{xx}/σ^0	1.376	1.409	1.373	1.398	1.367	1.387	0.9703	0.9623
σ_{yy}/σ^0	1.082	1.130	1.085	1.100	1.098	1.070	1.025	1.024
σ_{zz}/σ^0	1.376	1.407	1.371	1.399	1.354	1.390	0.9703	0.9652

$R_1 = R_2 = 1$, $d = 1.5$, $E_1 = E_2$ and $E_1/E_M = 0.5$, the stresses from the proposed formulation along y- and z-axes are displayed, respectively, in Figs. 6 and 7. It can be found that $\sigma_{xx} = \sigma_{zz}$ along y-axis due to the symmetry of problem. For $R_1 = R_2 = 1$, $d = 1.5$, $E_2/E_M = 2$ and $E_1/E_M = 0.5$, the stresses along y- and z-axes are

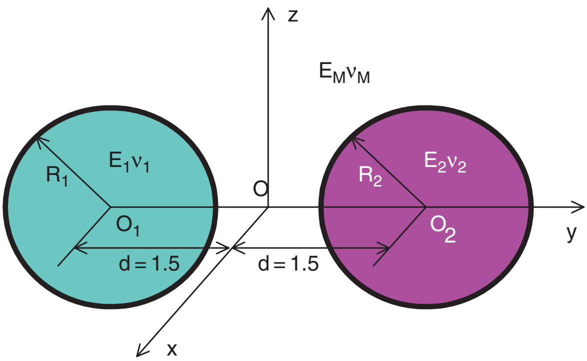

Fig. 5 Two spherical inhomogeneities in an infinite matrix

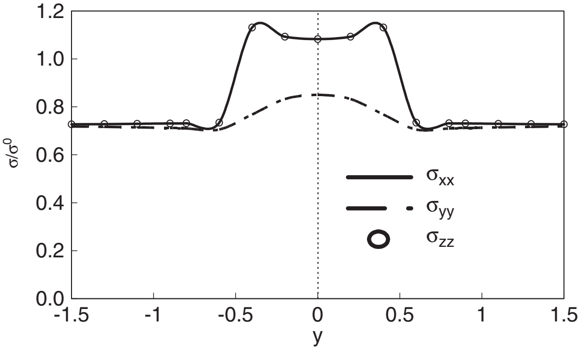

Fig. 6 Variation of the normalized stresses σ/σ^0 along y-axis ($E_1/E_M = 0.5$, $E_1 = E_2$)

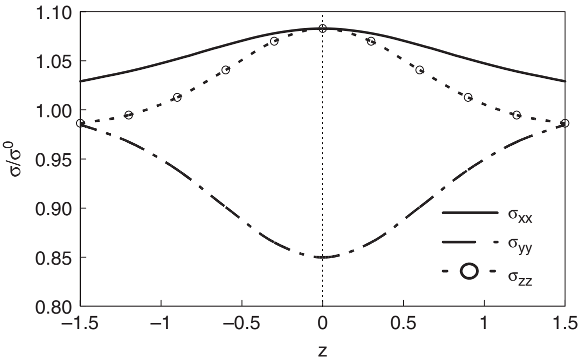

Fig. 7 Variation of the normalized stresses σ/σ^0 along z-axis ($E_1/E_M = 0.5$, $E_1 = E_2$)

displayed, respectively, in Figs. 8 and 9. Since the two spherical inhomogeneities now have different material properties, the stress distribution along y-axis is no longer symmetric about the vertical z-axis. However, the stress along z-axis is still symmetric about the vertical z-axis.

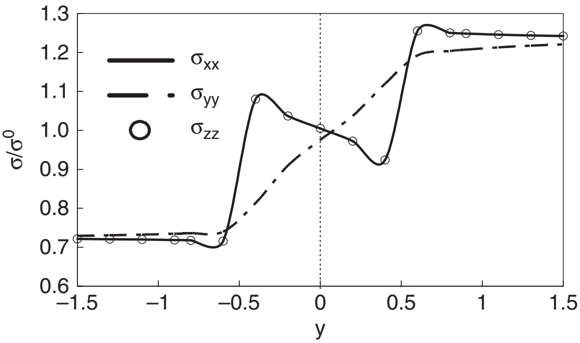

Fig. 8 Variation of the normalized stresses σ/σ^0 along y-axis ($E_1/E_M = 0.5$, $E_2/E_M = 2$)

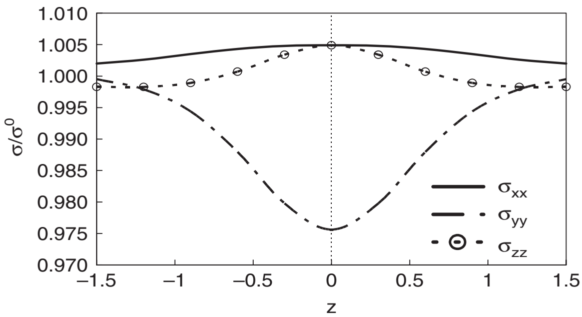

Fig. 9 Variation of the normalized stresses σ/σ^0 along z-axis ($E_1/E_M = 0.5$, $E_2/E_M = 2$)

4 Conclusions

An integral equation formulation for 3D inhomogeneity problems is derived based on the assumption that the matrix and inhomogeneity have the same Poisson's ratio. The proposed integral equation formulation only contains the displacements on the inhomogeneity-matrix interface. Therefore, the present formulation can easily be applied to solve various irregular inhomogeneity problems because it does not contain the interface-matrix traction (so that the corner singularity issue can be avoided). Three inhomogeneity problems are studied using the present formulation. The obtained results are in good agreement with those available in literature, and they can be considered as benchmark solutions for future investigation.

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