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Non-uniform Eshelby's tensor inside a spherical inclusion in a functionally graded space in transport phenomena

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ABSTRACT

Within the framework of thermal conduction, we consider a functionally graded isotropic infinite medium containing a spherical inclusion which undergoes prescribed uniform heat flux-free temperature gradient. In this research the thermal conductivity is assumed to be exponentially varied in space. Analytical expressions in series form for the temperature and the second-order Eshelby's conduction tensor inside and outside the spherical inclusion are obtained. Our analytical results indicate that the second-order Eshelby's conduction tensor is non-uniform within the spherical inclusion and that it is in general not symmetric. Furthermore our numerical results quantitatively demonstrate how the Eshelby's tensor within the spherical inclusion is non-uniformly distributed due to the spatially varying thermal conductivity.

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1. Introduction

Within the framework of steady thermal conduction, the Eshelby's inclusion problem (Eshelby, 1957) is defined as an infinite medium Ω containing a subdomain ω , called an inclusion, which undergoes a prescribed uniform heat flux-free temperature gradient (Quang et al., 2008), or "eigen thermal gradient" as termed by Hatta and Taya (1985). In addition the second-order tensor field relating the gradient of the temperature to the uniform heat flux-free temperature gradient is called Eshelby's conduction tensor (Hatta and Taya, 1985; Quang et al., 2008). In our understanding, the most attractive application of the study of Eshelby's inclusion problem in thermal conduction lies in that the Eshelby's conduction tensor can be further applied to predict the effective conductivity of fibrous composite by the so-called "equivalent inclusion method" proposed by Hatta and Taya (1985), which is an extension of Eshelby's equivalent inclusion method in elasticity (Eshelby, 1957).

We notice that in previous discussions on Eshelby's inclusion problem in thermal conduction (see for example, Hatta and Taya, 1985; Quang et al., 2008), the thermal conductivity tensor, no matter isotropic or anisotropic, was assumed to be constant over the infinite medium Ω . Functionally graded materials (FGMs), which have found many engineering applications such as thermal

* Corresponding author. E-mail address: xuwang_sun@hotmail.com (X. Wang). barrier coatings (Jarvis and Carter, 2002), are inhomogeneous materials with smoothly varying material properties.

This research focuses on the study of the 3D Eshelby's spherical inclusion problem in an infinite FGM, in which the isotropic thermal conductivity varies exponentially in space. Due to the mathematical similarity between anti-plane elasticity and 2D heat conduction, the simpler 2D Eshelby's circular inclusion problem in an FGM with exponentially varying thermal conductivity can be considered as solved (Wang et al., 2008). As well known (Hatta and Taya, 1985; Quang et al., 2008), the Eshelby's conduction tensor within an ellipsoidal inclusion (with the spherical inclusion as a special case) in a homogeneous medium is uniform. The main purpose of this study is to derive analytical expressions of the second-order Eshelby's conduction tensor and to quantitatively demonstrate how the Eshelby's conduction tensor is non-uniformly distributed within the spherical inclusion in an FGM. The results presented here for heat conduction are still valid for other transport phenomena, such as electrostatics, magnetostatics, electric conduction and diffusion (Ang et al., 1996; Sutradhar and Paulino, 2004; Quang et al., 2008).

2. Eshelby's spherical inclusion problem in an FGM

As shown in Fig. 1, we consider a 3D infinite domain Ω containing an Eshelby's spherical inclusion ω defined by $\omega: r = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq R$. The Fourier's law for an isotropic material can be expressed as

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Fig. 1. An infinite FGM containing a spherical inclusion ω : $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq R$ with uniform heat flux-free temperature gradient e_1^0, e_2^0, e_3^0 .

$$q_{1} = k \Big[e_{1} - e_{1}^{*}(\mathbf{x}) \Big], \qquad q_{2} = k \Big[e_{2} - e_{2}^{*}(\mathbf{x}) \Big],$$
$$q_{3} = k \Big[e_{3} - e_{3}^{*}(\mathbf{x}) \Big], \qquad (1)$$

where q_i and $e_i = -T_{,i}$ are the heat flux and the negative gradient of the temperature field *T*, respectively; $k = k(\mathbf{x})$ is the spatially varied thermal conductivity; $e_i^*(\mathbf{x})$ is the negative heat flux-free temperature gradient. More specifically $e_i^*(\mathbf{x})$ is uniform and is equal to e_i^0 inside the spherical inclusion ω and vanishes outside the spherical inclusion. In this research we assume that the thermal conductivity is exponentially varied in the x_3 direction, and is described by

$$k = \exp(2\beta x_3)k_0,\tag{2}$$

where the constant β is the exponential factor characterizing the degree of material gradient in the x_3 direction, and k_0 is also a material constant (or more specifically the thermal conductivity at origin).

In the case of steady thermal conduction with no heat source, the heat fluxes should satisfy the following divergence-free equation

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} = 0.$$
(3)

Consequently it follows from Eqs. (1)-(3) that the temperature field should satisfy the following partial differential equations within and outside the sphere

$$\nabla^2 T + 2\beta \frac{\partial I}{\partial x_3} = -2\beta e_3^0, \quad \text{when } r < R,$$

$$\nabla^2 T + 2\beta \frac{\partial T}{\partial x_3} = 0, \quad \text{when } r > R,$$
 (4)

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the three-dimensional (3D) Laplacian operator.

Next we introduce a new function *f* defined by

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$$T = \exp(-\beta x_3)f. \tag{5}$$

As a result Eq. (4) can be equivalently expressed in terms of f as

$$\nabla^2 f - \beta^2 f = -2\beta e_3^0 \exp(\beta x_3), \text{ when } r < R,$$

$$\nabla^2 f - \beta^2 f = 0, \quad \text{when } r > R \tag{6}$$

which means that the new function *f* satisfies a 3D inhomogeneous Helmholtz equation within the sphere and a 3D homogeneous Helmholtz equation outside the sphere whose general solutions can be easily obtained by separation of variables. Here it is of interest to point out that the idea of the introduction of a new function has been widely adopted when solving the governing partial differential equations in an FGM (see for example, Ang et al., 1996; Sutradhar and Paulino, 2004; Collet et al., 2006; Lazar, 2007; Wang et al., 2007). In the following we will separately address three typical cases for the prescribed heat flux-free temperature gradient: (i) $e_1^0 = e_2^0 = 0$, $e_3^0 \neq 0$; (ii) $e_2^0 = e_3^0 = 0$, $e_1^0 \neq 0$; and (iii) $e_1^0 = e_3^0 = 0$, $e_2^0 \neq 0$ in order to derive the second-order conduction tensor \mathbf{S}^{ω} . It will be observed that the expressions of the general solutions for the three cases are different.

2.1.
$$e_1^0 = e_2^0 = 0, e_3^0 \neq 0$$

For the case in which $e_1^0 = e_2^0 = 0$, $e_3^0 \neq 0$, the general solution to *f* inside and outside the spherical inclusion of radius *R* can be expressed as (Abramovitz and Stegun, 1972; Moon and Spencer, 1988)

$$f_{\rm in} = -e_3^0 x_3 \exp(\beta x_3) + e_3^0 \sum_{n=0}^{\infty} A_n i_n(|\beta|r) P_n(\cos\theta), \quad \text{for } r < R,$$
(7)

$$f_{\text{out}} = e_3^0 \sum_{n=0}^{\infty} B_n k_n (|\beta| r) P_n(\cos\theta), \quad \text{for } r > R,$$
(8)

where θ ($0 \le \theta \le \pi$) is the "cone angle" measured from the x_3 axis; A_n and B_n ($n = 0, 1, 2, 3, ..., +\infty$) are unknown constants to be determined; i_n and k_n are the *n*th order modified spherical Bessel functions of the first and second kinds, respectively; P_n is the Legendre polynomial of order *n*. It is noticed that the azimuthal angle φ is absent in the above expressions of the general solution.

The inclusion-matrix interface r = R is assumed to be perfectly bonded, and can be expressed in terms of the temperature T_{in} inside the inclusion and T_{out} outside as

$$T_{\rm in} = T_{\rm out},
\frac{\partial T_{\rm in}}{\partial r} + e_3^0 \cos\theta = \frac{\partial T_{\rm out}}{\partial r}, \quad \text{for } r = R.$$
(9)

The first equation in Eq. (9) states that the temperature is continuous across the interface r = R, whilst the second one in Eq. (9) states that the normal heat flux is also continuous across r = R. In view of Eq. (5), the above boundary conditions can also be expressed in terms of f and $\frac{\partial f}{\partial r}$ as

$$f_{\rm in} = f_{\rm out},
\frac{\partial f_{\rm in}}{\partial r} + e_3^0 \cos\theta \exp(\beta x_3) = \frac{\partial f_{\rm out}}{\partial r}, \quad \text{for } r = R,$$
(10)

where f_{in} is defined inside the spherical inclusion, whilst f_{out} is defined outside the spherical inclusion.

In order to satisfy the above boundary conditions at r = R, we first expand the exponential function $\exp(\beta x_3)$ as

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$$\exp(\beta x_3) = \sum_{n=0}^{+\infty} (2n+1)i_n(\beta r)P_n(\cos\theta), \qquad (11)$$

then the terms $\cos\theta \exp(\beta x_3)$ and $\cos^2\theta \exp(\beta x_3)$ can be expanded as

$$\cos\theta \exp(\beta x_3) = \sum_{n=0}^{+\infty} [ni_{n-1}(\beta r) + (n+1)i_{n+1}(\beta r)]P_n(\cos\theta),$$

$$\cos^{2}\theta \exp(\beta x_{3}) = \sum_{n=0}^{+\infty} \left[\frac{n(n-1)}{2n-1} i_{n-2}(\beta r) + \frac{4n^{3} + 6n^{2} - 1}{(2n-1)(2n+3)} i_{n}(\beta r) + \frac{(n+1)(n+2)}{2n+3} i_{n+2}(\beta r) \right] P_{n}(\cos\theta).$$
(12)

During the derivation of Eq. (12) from Eq. (11) we have utilized the identity $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$. By enforcing the boundary conditions across r = R in Eq. (10), we then obtain the following set of linear algebraic equations

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$$i_n(|\beta|R)A_n - k_n(|\beta|R)B_n = R[ni_{n-1}(\beta R) + (n+1)i_{n+1}(\beta R)],$$

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$$\begin{split} i'_{n}(|\beta|R)A_{n} - k'_{n}(|\beta|R)B_{n} &= \frac{\beta R}{|\beta|} \Big[\frac{n(n-1)}{2n-1} i_{n-2}(\beta R) \\ &+ \frac{4n^{3} + 6n^{2} - 1}{(2n-1)(2n+3)} i_{n}(\beta R) \\ &+ \frac{(n+1)(n+2)}{2n+3} i_{n+2}(\beta R) \Big], \\ (n &= 0, 1, 2, 3, \dots, +\infty), \end{split}$$

where the prime (') denotes the derivative with respect to the variable in the parenthesis, and

$$i'_n(x) = \frac{n}{2n+1}i_{n-1}(x) + \frac{n+1}{2n+1}i_{n+1}(x),$$

$$k'_{n}(x) = -\frac{n}{2n+1}k_{n-1}(x) - \frac{n+1}{2n+1}k_{n+1}(x).$$
(14)

As a result the unknowns A_n and B_n can be uniquely determined as

$$\begin{split} A_n &= R \bigg\{ k'_n(|\beta|R) [ni_{n-1}(\beta R) + (n+1)i_{n+1}(\beta R)] \\ &- k_n(|\beta|R) \frac{\beta}{|\beta|} \bigg[\frac{n(n-1)}{2n-1} i_{n-2}(\beta R) + \frac{4n^3 + 6n^2 - 1}{(2n-1)(2n+3)} i_n(\beta R) \\ &+ \frac{(n+1)(n+2)}{2n+3} i_{n+2}(\beta R) \bigg] \bigg\} \big[i_n(|\beta|R) k'_n(|\beta|R) \\ &- k_n(|\beta|R) i'_n(|\beta|R) \big]^{-1}, \end{split}$$

$$B_{n} = R \left\{ i'_{n}(|\beta|R)[ni_{n-1}(\beta R) + (n+1)i_{n+1}(\beta R)] - i_{n}(|\beta|R)\frac{\beta}{|\beta|} \left[\frac{n(n-1)}{2n-1}i_{n-2}(\beta R) + \frac{4n^{3} + 6n^{2} - 1}{(2n-1)(2n+3)}i_{n}(\beta R) + \frac{(n+1)(n+2)}{2n+3}i_{n+2}(\beta R) \right] \right\} \left[i_{n}(|\beta|R)k'_{n}(|\beta|R) - k_{n}(|\beta|R)i'_{n}(|\beta|R) \right]^{-1} (n = 0, 1, 2, 3, ..., +\infty).$$

$$(15)$$

Once A_n and B_n are determined, the temperature field inside and outside the spherical inclusion is uniquely given by

$$T_{\rm in} = -e_3^0 x_3 + e_3^0 \exp(-\beta x_3) \sum_{n=0}^{\infty} A_n i_n (|\beta| r) P_n(\cos\theta), \quad \text{for } r < R,$$
(16)

$$T_{\text{out}} = e_3^0 \exp(-\beta x_3) \sum_{n=0}^{\infty} B_n k_n (|\beta| r) P_n(\cos\theta), \quad \text{for } r > R.$$
(17)

In the spherical coordinate system, the negative gradient of the temperature inside the sphere is distributed as

$$\frac{e_r}{e_3^0} = \cos\theta + \exp(-\beta x_3) \bigg[\beta \cos\theta \sum_{n=0}^{\infty} A_n i_n(|\beta|r) P_n(\cos\theta) \\ - |\beta| \sum_{n=0}^{\infty} A_n i'_n(|\beta|r) P_n(\cos\theta) \bigg],$$

$$\frac{e_{\theta}}{e_{3}^{0}} = -\sin\theta + \exp(-\beta x_{3})\sin\theta \left[r^{-1} \sum_{n=0}^{\infty} A_{n} i_{n}(|\beta|r) P_{n}'(\cos\theta) -\beta \sum_{n=0}^{\infty} A_{n} i_{n}(|\beta|r) P_{n}(\cos\theta) \right],$$
(18)

and outside the sphere it is

$$\frac{e_r}{e_3^0} = \exp(-\beta x_3) \left[\beta \cos\theta \sum_{n=0}^{\infty} B_n k_n (|\beta|r) P_n(\cos\theta) - |\beta| \sum_{n=0}^{\infty} B_n k'_n (|\beta|r) P_n(\cos\theta) \right],$$

$$\frac{e_\theta}{e_3^0} = \exp(-\beta x_3) \sin\theta \left[r^{-1} \sum_{n=0}^{\infty} B_n k_n (|\beta|r) P'_n(\cos\theta) - \beta \sum_{n=0}^{\infty} B_n k_n (|\beta|r) P_n(\cos\theta) \right],$$
(19)

where the derivative of the Legendre polynomial can be obtained from the following identity

$$P'_n(x) = \frac{nxP_n(x) - nP_{n-1}(x)}{x^2 - 1}.$$
 (20)

The gradient of temperature in Eqs. (18) and (19) is expressed in the spherical coordinate system, it is not difficult to obtain the gradient in the Cartesian coordinate system as

$$e_1 = e_r \sin\theta \cos\varphi + e_\theta \cos\theta \cos\varphi,$$

 $e_2 = e_r \sin\theta \sin\varphi + e_\theta \cos\theta \sin\varphi,$

$$e_3 = e_r \cos\theta - e_\theta \sin\theta. \tag{21}$$

As a result the components of the Eshelby's conduction tensor due to e_3^0 are

$$S_{13}^{\omega} = \frac{e_1}{e_3^0}, \qquad S_{23}^{\omega} = \frac{e_2}{e_3^0}, \qquad S_{33}^{\omega} = \frac{e_3}{e_3^0}.$$
 (22)

It can then be easily observed from Eqs. (18), (21) and (22) that the components S_{13}^{ω} , S_{23}^{ω} , S_{33}^{ω} of the Eshelby's conduction tensor

(19)

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inside the spherical inclusion in an FGM space is intrinsically nonuniform. It should also be stressed that all the results derived in this subsection display a symmetry of revolution around the x_3 direction.

2.2.
$$e_2^0 = e_3^0 = 0, e_1^0 \neq 0$$

For the case in which $e_2^0 = e_3^0 = 0$, $e_1^0 \neq 0$, the general solution to *f* inside and outside the spherical inclusion of radius *R* can be expressed as

$$f_{\text{in}} = e_1^0 \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta) \cos\varphi, \quad \text{for } r < R,$$
(23)

$$f_{\text{out}} = e_1^0 \sum_{n=1}^{\infty} D_n k_n (|\beta| r) P_n^1(\cos\theta) \cos\varphi, \quad \text{for } r > R,$$
(24)

where C_n and D_n $(n = 1, 2, 3, ..., +\infty)$ are unknown constants to be determined; P_n^1 is the associated Legendre function of degree n and order 1. It should be noticed that the azimuthal angle φ is present in the above general solution expressions.

In this case the perfect inclusion-matrix interface r = R can be expressed in terms of T and $\frac{\partial T}{\partial r}$ as

$$T_{\rm in} = T_{\rm out},$$

$$\frac{\partial T_{\rm in}}{\partial r} + e_1^0 \cos\varphi \sin\theta = \frac{\partial T_{\rm out}}{\partial r},$$
 for $r = R$ (25)

or equivalently in terms of f and $\frac{\partial f}{\partial r}$

$$f_{\text{in}} = f_{\text{out}},$$

$$\frac{\partial f_{\text{in}}}{\partial r} + e_1^0 \cos\varphi \sin\theta \exp(\beta x_3) = \frac{\partial f_{\text{out}}}{\partial r},$$
 for $r = R.$ (26)

Due to the fact that the exponential function $\exp(\beta x_3)$ can be expanded into Eq. (11), then the term $\cos \varphi \sin \theta \exp(\beta x_3)$ appearing in Eq. (26)₂ can be expanded into

$$\cos\varphi\sin\theta\exp(\beta x_3) = \sum_{n=1}^{+\infty} [i_{n-1}(\beta r) - i_{n+1}(\beta r)]P_n^1(\cos\theta)\cos\varphi.$$
 (27)

During the above derivation we have utilized the identities $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$ and $P^1_n(x) = \sqrt{1-x^2}P'_n(x)$. By enforcing the boundary conditions across r = R in Eq. (26), we then obtain the following set of linear algebraic equations

$$i_{n}(|\beta|R)C_{n} - k_{n}(|\beta|R)D_{n} = 0,$$

$$i'_{n}(|\beta|R)C_{n} - k'_{n}(|\beta|R)D_{n} = -\frac{1}{|\beta|}[i_{n-1}(\beta R) - i_{n+1}(\beta R)]$$

$$(n = 1, 2, 3, ..., +\infty)$$
(28)

through which the unknowns C_n and D_n can be uniquely determined as

$$C_{n} = \frac{1}{|\beta|} \frac{k_{n}(|\beta|R)[i_{n-1}(\beta R) - i_{n+1}(\beta R)]}{i_{n}(|\beta|R)k'_{n}(|\beta|R) - k_{n}(|\beta|R)i'_{n}(|\beta|R)},$$

$$D_{n} = \frac{1}{|\beta|} \frac{i_{n}(|\beta|R)[i_{n-1}(\beta R) - i_{n+1}(\beta R)]}{i_{n}(|\beta|R)k'_{n}(|\beta|R) - k_{n}(|\beta|R)i'_{n}(|\beta|R)},$$

$$(n = 1, 2, 3, ..., +\infty).$$
(29)

Once C_n and D_n are determined, the temperature field inside and outside the spherical inclusion is uniquely given by

$$T_{\rm in} = e_1^0 \exp(-\beta x_3) \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta) \cos\varphi, \text{ for } r < R, \quad (30)$$

$$T_{\text{out}} = e_1^0 \exp(-\beta x_3) \sum_{n=1}^{\infty} D_n k_n (|\beta| r) P_n^1(\cos\theta) \cos\varphi, \text{ for } r > R.$$
(31)

In the spherical coordinate system, the negative gradient of the temperature inside the sphere is distributed as

$$\frac{e_r}{e_1^0} = \exp(-\beta x_3)\cos\varphi \left[\beta\cos\theta \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta) - |\beta| \sum_{n=1}^{\infty} C_n i'_n(|\beta|r) P_n^1(\cos\theta)\right],$$

$$\frac{e_{\theta}}{e_1^0} = \exp(-\beta x_3) \sin\theta \cos\varphi \left[r^{-1} \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^{1}'(\cos\theta) -\beta \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^{1}(\cos\theta) \right],$$

$$\frac{e_{\varphi}}{e_1^0} = \exp(-\beta x_3) \frac{\sin\varphi}{r\sin\theta} \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta),$$
(32)

and outside the sphere it is

$$\frac{e_r}{e_1^0} = \exp(-\beta x_3)\cos\varphi \left[\beta\cos\theta \sum_{n=1}^{\infty} D_n k_n(|\beta|r) P_n^1(\cos\theta) - |\beta| \sum_{n=1}^{\infty} D_n k'_n(|\beta|r) P_n^1(\cos\theta)\right],$$

$$\frac{e_{\theta}}{e_1^0} = \exp(-\beta x_3)\sin\theta\cos\varphi \left[r^{-1}\sum_{n=1}^{\infty} D_n k_n(|\beta|r) P_n^{1\,\prime}(\cos\theta) -\beta\sum_{n=1}^{\infty} D_n k_n(|\beta|r) P_n^{1}(\cos\theta)\right],$$

$$\frac{e_{\varphi}}{e_1^0} = \exp(-\beta x_3) \frac{\sin\varphi}{r\sin\theta} \sum_{n=1}^{\infty} D_n k_n (|\beta|r) P_n^1(\cos\theta),$$
(33)

where the derivative of the associated Legendre polynomial can be obtained from the following identity

$$P_n^{1\prime}(x) = -\frac{nP_n(x) + (n-1)xP_n'(x) - nP_{n-1}'(x)}{\sqrt{1-x^2}},$$
(34)

with $P'_n(x)$, the derivative of the Legendre polynomial having been given by Eq. (20).

The gradient of temperature in Eqs. (32) and (33) is expressed in the spherical coordinate system, it is not difficult to obtain the gradient in the Cartesian coordinate system as

$$e_1 = e_r \sin \theta \cos \varphi + e_\theta \cos \theta \cos \varphi - e_\varphi \sin \varphi,$$

 $e_2 = e_r \sin \theta \sin \varphi + e_\theta \cos \theta \sin \varphi + e_\varphi \cos \varphi,$

$$e_3 = e_r \cos \theta - e_\theta \sin \theta. \tag{35}$$

As a result the components of the Eshelby's conduction tensor due to e_1^0 are

$$S_{11}^{\omega} = \frac{e_1}{e_1^0}, \qquad S_{21}^{\omega} = \frac{e_2}{e_1^0}, \qquad S_{31}^{\omega} = \frac{e_3}{e_1^0}.$$
 (36)

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It can then be easily observed from Eqs. (32), (35) and (36) that the components S_{11}^{ω} , S_{21}^{ω} , S_{31}^{ω} of the Eshelby's conduction tensor inside the spherical inclusion in an FGM space are intrinsically non-uniform.

2.3.
$$e_1^0 = e_3^0 = 0, e_2^0 \neq 0$$

Here it should be pointed out that the discussion for the case $e_1^0 = e_3^0 = 0$, $e_2^0 \neq 0$ is very similar to that for $e_2^0 = e_3^0 = 0$, $e_1^0 \neq 0$. In the following we only list the main results. In this case, the temperature field inside and outside the spherical inclusion is given by

$$T_{\rm in} = e_2^0 \exp(-\beta x_3) \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta) \sin\phi, \text{ for } r < R, \quad (37)$$

$$T_{\text{out}} = e_2^0 \exp(-\beta x_3) \sum_{n=1}^{\infty} D_n k_n (|\beta| r) P_n^1(\cos\theta) \sin\varphi, \text{ for } r > R, \quad (38)$$

where C_n and D_n ($n = 1, 2, 3, ..., +\infty$) are uniquely determined by Eq. (29) so as to satisfy the following boundary conditions at the perfect inclusion-matrix interface r = R

$$T_{\rm in} = T_{\rm out},$$

$$\frac{\partial T_{\rm in}}{\partial r} + e_2^0 \sin \varphi \sin \theta = \frac{\partial T_{\rm out}}{\partial r}, \quad \text{for } r = R.$$
(39)

In the spherical coordinate system, the negative gradient of the temperature inside the sphere is distributed as

$$\frac{e_r}{e_2^0} = \exp(-\beta x_3) \sin\varphi \left[\beta \cos\theta \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta) - |\beta| \sum_{n=1}^{\infty} C_n i'_n(|\beta|r) P_n^1(\cos\theta) \right],$$

$$\frac{e_\theta}{e_2^0} = \exp(-\beta x_3) \sin\theta \sin\varphi \left[r^{-1} \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^{1\,\prime}(\cos\theta) - \beta \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta) \right],$$

$$e_n = \cos\varphi e_n^{\infty}$$

$$\frac{e_{\varphi}}{e_2^0} = -\exp(-\beta x_3) \frac{\cos\varphi}{r\sin\theta} \sum_{n=1}^{\infty} C_n i_n(|\beta|r) P_n^1(\cos\theta), \tag{40}$$

and outside the sphere it is

$$\frac{e_r}{e_2^0} = \exp(-\beta x_3) \sin\varphi \left[\beta \cos\theta \sum_{n=1}^{\infty} D_n k_n(|\beta|r) P_n^1(\cos\theta) - |\beta| \sum_{n=1}^{\infty} D_n k'_n(|\beta|r) P_n^1(\cos\theta)\right],$$

$$\frac{e_{\theta}}{e_{2}^{0}} = \exp(-\beta x_{3})\sin\theta\sin\varphi \left[r^{-1}\sum_{n=1}^{\infty}D_{n}k_{n}(|\beta|r)P_{n}^{1}(\cos\theta) -\beta\sum_{n=1}^{\infty}D_{n}k_{n}(|\beta|r)P_{n}^{1}(\cos\theta)\right],$$

$$e_{\theta} = \exp(-\beta x_{3})\cos\theta \sum_{n=1}^{\infty}D_{n}k_{n}(|\beta|r)P_{n}^{1}(\cos\theta)$$

$$P_{2}^{\underline{\rho}} = -\exp(-\beta x_{3}) \frac{\cos\varphi}{r\sin\theta} \sum_{n=1}^{\infty} D_{n} k_{n}(|\beta|r) P_{n}^{1}(\cos\theta).$$
(41)

The gradient of temperature in the Cartesian coordinate system is determined by Eq. (35) once we obtain the gradient of temperature in the spherical coordinate system. Consequently the components of the Eshelby's conduction tensor due to e_2^0 are

$$S_{12}^{\omega} = \frac{e_1}{e_2^0}, \qquad S_{22}^{\omega} = \frac{e_2}{e_2^0}, \qquad S_{32}^{\omega} = \frac{e_3}{e_2^0}.$$
 (42)

It can then be easily observed from Eqs. (40), (35) and (42) that the components S_{12}^{ω} , S_{22}^{ω} , S_{32}^{ω} of the Eshelby's conduction tensor inside the spherical inclusion in an FGM space are intrinsically non-uniform.

Up to now we have obtained the expressions for all the components S_{ij}^{ω} of the second-order Eshelby's conduction tensor \mathbf{S}^{ω} . The previous results show that: (i) all the components of \mathbf{S}^{ω} are non-uniform within the spherical inclusion; (ii) all the non-diagonal components of \mathbf{S}^{ω} are non-zero; (iii) \mathbf{S}^{ω} is not symmetric due to the fact that $S_{13}^{\omega} \neq S_{31}^{\omega}$ and $S_{23}^{\omega} \neq S_{32}^{\omega}$; (iv) there exist in total five independent components in \mathbf{S}^{ω} since the fact that $S_{12}^{\omega} = S_{21}^{\omega} = (S_{11}^{\omega} - S_{22}^{\omega}/2) \tan 2\varphi$, $S_{23}^{\omega} = S_{13}^{\omega} \tan \varphi$ and $S_{32}^{\omega} = S_{31}^{\omega} \tan \varphi$. By noticing that for a homogeneous material the Eshelby's conduction tensor within the spherical inclusion is $\mathbf{S}^{\omega} = \mathbf{I}/3$ (Hatta and Taya, 1985; Quang et al., 2008), then an Eshelby's spherical inclusion in an FGM will cause an Eshelby's conduction tensor \mathbf{S}^{ω} which is a far cry from that for a homogeneous material. In the following we present numerical results to quantitatively demonstrate the non-uniform Eshelby's conduction tensor inside the spherical inclusion.



Fig. 2. Distributions of the dimensionless temperature field $\tilde{T} = -T/Re_3^0$ along the x_3 -axis for different values of the gradient parameter βR . $e_1^0 = e_2^0 = 0$, $e_3^0 \neq 0$.

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Fig. 3. Distributions of the Eshelby's tensor component $S_{33}^{\omega} = e_3/e_3^0$ along the x_3 -axis for different values of the gradient parameter βR . $e_1^0 = e_2^0 = 0$, $e_3^0 \neq 0$.

3. Numerical results

In the following calculations, the heat flux-free temperature gradient is always chosen as $e_1^0 = e_2^0 = 0$, $e_3^0 \neq 0$. We demonstrate in Figs. 2 and 3 the distributions of the dimensionless temperature $\tilde{T} = -T/(Re_3^0)$ and the Eshelby's tensor component $S_{33}^{\omega} = e_3/e_3^0$ along the x_3 -axis for different values of the gradient parameter βR . During the calculations the series is truncated at n = 20 in order to obtain sufficiently accurate results with the relative errors less than 0.01%. When $\beta R \rightarrow 0$, the calculated temperature field and the Eshelby's tensor component S_{33}^{ω} are in complete agreement with the following exact results for a homogeneous material

$$\tilde{T} = \begin{cases} 1/(3\tilde{x}_3^2), & \tilde{x}_3 > 1, \\ \tilde{x}_3/3, & -1 < \tilde{x}_3 < 1, \\ -1/(3\tilde{x}_3^2), & \tilde{x}_3 < -1, \end{cases}$$
(43)

$$S_{33}^{\omega} = \begin{cases} -2/(3|\tilde{x}_3|^3), & |\tilde{x}_3| > 1, \\ 1/3, & |\tilde{x}_3| < 1, \end{cases}$$
(44)

where $\bar{x}_3 = x_3/R$. Our numerical results also verify that the temperature is indeed continuous at the interface r = R and that the jump in the Eshelby's tensor component S_{33}^{ω} at the interface r = R satisfies the following condition

$$(S_{33}^{\omega})^{\rm in} - (S_{33}^{\omega})^{\rm out} = 1, \qquad \tilde{x}_3 = \pm 1$$
 (45)

which can be easily deduced from Eq. $(9)_2$. Thus the results presented here can be considered as confirmed.

It is found that the gradient parameter has a significant influence on the distributions of the temperature field and the Eshelby's conduction tensor. More specifically the following can be observed from Fig. 2 for the temperature distribution:

- (i) The normalized temperature *T* always reaches its maximum at x₃ = R and gets its minimum at x₃ = −R for a fixed value of the gradient parameter β (≥0);
- (ii) The temperature within the spherical inclusion is no longer a linear function of the coordinate x_3 for a non-zero gradient parameter β ; the magnitudes of the temperature for $x_3 > R$ outside the inclusion are always reduced, while those for $x_3 < -R$ are always increased when the gradient parameter βR increases.

In addition the following can be observed from Fig. 3 for the Eshelby's tensor component S_{33}^{ω} :

- (i) The Eshelby's tensor component S_{33}^{ω} reaches its maximum value at the inclusion side of the interface $x_3 = R$, and gets its minimum value at the matrix side of the interface $x_3 = -R$ for a fixed value of the gradient parameter β (≥ 0);
- (ii) The Eshelby's tensor component S_{33}^{ω} within the spherical inclusion is no longer uniform for non-zero gradient parameter β , and the non-uniformity of S_{33}^{ω} within the spherical inclusion is more apparent for $\beta R=2-4$ (for example when $\beta R=4$, the internal S_{33}^{ω} is 0.82 at $x_3=R$ whereas it is reduced to 0.35 for $x_3=-R$); our results show that when βR is large enough ($\beta R>8$), the internal Eshelby's tensor component S_{33}^{ω} within the sphere becomes *nearly uniform* along the x_3 -axis.

Furthermore it is also observed that the distribution trends for the temperature and Eshelby's tensor induced by a 3D Eshelby's spherical inclusion are qualitatively in agreement with those induced by a 2D Eshelby's circular inclusion (Wang et al., 2008).

Finally we illustrate in Fig. 4 the contour plots of the dimensionless temperature field \tilde{T} in the plane $x_2 = 0$ with $\beta R = 0.6$. This is to show an overall picture of the distributions of the



Fig. 4. Contour plots of the dimensionless temperature field $\tilde{T} = -T/(Re_3^0)$ in the plane $x_2 = 0$ with $\beta R = 0.6$. $e_1^0 = e_2^0 = 0$, $e_3^0 \neq 0$.

temperature field due to a uniform heat flux-free temperature gradient e_3^0 prescribed within the sphere with exponentially varying thermal conductivity. It is observed that the temperature contours for FGM are quite different from those for homogeneous materials when $\beta = 0$. For example, the temperature contours within the spherical inclusion for a homogeneous material are parallel straight lines, while those for an FGM are curved lines as demonstrated in Fig. 4.

4. Conclusions

In this paper, the 3D Eshelby's spherical inclusion problem in an FGM with exponentially varying thermal conductivity was addressed in detail. The temperature field was presented in Eqs. (16) and (17) due to e_3^0 , Eqs. (30) and (31) due to e_1^0 , Eqs. (37) and (38) due to e_2^0 while the Eshelby's conduction tensor, which is nonuniform inside the spherical inclusion, was given by Eqs. (22), (36) and (42). The analysis of an FGM containing a spherical inclusion revealed that: (i) all the components of S^{ω} are non-uniform within the spherical inclusion; (ii) all the non-diagonal components of S^{ω} are non-zero; (iii) S^{ω} in general is not symmetric; (iv) there exist in total five independent components in \mathbf{S}^{ω} since $S_{12}^{\omega} = S_{21}^{\omega} = (S_{11}^{\omega} - S_{22}^{\omega}/2)\tan 2\varphi, \ S_{23}^{\omega} = S_{13}^{\omega}\tan\varphi, \ S_{32}^{\omega} = S_{31}^{\omega}\tan\varphi.$ Our numerical results also clearly demonstrated how the temperature and the Eshelby's conduction tensor are influenced by the non-zero gradient parameter β . It is expected that the discussions of a spherical inclusion in an FGM with some special non-exponential variations (Sutradhar and Paulino, 2004; Collet et al., 2006) can also be carried out.

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