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Some basic problems in anisotropic bimaterials with a viscoelastic interface

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ABSTRACT

This research is devoted to the study of anisotropic bimaterials with Kelvin-type viscoelastic interface under antiplane deformations. First we derive the Green's function for a bimaterial with a Kelvin-type viscoelastic interface subjected to an antiplane force and a screw dislocation by means of the complex variable method. Explicit expressions are derived for the time-dependent stress field induced by the antiplane force and screw dislocation. Also presented is the time-dependent image force acting on the screw dislocation due to its interaction with the Kelvin-type viscoelastic interface. Second we investigate a rectangular inclusion with uniform antiplane eigenstrains embedded in one of the two bonded anisotropic half-planes by virtue of the derived Green's function for a line force. The explicit expressions for the time-dependent stress field induced by the rectangular inclusion are obtained in terms of the simple logarithmic and exponential integral functions. It is observed that in general the stresses exhibit the logarithmic singularity at the four corners of the rectangular inclusion. Our results also show that when one side of the rectangular inclusion lies on the viscoelastic interface, the interfacial tractions are still regular at the two corners of the inclusion which are located on the interface. Last we address a finite Griffith crack normal to the viscoelastic interface by means of the obtained Green's function for a screw dislocation. The crack problem is formulated in terms of a resulting singular integral equation which is solved numerically. The time-dependent stress intensity factors at the two crack tips are obtained and some interesting features are discussed.

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1. Introduction

Nowadays composite structures have found many important applications. It has been identified that the interface separating different phases in composites is critical in determining the local and overall behaviors of the composites. At high temperatures the time-dependent viscoelastic behavior of the interface should be taken into consideration (Fan and Wang, 2003).

In this research we will discuss some typical micromechanics and fracture mechanics issues in anisotropic elastic bimaterials with viscoelastic interface. In previous works on Green's functions only the isotropic properties of the bimaterials were considered (Fan and Wang, 2003; Ang and Fan, 2004; Wang and Pan, 2008a). On the other hand discussions on Eshelby's inclusion and crack problems in composites with viscoelastic interface are still very rare in the literature. Based on these considerations, this research is concerned with some quasi-static antiplane problems of an anisotropic elastic bimaterial with a Kelvin-type viscoelastic interface in which the linear spring and the linear dashpot are connected in parallel. In Section 2 we present the basic formulations for the problem. In Section 3 we first derive the time-dependent Green's function for an anisotropic bimaterial with a viscoelastic interface subjected to an antiplane force and a screw dislocation. In Section 4 the obtained Green's function for an antiplane force is then employed to address a rectangular inclusion with uniform antiplane eigenstrains in one of the two bonded anisotropic halfplanes. In Section 5 the obtained Green's function for a screw dislocation is employed to solve the problem of a crack normal to the viscoelastic interface in the upper half-plane. Conclusions are drawn in Section 6.

2. Basic formulations

In a fixed rectangular coordinate system x_i (i = 1, 2, 3), we let u_i and σ_{ij} be the elastic displacement and stress, respectively. If the material possesses a symmetry plane at $x_3 = 0$, then the stress-strain relation for the antiplane deformation is

$$\sigma_{31} = C_{55}u_{,1} + C_{45}u_{,2}, \qquad (1)$$

$$\sigma_{32} = C_{44}u_{,2} + C_{45}u_{,1}, \qquad (1)$$

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where $u = u_3$, C_{44} , C_{45} , C_{55} are elastic constants and the subscript ",i" stands for differentiation with respect to x_i . The positive definiteness of the strain energy density requires that

$$C_{44} > 0, \quad C_{55} > 0, \quad C_{44}C_{55} - C_{45}^2 > 0. \tag{2}$$

For the special case of an orthotropic material with its material orthotropic axes coincident with the reference axes, one has $C_{45} = 0$.

The equation of equilibrium in terms of the displacement is

$$C_{55}u_{,11} + 2C_{45}u_{,12} + C_{44}u_{,22} = 0.$$
(3)

In writing Eq. (3) we have assumed that the inertia effect can be neglected. The general solution of (3) can be expressed in terms of a single analytic function $f(z_p, t)$ as (Ting, 1996)

$$u = \text{Im}\{f(z_p, t)\}, \quad z_p = x_1 + px_2,$$
 (4)

where

$$p = \frac{-C_{45} + i\sqrt{C_{44}C_{55} - C_{45}^2}}{C_{44}}.$$
(5)

It should be noted that the appearance of the real time t in the analytic function f is due solely to the influence of the viscoelastic interface.

The stresses σ_{31} , σ_{32} and the stress function ϕ are then given by $\tau_{31} + n\sigma_{32} - i\mu Im \{n\} \overline{f'(z_n, t)}$ (6)

$$\phi = \mu \operatorname{Re}\{f(z_p, t)\},\tag{0}$$

where the prime denotes differentiation with respect to the complex variable z_p , $\mu = \sqrt{C_{44}C_{55} - C_{45}^2}$, and the stresses σ_{31}, σ_{32} are related to the stress function ϕ through

$$\sigma_{31} = -\phi_2, \quad \sigma_{32} = \phi_1. \tag{8}$$

3. Green's functions for anisotropic bimaterials subjected to an antiplane force and a screw dislocation

At the initial moment t = 0, we introduce an antiplane line force q and a screw dislocation with magnitude b and fix them at the position $(x_1, x_2) = (x_0, y_0)$, $(y_0 > 0)$ in the upper anisotropic half-plane which is bonded to another lower anisotropic half-plane through a Kelvin-type viscoelastic interface $x_2 = 0$, as shown in Fig. 1. In what follows the subscripts 1 and 2 (or the superscripts (1) and

(2)) are used to denote, respectively, the quantities in the upper and lower half-planes. Due to the fact that $z_{p1} = z_{p2} = z = x_1 + ix_2$ on $x_2 = 0$, we will replace z_{p1} and z_{p2} by the common variable z. When the analysis is finished the complex variable should be changed back to the corresponding complex variables z_{p1} for the upper half-plane and z_{p2} for the lower half-plane. The boundary conditions on a Kelvin-type viscoelastic interface are given by

$$\sigma_{32}^{(1)} = \sigma_{32}^{(2)} = k(u^{(1)} - u^{(2)}) + \eta(\dot{u}^{(1)} - \dot{u}^{(2)}) \quad \text{on } x_2 = 0, \tag{9}$$

where a dot over the quantity denotes differentiation with respect to real time *t*, *k* is the "spring constant" of the interface and η is the viscosity coefficient. In the Kelvin model the linear spring and the linear dashpot are connected in parallel (Fan and Wang, 2003).

The above boundary conditions can be equivalently expressed in terms of the two analytic functions, $f_1(z, t)$ defined in the upper half-plane and $f_2(z, t)$ defined in the lower half-plane, as

$$\begin{split} & \mu_1 f_1^+(x_1,t) + \mu_1 f_1^-(x_1,t) = \mu_2 f_2^-(x_1,t) + \mu_2 f_2^+(x_1,t), \\ & k \big[f_1^+(x_1,t) - \bar{f}_1^-(x_1,t) - f_2^-(x_1,t) + \bar{f}_2^+(x_1,t) \big] \\ & + \eta \big[\dot{f}_1^+(x_1,t) - \dot{\bar{f}}_1^-(x_1,t) - \dot{f}_2^-(x_1,t) + \dot{\bar{f}}_2^+(x_1,t) \big] \\ & = i \mu_2 \big[f_2'^-(x_1,t) + \bar{f}_2'^+(x_1,t) \big], \quad x_2 = 0, \end{split}$$
(10)

where the superscripts "+" and "-" denote the limit values from the upper and lower sides of the interface $x_2 = 0$, respectively. It follows from $(10)_1$ that

$$f_{1}(z,t) = \frac{\mu_{2}}{\mu_{1}}\bar{f}_{2}(z,t) + f_{0}(z) - \bar{f}_{0}(z),$$

$$\bar{f}_{1}(z,t) = \frac{\mu_{2}}{\mu_{1}}f_{2}(z,t) - f_{0}(z) + \bar{f}_{0}(z),$$

(11)

where $f_0(z) = \frac{b-i\mu_1^{-1}q}{2\pi} \ln(z-z_0)$, $(z_0 = x_0 + p_1y_0)$ is the potential for a line force and line dislocation at $(x_1, x_2) = (x_0, y_0)$ in an infinite homogeneous material with the material properties $C_{44}^{(1)}, C_{45}^{(1)}, C_{55}^{(1)}$.

Substituting Eq. (11) into Eq. $(10)_2$, we arrive at

$$k\frac{\mu_{1}+\mu_{2}}{\mu_{1}}\bar{f}_{2}^{+}(x_{1},t) - i\mu_{2}\bar{f}_{2}^{\prime+}(x_{1},t) + \eta\frac{\mu_{1}+\mu_{2}}{\mu_{1}}\bar{f}_{2}^{+}(x_{1},t) - 2k\bar{f}_{0}(x_{1})$$

$$= k\frac{\mu_{1}+\mu_{2}}{\mu_{1}}f_{2}^{-}(x_{1},t) + i\mu_{2}f_{2}^{\prime-}(x_{1},t) + \eta\frac{\mu_{1}+\mu_{2}}{\mu_{1}}\dot{f}_{2}^{-}(x_{1},t) - 2kf_{0}(x_{1})$$
on $x_{2} = 0.$
(12)

It is apparent that the left hand side of Eq. (12) is analytic in the upper half-plane, whilst the right hand side of Eq. (12) is analytic in the lower half-plane. Consequently the continuity condition in



Fig. 1. An antiplane line force q and a screw dislocation b located at $(x_1, x_2) = (x_0, y_0)$, $(y_0 > 0)$ in the upper anisotropic half-plane which is bonded to a lower anisotropic half-plane through a Kelvin-type viscoelastic interface $x_2 = 0$.

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Eq. (12) implies that the left and right sides of Eq. (12) are identically zero in the upper and lower half-planes, respectively. It follows that

$$-i\chi f_{2}(z,t) + f_{2}'(z,t) - i\gamma \dot{f}_{2}(z,t) = -\frac{\chi(q+i\mu_{1}b)}{\pi(\mu_{1}+\mu_{2})}\ln(z-z_{0}), \quad x_{2} \leq 0,$$
(13)

where $\chi = k \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}$, $\gamma = \eta \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}$. At t = 0 when the dislocation and the line force are just introduced into the upper half-plane, the displacement across the interface has no time to have a jump due to the dashpot. Therefore the displacement is continuous across the interface at t = 0 (i.e., the interface is perfect when t = 0), and then the following initial condition holds

$$f_2'(z,0) = \frac{\mu_1 b - iq}{\pi(\mu_1 + \mu_2)} \frac{1}{z - z_0}.$$
(14)

When $t \to \infty$, the interface should be at a steady state and there is no time effect. In this case it follows from Eq. (13) that

$$-i\chi f_2(z,\infty) + f'_2(z,\infty) = -\frac{\chi(q+i\mu_1b)}{\pi(\mu_1+\mu_2)}\ln(z-z_0).$$
 (15)

The solution to the above is easy to be found as

$$f_2'(z,\infty) = \frac{\chi(q+i\mu_1 b)}{\pi(\mu_1+\mu_2)} \exp\left[i\chi(z-z_0)\right] E_1[i\chi(z-z_0)],$$
(16)

where the exponential integral is defined by

$$E_1(z) = -\int_{\infty}^{z} \frac{e^{-\xi}}{\xi} d\xi.$$
(17)

In addition

$$f_2'(z,t) \to 0 \quad \text{as } z \to \infty$$
 (18)

due to the fact that at far field the stresses should approach zero. In view of the initial state equation (14), the steady state equation (16) and far field condition equation (18), the solution to Eq. (13) can be given by

$$f_{2}'(z,t) = \frac{\chi(q+i\mu_{1}b)}{\pi(\mu_{1}+\mu_{2})} \exp [i\chi(z-z_{0})] \{E_{1}[i\chi(z-z_{0})] - E_{1}[i\chi(z-z_{0}-it/\gamma)]\} + \frac{(\mu_{1}b-iq)\exp(-\chi t/\gamma)}{\pi(\mu_{1}+\mu_{2})(z-z_{0}-it/\gamma)}.$$
(19)

It can be checked from the above expression that the conditions, Eqs. (14), (16) and (18), are satisfied. Here it shall be mentioned that the term $\frac{\chi(q+i\mu_1b)}{\pi(\mu_1+\mu_2)} \exp [i\lambda(z-z_0)]E_1[i\lambda(z-z_0)]$ in Eq. (19) is a particular solution to Eq. (13) while the other two terms $h_1(z,t) = \exp[i\chi(z-z_0)]E_1[i\chi(z-z_0-it/\gamma)]$ and $h_2(z,t) = \frac{\exp(-\chi t/\gamma)}{z-z_0-it/\gamma}$ in Eq. (19) are two homogeneous solutions to Eq. (13), i.e.,

$$-i\chi h_j(z,t) + h'_j(z,t) - i\gamma \dot{h}_j(z,t) = 0, \quad j = 1, 2.$$
(20)

In fact the above discussion on a line force and a dislocation can be easily extended to any types of singularities such as concentrated couples interacting with the viscoelastic interface. Once we know $f_2(z,0)$ for a perfect interface and $f_2(z,\infty)$ for a linear spring interface, the solution at any time can be conveniently written down as

$$f_2(z,t) = f_2(z,\infty) + \exp(-\chi t/\gamma)[f_2(z-it/\gamma,0) - f_2(z-it/\gamma,\infty)].$$
(21)

It follows from Eqs. (11) and (19) that $f_1(z,t)$ can be determined as

$$\begin{aligned} f_1'(z,t) &= \frac{\chi \mu_2(\mu_1^{-1}q - ib)}{\pi(\mu_1 + \mu_2)} \exp\left[-i\chi(z - \bar{z}_0)\right] \\ &\{E_1[-i\chi(z - \bar{z}_0)] - E_1[-i\chi(z - \bar{z}_0 + it/\gamma)]\} \\ &+ \frac{\mu_2(b + i\mu_1^{-1}q) \exp(-\chi t/\gamma)}{\pi(\mu_1 + \mu_2)(z - \bar{z}_0 + it/\gamma)} - \frac{b + i\mu_1^{-1}q}{2\pi(z - \bar{z}_0)} + \frac{b - i\mu_1^{-1}q}{2\pi(z - z_0)}. \end{aligned}$$
(22)

In addition it is not difficult to write down the full-field expressions as follows

$$f_{1}'(z_{p1},t) = \frac{\chi \mu_{2}(\mu_{1}^{-1}q - ib)}{\pi(\mu_{1} + \mu_{2})} \exp\left[-i\chi(z_{p1} - \bar{z}_{0})\right] \\ \left\{E_{1}\left[-i\chi(z_{p1} - \bar{z}_{0})\right] - E_{1}\left[-i\chi(z_{p1} - \bar{z}_{0} + it/\gamma)\right]\right\} \\ + \frac{\mu_{2}(b + i\mu_{1}^{-1}q)\exp(-\chi t/\gamma)}{\pi(\mu_{1} + \mu_{2})(z_{p1} - \bar{z}_{0} + it/\gamma)} - \frac{b + i\mu_{1}^{-1}q}{2\pi(z_{p1} - \bar{z}_{0})} + \frac{b - i\mu_{1}^{-1}q}{2\pi(z_{p1} - z_{0})},$$
(23)

$$f_{2}'(z_{p2},t) = \frac{\chi(q+i\mu_{1}b)}{\pi(\mu_{1}+\mu_{2})} \exp\left[i\chi(z_{p2}-z_{0})\right] \left\{E_{1}\left[i\chi(z_{p2}-z_{0})\right] - E_{1}\left[i\chi(z_{p2}-z_{0}-it/\gamma)\right]\right\} + \frac{(\mu_{1}b-iq)\exp(-\chi t/\gamma)}{\pi(\mu_{1}+\mu_{2})(z_{p2}-z_{0}-it/\gamma)}.$$
(24)

Consequently the time-dependent stresses induced by the antiplane force and screw dislocation are given by

$$\begin{aligned} \sigma_{31}^{(1)} + \bar{p}_{1}\sigma_{32}^{(1)} &= \frac{\chi\mu_{1}\mu_{2}(\mu_{1}b + iq)}{\pi C_{44}^{(1)}(\mu_{1} + \mu_{2})} \exp\left[-i\chi(z_{p1} - \bar{z}_{0})\right] \\ &\left\{ E_{1}\left[-i\chi(z_{p1} - \bar{z}_{0} + it/\gamma)\right] - E_{1}\left[-i\chi(z_{p1} - \bar{z}_{0})\right] \right\} \\ &+ \frac{\mu_{1}\mu_{2}(q - i\mu_{1}b)\exp(-\chi t/\gamma)}{\pi C_{44}^{(1)}(\mu_{1} + \mu_{2})(z_{p1} - \bar{z}_{0} + it/\gamma)} - \frac{\mu_{1}(q - i\mu_{1}b)}{2\pi C_{44}^{(1)}(z_{p1} - \bar{z}_{0})} \\ &- \frac{\mu_{1}(q + i\mu_{1}b)}{2\pi C_{44}^{(1)}(z_{p1} - z_{0})}, \end{aligned}$$
(25)

in the upper half-plane, and

$$\sigma_{31}^{(2)} + \bar{p}_{2}\sigma_{32}^{(2)} = \frac{\chi \mu_{2}^{2}(\mu_{1}b - iq)}{\pi C_{44}^{(2)}(\mu_{1} + \mu_{2})} \exp\left[i\chi(z_{p2} - z_{0})\right] \\ \left\{ E_{1}\left[i\chi(z_{p2} - z_{0})\right] - E_{1}\left[i\chi(z_{p2} - z_{0} - it/\gamma)\right] \right\} \\ - \frac{\mu_{2}^{2}(q + i\mu_{1}b)\exp(-\chi t/\gamma)}{\pi C_{44}^{(2)}(\mu_{1} + \mu_{2})(z_{p2} - z_{0} - it/\gamma)},$$
(26)

in the lower half-plane.

In addition the time-dependent interfacial traction is distributed on the real x_1 -axis as follows

$$\sigma_{32}^{(1)} = \sigma_{32}^{(2)} = \frac{\chi\Gamma}{2\pi} \operatorname{Re}\left\{ (q + i\mu_1 b) \left[\frac{g\left(\frac{\lambda + i\tilde{\chi}}{2}\right)}{\lambda + i\tilde{\chi}} + \frac{\exp(-\lambda\tilde{t})\left[1 - g\left(\frac{\lambda + \lambda\tilde{t} + i\tilde{\chi}}{2}\right)\right]}{\lambda + \lambda\tilde{t} + i\tilde{\chi}} \right] \right\},$$
(27)

where

$$\tilde{x} = \chi[x_1 - x_0 - y_0 \operatorname{Re}\{p_1\}], \quad \tilde{t} = t/t_0,$$

$$\lambda = y_0 \chi \operatorname{Im}\{p_1\} = \frac{2y_0 k}{C_{44}^{(1)} \Gamma}, \quad t_0 = y_0 \gamma \operatorname{Im}\{p_1\} = \frac{2y_0 \eta}{C_{44}^{(1)} \Gamma},$$

$$\Gamma = \frac{2\mu_2}{\mu_1 + \mu_2},$$
(29)

and

$$g(z) = 2z \exp(2z)E_1(2z).$$
 (30)

In the above expressions, Γ ($0 \le \Gamma \le 2$) is the two-phase mismatch parameter, λ is a dimensionless parameter measuring the interface "rigidity" and t_0 is the relaxation time.

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The image force on the screw dislocation with the line force q = 0 can be determined as

$$F_{y} = -\frac{b^{2}\mu_{1}}{4\pi y_{0}} \left\{ 1 - \Gamma \left[g(\lambda) + \left(1 + \frac{\tilde{t}}{2} \right)^{-1} \exp\left(-\lambda \tilde{t} \right) \left[1 - g\left(\lambda + \frac{\lambda \tilde{t}}{2} \right) \right] \right\} \right\}$$

$$F_{x} = 0,$$
(31)

where F_x and F_y are respectively the horizontal and vertical components of the image force. Interestingly it is observed from the above expression that once the two-phase mismatch parameter Γ , the interface "rigidity" λ and the relaxation time t_0 are properly defined, the expression of the image force on a screw dislocation in an anisotropic bimaterial with a Kelvin-type viscoelastic interface is nominally identical to that in an isotropic bimaterial with a Kelvin-type viscoelastic interface (Fan and Wang, 2003; Wang and Pan, 2008a). The influence of the anisotropic effect is reflected in the above definitions of the two-phase mismatch parameter Γ , the interface "rigidity" λ and the relaxation time t_0 . When k = 0 for a viscous interface, it is found that

$$f_{1}'(z_{p1},t) = \frac{\mu_{2}(b+i\mu_{1}^{-1}q)}{\pi(\mu_{1}+\mu_{2})(z_{p1}-\bar{z}_{0}+it/\gamma)} - \frac{b+i\mu_{1}^{-1}q}{2\pi(z_{p1}-\bar{z}_{0})} + \frac{b-i\mu_{1}^{-1}q}{2\pi(z_{p1}-z_{0})},$$
(32)

$$f_{2}'(z_{p2},t) = \frac{\mu_{1}b - iq}{\pi(\mu_{1} + \mu_{2})} \frac{1}{z_{p2} - z_{0} - it/\gamma},$$
(33)

which approach the results for a completely debonded interface as $t \to \infty$. The image force on the dislocation due to its interaction with a viscous interface thus is

$$F_{y} = -\frac{b^{2}\mu_{1}}{4\pi y_{0}} \left[1 - \Gamma \left(1 + \frac{t}{2t_{0}} \right)^{-1} \right].$$
(34)

In the following two sections we will demonstrate that the obtained Green's function due to a line force and a screw dislocation can be used to investigate a rectangular inclusion with uniform eigenstrain and a crack normal to the viscoelastic interface.

4. The time-dependent elastic field induced by a rectangular inclusion in an anisotropic bimaterial with a viscoelastic interface

Rectangular inclusions are very common in strained quantum wire structures (i.e., Pan, 2004; Jiang and Pan, 2004). In this section, therefore, we consider a rectangular inclusion Ω with uniform antiplane eigenstrains ε_{31}^* , ε_{32}^* (Fig. 2) in the upper half-plane which is bonded to a lower half-plane through a Kelvin-type viscoelastic interface described by Eq. (9). The coordinates of the four corners of the rectangular inclusion are: the upper left corner: $x_1 = a_1, x_2 = b_2$; the upper right corner: $x_1 = a_2, x_2 = b_2$; the lower right corner: $x_1 = a_1, x_2 = b_1$. Both the upper and the lower half-planes are anisotropic materials having the symmetry plane at $x_3 = 0$.

According to Mura (1987), the total displacement w induced by the uniform eigenstrains imposed on the rectangular inclusion can be expressed as

$$w = 2(C_{55}^{(1)}\varepsilon_{31}^* + C_{45}^{(1)}\varepsilon_{32}^*) \int_{\partial\Omega} G(\mathbf{x}, \mathbf{x}')n_1 \, dl + 2(C_{45}^{(1)}\varepsilon_{31}^* + C_{44}^{(1)}\varepsilon_{32}^*) \times \int_{\partial\Omega} G(\mathbf{x}, \mathbf{x}')n_2 \, dl,$$
(35)

where $\partial \Omega$ is the boundary of the inclusion Ω and n_i is the outward normal on the boundary of the inclusion. The elastic displacement u = w outside the inclusion and $u = w - 2\varepsilon_{31}^* x_1 - 2\varepsilon_{32}^* x_2$ inside the

inclusion. The integration in Eq. (35) is with respect to the source point \mathbf{x}' of the Green's function $G(\mathbf{x}, \mathbf{x}')$ for a line force obtained in the previous section. The above expression indicates that once the Green's function for the problem is derived, the induced displacement can be found by simply carrying out the line integrals in (35).

Making use of these derivatives of the Green's functions for an antiplane force and carrying out the line integrals in (35), it is finally found that the two analytic functions for the corresponding rectangular inclusion problem are given by

$$f_{1}'(z_{p1},t) = \frac{i(C_{44}^{(1)}\varepsilon_{32}^{*} + C_{45}^{(1)}\varepsilon_{31}^{*})}{\pi\mu_{1}} \sum_{j=1}^{4} (-1)^{j} \{\ln(z_{p1} - z_{j}) + (1 - \Gamma) \\ \times \ln(z_{p1} - \bar{z}_{j}) - \Gamma \exp\left[-i\chi(z_{p1} - \bar{z}_{j})\right] \{E_{1}\left[-i\chi(z_{p1} - \bar{z}_{j})\right] \\ - E_{1}\left[-i\chi(z_{p1} - \bar{z}_{j} + it/\gamma)\right] \} \} + \frac{i(C_{55}^{(1)}\varepsilon_{31}^{*} + C_{45}^{(1)}\varepsilon_{32}^{*})}{\pi\mu_{1}} \\ \times \sum_{j=1}^{4} (-1)^{j} \{\frac{\ln(z_{p1} - z_{j})}{p_{1}} + \frac{1 - \Gamma}{\bar{p}_{1}} \ln(z_{p1} - \bar{z}_{j}) \\ - \frac{\Gamma}{\bar{p}_{1}} \exp\left[-i\chi(z_{p1} - \bar{z}_{j})\right] \{E_{1}\left[-i\chi(z_{p1} - \bar{z}_{j})\right] \\ - E_{1}\left[-i\chi(z_{p1} - \bar{z}_{j} + it/\gamma)\right] \} (x_{2} \ge 0)$$
(36)

$$f_{2}'(z_{p2},t) = \frac{2I[(C_{55}^{c} + p_{1}C_{45}^{c})\hat{z}_{31}^{c} + (C_{45}^{c} + p_{1}C_{44}^{c})\hat{z}_{32}^{c}]}{\pi p_{1}(\mu_{1} + \mu_{2})} \\ \times \sum_{j=1}^{4} (-1)^{j} \{ \ln(z_{p2} - z_{j}) + \exp\left[i\chi(z_{p2} - z_{j})\right] \\ \times \{ E_{1}[i\chi(z_{p2} - z_{j})] - E_{1}[i\chi(z_{p2} - z_{j} - it/\gamma)] \} \} \quad (x_{2} \leq 0)$$
(37)

where z_i (j = 1 - 4) are defined by

$$z_1 = a_1 + p_1 b_2, \quad z_2 = a_2 + p_1 b_2, \quad z_3 = a_2 + p_1 b_1, \quad z_4 = a_1 + p_1 b_1.$$
(38)

It can be easily checked that the above expressions of $f'_1(z_{p1}, 0)$ and $f'_2(z_{p2}, 0)$ are just those for a perfect interface, whilst those of $f'_1(z_{p1}, \infty)$ and $f'_2(z_{p2}, \infty)$ are for a spring-type imperfect interface. Particularly when the interface is a viscous one with k = 0, the two analytic functions for the corresponding rectangular inclusion problem are given by

$$f_{1}'(z_{p1},t) = \frac{i(C_{44}^{(1)}\varepsilon_{32}^{*} + C_{45}^{(1)}\varepsilon_{31}^{*})}{\pi\mu_{1}} \sum_{j=1}^{4} (-1)^{j} [\ln(z_{p1} - z_{j}) + \ln(z_{p1} - \bar{z}_{j}) - \Gamma \ln(z_{p1} - \bar{z}_{j} + it/\gamma)] + \frac{i(C_{55}^{(1)}\varepsilon_{31}^{*} + C_{45}^{(1)}\varepsilon_{32}^{*})}{\pi\mu_{1}} \times \sum_{j=1}^{4} (-1)^{j} [\frac{\ln(z_{p1} - z_{j})}{p_{1}} + \frac{1}{\bar{p}_{1}} \ln(z_{p1} - \bar{z}_{j}) - \frac{\Gamma}{\bar{p}_{1}} \ln(z_{p1} - \bar{z}_{j} + it/\gamma)] \quad (x_{2} \ge 0)$$
(39)

$$f_{2}'(z_{p2},t) = \frac{2i\left[(C_{55}^{(1)} + p_{1}C_{45}^{(1)})\varepsilon_{31}^{*} + (C_{45}^{(1)} + p_{1}C_{44}^{(1)})\varepsilon_{32}^{*}\right]}{\pi p_{1}(\mu_{1} + \mu_{2})} \\ \times \sum_{j=1}^{4} (-1)^{j} \ln(z_{p2} - z_{j} - it/\gamma) \quad (x_{2} \leq 0)$$
(40)

It shall be mentioned that the above expressions (36) and (39) for $f'_1(z_{p1},t)$ in the upper half-plane are valid both inside and outside the rectangular inclusion. The multi-valued logarithmic function term $h(z_{p1}) = \sum_{j=1}^{4} (-1)^j \ln(z_{p1} - z_j)$ in $f'_1(z_{p1},t)$ is chosen in such a way that when $\varepsilon^*_{31} \neq 0$, $\varepsilon^*_{32} = 0$, $h(z_{p1})$ has one cut along the

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Fig. 2. A rectangular inclusion with uniform antiplane eigenstrains embedded in the upper anisotropic half-plane which is bonded to a lower anisotropic half-plane through a viscoelastic interface.

line segment connecting $z_{p1} = z_1$ and $z_{p1} = z_2$, and has another cut along the line segment connecting $z_{p1} = z_3$ and $z_{p1} = z_4$. When $\varepsilon_{31}^* = 0$, $\varepsilon_{32}^* \neq 0$, $h(z_{p1})$ has one cut along the line segment connecting $z_{p1} = z_1$ and $z_{p1} = z_4$, and has another cut along the line segment connecting $z_{p1} = z_2$ and $z_{p1} = z_3$. In doing so it can be verified from Eqs. (36) and (39) that when crossing the two interfaces $x_2 = b_1, a_1 < x_1 < a_2$ and $x_2 = b_2, a_1 < x_1 < a_2$ the stress component σ_{31} undergoes the following jump

$$\sigma_{31}^{\text{inclusion}} - \sigma_{31}^{\text{matrix}} = -\frac{2\mu_1^2}{C_{44}^{(1)}} \varepsilon_{31}^*.$$
(41)

Similarly when crossing the two interfaces $x_1 = a_1, b_1 < x_2 < b_2$ and $x_1 = a_2, b_1 < x_2 < b_2$ the stress component σ_{32} undergoes the following jump

$$\sigma_{32}^{\text{inclusion}} - \sigma_{32}^{\text{matrix}} = -\frac{2\mu_1^2}{C_{55}^{(1)}} \varepsilon_{32}^*.$$
(42)

The above jump properties can also be deduced from the linear constitutive equation (1). In general, the stress field in the anisotropic bimaterial can be obtained by using Eqs. (6), (36) and (37) [or Eqs. (39) and (40) for a viscous interface]. Furthermore, for the general eigenstrain case where $\varepsilon_{31}^* \neq 0$, $\varepsilon_{32}^* \neq 0$, one can utilize the method of superposition to find the corresponding stress field.

It is easily observed from Eq. (36) that in general the stresses exhibit logarithmic singularity at the four corners of the rectangular inclusion. Particularly when one side of the rectangular inclusion lies on the viscoelastic interface, i.e., $b_1 = 0$, it is observed from Eq. (36) or Eq. (37) that the traction σ_{32} for t > 0 along the interface is still *regular* by noticing the fact that $E_1(z) = -\ln z - \gamma + o(z)$ when $z \rightarrow 0$ with $\gamma \approx 0.577$ being the Euler-Mascheroni constant. Physically in our viscoelastic interface model (9) the interface traction is connected to the displacement jump and the velocity jump on the interface must be *finite*, consequently the interface traction should also be finite.

In order to demonstrate the obtained solutions, we consider a rectangular inclusion in the upper S-Glass Epoxy half-plane which is bonded to the lower AS4/8552 half-plane through a viscoelastic interface. The material constants of S-Glass Epoxy are $C_{44} = 4.8$ GPa, $C_{55} = 5.5$ GPa and $C_{45} = 0$, and those of AS4/8552 are $C_{44} = 10.5628$ GPa, $C_{55} = 7.17055$ GPa and $C_{45} = 0$. In addition the rectangular inclusion has the dimension $2a \times a$ with $a_1 = -a$, $a_2 = a$ and $b_1 = 0$, $b_2 = a$. This configuration represents the extreme case where the lower side of the inclusion just lies on the viscoelastic interface separating the two half-planes. Here we are particularly interested in the interfacial stresses caused by the inclusion due to the fact that the interface is the weakest part of the composite. In addition we will consider the eigenstrains $\varepsilon_{31}^* \neq 0$, $\varepsilon_{32}^* = 0$.

Figs. 3–5 demonstrate the distributions of the interface traction $\sigma_{32}^{(1)}=\sigma_{32}^{(2)}=\sigma_{32}$ and the interface stresses $\sigma_{31}^{(1)},\sigma_{31}^{(2)}$ on the interface $x_2 = 0$ caused by nonzero ε_{31}^* at four different normalized times $\tilde{t} = t\chi/\gamma = 0, 0.05, 0.2, \infty$ with $a\chi = 0.5$. Here $\sigma_{31}^{(1)}$ within the section $-a \leq x_1 \leq a$ should be understood as the stress component within the rectangular inclusion. It is observed from Fig. 3 that σ_{32} at time t = 0 for a perfect interface is singular at the two corners $x_1 = \pm a$, which is in agreement with our previous observation (Wang and Pan, 2008b). When t > 0 the traction σ_{32} along the interface becomes regular, as predicted above. Furthermore the magnitude of σ_{32} is a monotonically decreasing function of the time. It is observed from Figs. 4 and 5 that $\sigma_{31}^{(1)}$ has the same negative sign both inside and outside the inclusion along the interface whilst $\sigma_{31}^{(2)}$ changes its sign along the interface. The maximum magnitude of $\sigma_{31}^{(1)}$ at any time always occurs at $x_1 = 0$, whilst the location of the maximum magnitude of $\sigma_{31}^{(2)}$ depends on the time. In this example when $\tilde{t} < 0.10$ the maximum magnitude of $\sigma_{31}^{(2)}$ occurs in the vicinity of the two corners, namely around points $x_1 = \pm \delta a$ with $0 < \delta < 1$; on the other hand when $\tilde{t} \ge 0.10$ the maximum magnitude of $\sigma_{31}^{(2)}$ always occurs at $x_1 = 0$, the center of the lower side of the inclusion.

Before ending this section we add that the problem of an arbitrary shaped polygonal inclusion (Pan, 2004; Jiang and Pan, 2004) can also be solved by carrying out the line integrals in Eq.



Fig. 3. Distribution of the interfacial traction $\sigma_{32}^{(1)} = \sigma_{32}^{(2)} = \sigma_{32}$ (on $x_2 = 0$) caused by eigenstrain ε_{31}^* at four different normalized times $\tilde{t} = t\chi/\gamma = 0, 0.05, 0.2, \infty$ with $a_1 = -a, a_2 = a, b_1 = 0, b_2 = a$ and $a\chi = 0.5$.

(35). As expected the resulting expressions for an arbitrary shaped polygonal inclusion will become more complicated than those for a rectangular inclusion obtained in this section. Alternatively, for inclusions with rounded corners (for example elliptical inclusions), Ru's method of analytical continuation (Ru, 2000, 2001; Wang et al., 2007) can be employed. In Appendix A, we present an analytical treatment of inclusions with rounded corners interacting with a viscoelastic interface.

5. A finite Griffith crack normal to the viscoelastic interface

In this section we consider a finite Griffith crack along the x_2 -axis in the upper half-plane, as illustrated in Fig. 6. The two tips of the crack are located at $(0,y_1)$ and $(0,y_2)$, $(y_2 > y_1 > 0)$, respectively. On the crack surface the following constant traction condition is imposed

$$\sigma_{31}^{(1)} = -\sigma_0 \quad (x_1 = 0, y_1 < x_2 < y_2). \tag{43}$$

In order to solve this crack problem we can resort to the obtained Green's function for a screw dislocation. Furthermore in order to simplify the analysis involved we assume that the upper half-plane is orthotropic such that $p_1 = i\delta$ with δ being a positive real value. The resulting singular integral equation can be finally obtained as

$$\frac{1}{\pi} \int_{y_1}^{y_2} \frac{B(\xi, \tilde{t})}{\xi - y} d\xi + \frac{1}{\pi} \int_{y_1}^{y_2} k(\xi, y, \tilde{t}) B(\xi, \tilde{t}) d\xi = -\frac{2\sigma_0}{\mu_1} \quad (y_1 \leqslant y \leqslant y_2),$$
(44)

where $B(\xi, \tilde{t})$ is the unknown time-dependent dislocation density in the crack site, and the time-dependent kernel $k(\xi, y, \tilde{t})$ is given by



Fig. 4. Distribution of the interfacial stress $\sigma_{31}^{(1)}$ on the upper side of the interface $x_2 = 0$ caused by eigenstrain ε_{31}^* at four different normalized times $\tilde{t} = t\chi/\gamma = 0, 0.05, 0.2, \infty$ with $a_1 = -a, a_2 = a, b_1 = 0, b_2 = a$ and $a\chi = 0.5$.



Fig. 5. Distribution of the interfacial stress $\sigma_{31}^{(2)}$ on the lower side of the interface $x_2 = 0$ caused by eigenstrain ε_{31}^* at four different normalized times $\tilde{t} = t\chi/\gamma = 0, 0.05, 0.2, \infty$ with $a_1 = -a, a_2 = a, b_1 = 0, b_2 = a$ and $a\chi = 0.5$.

$$k(\xi, y, \tilde{t}) = \frac{1}{\xi + y} - \frac{\Gamma \exp(-t)}{\xi + y + \tilde{t}/\chi'} + \chi' \Gamma \exp\left[\chi'(\xi + y)\right] \left\{ E_1 \left[\chi'(\xi + y) + \tilde{t}\right] - E_1 \left[\chi'(\xi + y)\right] \right\},$$
(45)

with $\chi' = \delta \chi, \gamma' = \delta \gamma, \tilde{t} = t \chi / \gamma$.

In addition the following single valuedness condition for the displacement must be imposed for a Griffith crack

$$\int_{y_1}^{y_2} B(\xi, \tilde{t}) d\xi = 0.$$
(46)

Eqs. (45) and (46) are solved numerically using the method developed by Erdogan and Gupta (1972). First the limits of integration are changed to (-1,1) by the substitution

$$y = \frac{y_2 - y_1}{2}s + \frac{y_2 + y_1}{2}, \quad \xi = \frac{y_2 - y_1}{2}q + \frac{y_2 + y_1}{2}.$$
 (47)

Then, assuming square root singularities at both ends of the crack, the unknown dislocation density can be expressed as

$$B(q,\tilde{t}) = F(q,\tilde{t})(1-q^2)^{-1/2},$$
(48)

where $F(q, \tilde{t})$ is regular in the interval (-1, 1).

Finally Eqs. (45) and (46) are discretized, and the resulting system of linear simultaneous equations is solved to obtain $g(q, \tilde{t})$ at selected collocation points in the interval (-1,1).

Once Eqs. (45) and (46) are solved for the unknown dislocation density $B(\xi, \tilde{t})$, the time-dependent Mode-III stress intensity factors (SIFs) are calculated from the expressions given by Erdogan (1983) as

$$\begin{split} & \mathcal{K}_{\mathrm{III}}^{\mathrm{L}}(\tilde{t}) = \frac{\mu_{1}}{2} \lim_{\xi \to y_{1}} \sqrt{2(\xi - y_{1})} B(\xi, \tilde{t}) = \frac{\mu_{1}}{2} \sqrt{(y_{2} - y_{1})/2} F(-1, \tilde{t}), \\ & \mathcal{K}_{\mathrm{III}}^{\mathrm{U}}(\tilde{t}) = -\frac{\mu_{1}}{2} \lim_{\xi \to y_{2}} \sqrt{2(y_{2} - \xi)} B(\xi, \tilde{t}) = -\frac{\mu_{1}}{2} \sqrt{(y_{2} - y_{1})/2} F(1, \tilde{t}), \end{split}$$

$$(49)$$

where the superscripts L and U represent the lower and upper crack tips, respectively.

In the following numerical example we assume that $y_2 = 6y_1$, $\chi' y_1 = 0.5$ and $\Gamma = 1.5$. Fig. 7 demonstrates the variations of the normalized SIFs $K^{\rm L}(\tilde{t}) = \frac{K_{\rm III}^{\rm L}(\tilde{t})}{\sigma_0 \sqrt{(y_2 - y_1)/2}}$ and $K^{\rm U}(\tilde{t}) = \frac{K_{\rm III}^{\rm U}(\tilde{t})}{\sigma_0 \sqrt{(y_2 - y_1)/2}}$ as a function of the normalized time $\tilde{t} = t\chi/\gamma$. It can be observed from Fig. 7 that: (i) at the initial time when the interface is perfect, $K^{\rm L}(0) < K^{\rm U}(0) < 1$; (ii) as time evolves both the two SIFs



Fig. 6. A finite crack along the x_2 -axis in the upper half-plane of the anisotropic bimaterial.



Fig. 7. Variation of the normalized SIFs $K^{L}(\tilde{t}) = \frac{K_{III}^{L}(\tilde{t})}{\sigma_{0}\sqrt{(y_{2}-y_{1})/2}}$ and $K^{U}(\tilde{t}) = \frac{K_{IIII}^{U}(\tilde{t})}{\sigma_{0}\sqrt{(y_{2}-y_{1})/2}}$ as a function of the normalized time $\tilde{t} = t\chi/\gamma$ with $y_{2} = 6y_{1}, \chi'y_{1} = 0.5$ and $\Gamma = 1.5$. The upper half-plane is orthotropic.

monotonically increase with time, while $K^{L}(\tilde{t})$ increases much faster than $K^{U}(\tilde{t})$ and consequently at the time $\tilde{t} = 0.4767$, $K^{L}(\tilde{t}) = K^{U}(\tilde{t}) = 0.9868$; (iii) when $\tilde{t} > 0.4767$, $K^{L}(\tilde{t}) > K^{U}(\tilde{t})$; (iv) when $\tilde{t} \ge 5$, the SIFs approach the steady state values for a crack interacting with a linear spring interface.

6. Conclusions

We first derived explicit expressions of the Green's function for an anisotropic elastic half-plane subjected to an antiplane force qand a screw dislocation with the magnitude b. The time-dependent stresses induced by the force and dislocation and the image force acting on the screw dislocation were also obtained. It is observed that the image force is totally controlled by the two-phase mismatch parameter Γ ($0 \le \Gamma \le 2$), the interfacial rigidity λ and the relaxation time t_0 (see Eq. (31)).

Next the obtained Green's function for a line force was employed to investigate a rectangular inclusion with uniform eigenstrains in the upper half-plane of the two bonded half-planes. It is observed that the obtained solutions in the complex form are strikingly concise (see Eqs. (36) and (37)). Contrary to the case of a perfect interface (Wang and Pan, 2008b), it seems impossible to write down the real form expressions when the interface is viscoelastic due to the existence of the exponential integrals in the expressions.

Last the obtained Green's function for a screw dislocation was employed to construct the singular integral equation (44) for a finite Griffith crack normal to the viscoelastic interface. The timedependent SIFs at the two crack tips can be easily calculated through the numerical solution of the obtained singular integral equation. One unique property in the kernel $k(\xi, y, \tilde{t})$ of the singular integral equation is that it is time dependent due to the influence of the viscoelastic interface.

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Appendix A. Treatment of inclusions with rounded corners

Here we consider an Eshelby's inclusion (with uniform antiplane eigenstrains ε_{31}^* , ε_{32}^*) of arbitrary shape described by curve Γ in the upper half-plane of the anisotropic bimaterial with a viscoelastic interface. By employing the method of analytical continuation (Ru, 2000, 2001; Wang et al., 2007), the two analytical functions $f_0(z,t)$ within the inclusion and $f_1(z,t)$ outside the inclusion within the upper half-plane can be expressed in terms of $f_2(z,t)$ defined in the lower half-plane as follows

$$\begin{split} f_0(z,t) &= \frac{\mu_2}{\mu_1} \bar{f}_2(z,t) - \bar{d} \big[\bar{D}(z) - \bar{P}(z) \big] - cz - dP(z), \\ f_1(z,t) &= \frac{\mu_2}{\mu_1} \bar{f}_2(z,t) - \bar{d} \big[\bar{D}(z) - \bar{P}(z) \big] + d[D(z) - P(z)], \end{split}$$
(A1)

where

$$c = \frac{2i(\varepsilon_{32}^* - \bar{p}_1\varepsilon_{31}^*)}{p_1 - \bar{p}_1}, \quad d = \frac{2i(p_1\varepsilon_{31}^* - \varepsilon_{32}^*)}{p_1 - \bar{p}_1}.$$
 (A2)

In Eq. (A1), $\bar{z} = D(z)$ is analytic in the exterior of $\Gamma_1 = (x_1 + p_1 x_2, \text{where } z = x_1 + i x_2 \in \Gamma)$, except at infinity where it has pole of finite degree determined by the asymptotic behavior D(z) = P(z) + o(1) as $|z| \to \infty$; P(z) is a polynomial in z of finite degree (Ru, 2000, 2001); and $f_2(z, t)$ satisfies the following partial differential equation

$$-i\chi f_2(z,t) + f_2'(z,t) - i\gamma \dot{f}_2(z,t) = -\frac{2i\mu_1\chi d}{\mu_1 + \mu_2} [D(z) - P(z)], \quad x_2 \le 0.$$
(A3)

According to Eq. (21), it is enough to derive $f_2(z,0)$ for a perfect interface and $f_2(z,\infty)$ for a linear spring interface to arrive at $f_2(z,t)$ for any other time. $f_2(z,0)$ for a perfect interface can be simply derived as

$$f_2(z,0) = \frac{2\mu_1 d}{\mu_1 + \mu_2} [D(z) - P(z)].$$
(A4)

On the other hand $f_2(z,\infty)$ satisfies the following differential equation

$$-i\chi f_{2}(z,\infty) + f_{2}'(z,\infty) = -\frac{2i\mu_{1}\chi d}{\mu_{1} + \mu_{2}}[D(z) - P(z)], \quad x_{2} \leq 0$$
(A5)

whose solution can be conveniently expressed as (Wang et al., 2007)

$$f_2(z,\infty) = -\frac{2i\mu_1\chi d}{\mu_1 + \mu_2} \exp(i\chi z) \int_{-\infty i}^{z} [D(\xi) - P(\xi)] \exp(-i\chi\xi) d\xi \quad (x_2 \leq 0).$$
(A6)

Next we illustrate the above solution through an example of an elliptical inclusion with semi-major and semi-minor axes *a* and *b*. We further assume that the center of the ellipse is located at $x_1 = 0$ and $x_2 = \delta(\delta > 0)$ and the principal axes are parallel to the coordinate axes. In this case D(z), P(z) and D(z)-P(z) can be explicitly determined as

$$D(z) = \frac{a^2 + |p_1|^2 b^2}{a^2 + p_1^2 b^2} (z - p_1 \delta) + \bar{p}_1 \delta + \frac{i(p_1 - \bar{p}_1)ab}{a^2 + p_1^2 b^2} \times \sqrt{(z - p_1 \delta)^2 - (a^2 + p_1^2 b^2)},$$
(A7)

$$P(z) = \frac{a - i\bar{p}_1 b}{a - ip_1 b} z - \frac{a(p_1 - \bar{p}_1)}{a - ip_1 b} \delta,$$
(A8)

$$D(z) - P(z) = \frac{1(p_1 - p_1)ab}{(z - p_1\delta) + \sqrt{(z - p_1\delta)^2 - (a^2 + p_1^2b^2)}}.$$
 (A9)

Consequently it follows from Eqs. (A6) and (A9) that

$$f_{2}(z,\infty) = \frac{(p_{1} - \bar{p}_{1})ab\mu_{1}\chi d}{\mu_{1} + \mu_{2}} \exp\left[i\chi(z - p_{1}\delta)\right] \\ \times \int_{i\chi(z - p_{1}\delta)}^{\infty} \frac{2\exp(-\xi)}{\xi + \sqrt{\xi^{2} + \chi^{2}(a^{2} + p_{1}^{2}b^{2})}} d\xi.$$
(A10)

In the general situation in which $a^2 + p_1^2 b^2 \neq 0$, the integral in Eq. (A10) can be numerically carried out very easily (Wang et al.,

2007). When $p_1 = ia/b$ (or $a^2 + p_1^2 b^2 = 0$), the integral in Eq. (A10) reduces to the exponential integral defined by Eq. (17) such that

$$f_2(z,\infty) = \frac{2ia^2\mu_1\chi d}{\mu_1 + \mu_2} \exp{[i\chi(z - p_1\delta)]E_1[i\chi(z - p_1\delta)]}.$$
 (A11)

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