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Thermal Green's functions in plane anisotropic bimaterials with spring-type and Kapitza-type imperfect interface

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Abstract By means of the extended version of the Stroh formalism for uncoupled thermo-anisotropic elasticity, two-dimensional Green's function solutions in terms of exponential integrals are derived for the thermoelastic problem of a line heat source and a temperature dislocation near an imperfect interface between two different anisotropic half-planes with different thermo-mechanical properties. The imperfect interface investigated here is modeled as a generalized spring layer with vanishing thickness: (1) the normal heat flux is continuous at the interface, whereas the temperature field undergoes a discontinuity which is proportional to the normal heat flux; (2) the tractions are continuous across the interface, whereas the displacements undergo jumps which are proportional to the interface tractions. This kind of imperfect interface can be termed a thermally weakly conducting and mechanically compliant interface. In the Appendix we also present the isothermal Green's functions in anisotropic bimaterials with an elastically stiff interface to demonstrate the basic ingredients in the analyses of a stiff interface.

1 Introduction

Up to now various imperfect interface models with vanishing thickness have been proposed to account for the thin interphase layer with finite thickness between two different phases [1–7]. There are mainly two types of imperfect interfaces in the context of heat conduction [1–3], namely the weakly conducting interface (or the well known Kapitza thermal contact resistance model) and the highly conducting interface. At a weakly conducting interface, the normal heat flux is continuous across the interface but the temperature undergoes a jump which is proportional to the normal heat flux. On the other hand, at a highly conducting interface, the temperature is continuous across the interface, and the normal heat flux exhibits a discontinuity proportional to a surface differential operator of the temperature. Similarly there are also mainly two types of imperfect interface in the context of elasticity [1,4–7], namely the compliant spring-type interface and the stiff interface (or the generalized Young-Laplace model [1]). At a compliant spring-type interface, the tractions are continuous across the interface, whereas the displacements undergo jumps which are proportional to the interface tractions. At a stiff interface, the displacements are continuous across the interface, whereas the tractions undergo jumps which are proportional to certain surface differential operators of the interface displacements.

In this work, we investigate the two-dimensional temperature and the thermoelastic fields induced by a steady line heat source and a temperature dislocation interacting with a planar imperfect interface between two

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anisotropic half-planes by means of the extended version of the Stroh formalism for two-dimensional problems in uncoupled thermoelasticity [8,9]. Recently Kattis et al. [9] discussed the thermal Green's functions in anisotropic bimaterials with a perfect interface across which the temperature, normal heat flux, displacements and tractions are all continuous (see Eqs. (67), (72) in [9]). Here we confine our attention to a thermally weakly conducting and mechanically compliant imperfect interface [1,7]. Full-field solutions of the two analytic functions and the two analytic function vectors characterizing the temperature and the thermoelastic fields in the two half-planes are derived in terms of the exponential integrals [10]. The obtained thermal Green's functions together with the isothermal Green's functions for a line force and a line dislocation can be further employed to study cracks interacting with the spring-type imperfect interface under thermo-mechanical loadings [9,11,12].

2 Basic formulations

The basic equations for thermo-anisotropic elasticity, which include the stress-strain laws, the equilibrium equations, the heat conduction equation and the balance of energy, are [8]

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}u_{k,l} - \beta_{ij}T, \\ \sigma_{ij,j} &= 0, \\ h_i &= -k_{ij}T_{,j}, \quad h_{i,i} = 0, \end{aligned} \tag{1}$$

where repeated indices mean summation, a comma followed by i $(i=1,2,3)$ stands for the derivative with respect to the ith spatial coordinate; u_i, σ_{ij}, h_i, and T are, respectively, the displacement, stress, heat flux and temperature; C_{ijkl} and β_{ij} are, respectively, the elastic and stress-temperature coefficients; $k_{ij} = k_{ji}$ is the heat conduction coefficient. For two-dimensional problems in which all quantities depend only on x_1 and x_2, the general solutions can be expressed as [8,9]

$$\begin{aligned} T &= g'(z_0) + \overline{g'(z_0)}, \\ \Theta &= \mathrm{i}\gamma\left[g'(z_0) - \overline{g'(z_0)}\right], \\ \mathbf{u} &= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T = \mathbf{A}\mathbf{f}(z) + \bar{\mathbf{A}}\overline{\mathbf{f}(z)} + \mathbf{c}g(z_0) + \bar{\mathbf{c}}\overline{g(z_0)}, \\ \mathbf{\Phi} &= \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \end{bmatrix}^T = \mathbf{B}\mathbf{f}(z) + \bar{\mathbf{B}}\overline{\mathbf{f}(z)} + \mathbf{d}g(z_0) + \bar{\mathbf{d}}\overline{g(z_0)}, \end{aligned} \tag{2}$$

where the prime denotes differentiation with respect to the complex variable, and

$$\begin{aligned} z_0 &= x_1 + p_0x_2, \quad p_0 = \frac{-k_{12} + \mathrm{i}\gamma}{k_{22}}, \quad \gamma = \sqrt{k_{11}k_{22} - k_{12}^2} > 0, \\ \mathbf{A} &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}, \\ \mathbf{f}(z) &= \begin{bmatrix} f_1(z_1) & f_2(z_2) & f_3(z_3) \end{bmatrix}^T, \\ z_i &= x_1 + p_ix_2, \quad \mathrm{Im}\{p_i\} > 0, \quad (i=1\text{–}3), \end{aligned} \tag{3}$$

with

$$\begin{aligned} &\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}\begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} = p_i\begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix}, \quad (i=1\text{–}3), \\ &\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}\begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = p_0\begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{N}_2 \\ \mathbf{I} & \mathbf{N}_1^T \end{bmatrix}\begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix}, \end{aligned} \tag{4}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}, \tag{5}$$

$$\boldsymbol{\beta}_1 = \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \end{bmatrix}^T, \quad \boldsymbol{\beta}_2 = \begin{bmatrix} \beta_{12} & \beta_{22} & \beta_{32} \end{bmatrix}^T, \tag{6}$$

and

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{7}$$

p_0 in Eq. (3) can be called the thermal eigenvalue, while p_k ($k=1\ldots3$) in Eq. (3) are the Stroh eigenvalues [8]. The above general solution is valid provided that $p_0 \neq p_1 \neq p_2 \neq p_3$. In addition the heat flux function Θ is defined in terms of the heat fluxes h_1 and h_2 as

$$h_1 = \Theta_{,2}, \quad h_2 = -\Theta_{,1}, \tag{8}$$

and the stress function vector $\mathbf{\Phi}$ is defined, in terms of the stresses, as follows:

$$\sigma_{i1} = -\Phi_{i,2}, \quad \sigma_{i2} = \Phi_{i,1}, \quad (i=1\text{–}3). \tag{9}$$

Due to the fact that the two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following normalized orthogonal relationship [8]

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \mathbf{I}, \tag{10}$$

then three real Barnett-Lothe tensors $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ can be introduced [8]

$$\mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -2\mathrm{i}\mathbf{B}\mathbf{B}^T. \tag{11}$$

Furthermore, the two matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric, while $\mathbf{SH}$, $\mathbf{LS}$, $\mathbf{H}^{-1}\mathbf{S}$, $\mathbf{SL}^{-1}$ are anti-symmetric.

3 Thermal Green's functions for anisotropic bimaterials with imperfect interface

3.1 Problem description

Two jointed semi-infinite anisotropic thermoelastic solids consist of solid 1 in the upper half-plane ($x_2 > 0$) and solid 2 in the lower half-plane ($x_2 < 0$). Assume that a line heat source of strength Q_0 and a temperature dislocation of discontinuity T_0 are both located at $\left[\hat{x}_1, \hat{x}_2\right]$, ($\hat{x}_2 > 0$) in the upper half-plane of the anisotropic bimaterial. Throughout this paper, the subscripts 1 and 2 (or the superscripts 1, 2) are used to identify the respective quantities in the upper and lower half-planes, respectively.

In this research, the two half-planes are assumed to be bonded imperfectly through the real axis $x_2 = 0$. Using the weakly conducting and compliant imperfect interface model described in the Introduction, the boundary conditions on the imperfect interface $x_2 = 0$ take the following form:

$$\begin{aligned} &h_2^{(1)} = h_2^{(2)}, \quad T_1 - T_2 = -\alpha_0 h_2^{(2)}, \\ &\sigma_{12}^{(1)} = \sigma_{12}^{(2)}, \quad \sigma_{22}^{(1)} = \sigma_{22}^{(2)}, \quad \sigma_{32}^{(1)} = \sigma_{32}^{(2)}, \quad x_2 = 0, \end{aligned} \tag{12}$$

$$\begin{bmatrix} u_1^{(1)} - u_1^{(2)} \\ u_2^{(1)} - u_2^{(2)} \\ u_3^{(1)} - u_3^{(2)} \end{bmatrix} = \mathbf{\Lambda} \begin{bmatrix} \sigma_{12}^{(2)} \\ \sigma_{22}^{(2)} \\ \sigma_{32}^{(2)} \end{bmatrix}, \quad x_2 = 0, \tag{13}$$

where $\alpha_0 \geq 0$, and

$$\mathbf{\Lambda} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix} \tag{14}$$

is real, symmetric and positive semidefinite (in the following analysis we treat $\mathbf{\Lambda}$ as positive definite). In writing Eq. (13), we have adopted the anisotropic spring-type interface model derived by Benveniste [1, Eq. (6.4)]. When $\alpha_0 = 0$ the interface is thermally perfect, whereas $\alpha_0 \to \infty$ stands for adiabatic contact. On the other hand when $\mathbf{\Lambda} = \mathbf{0}$ the interface is mechanically perfect, whereas $\mathbf{\Lambda} \to \infty$ means that the interface is a traction-free surface.

In view of Eqs. (8) and (9), the thermo-mechanical boundary conditions in Eqs. (12) and (13) can also be equivalently expressed in terms of T, Θ, $\mathbf{u}$ and $\mathbf{\Phi}$ as

$$\Theta_1 = \Theta_2, \quad T_1 - T_2 = \alpha_0 \Theta_{2,1}, \quad x_2 = 0, \tag{15}$$

$$\mathbf{\Phi}_1 = \mathbf{\Phi}_2, \quad \mathbf{u}_1 - \mathbf{u}_2 = \mathbf{\Lambda}\Phi_{2,1}, \quad x_2 = 0. \tag{16}$$

Due to the fact that on the interface $x_2 = 0$ we have $z_0 = z_1 = z_2 = z_3 = z$, ($z = x_1 + \mathrm{i}x_2$), then during the following analysis we can first replace z_k ($k = 0, 1, 2, 3$) by the common complex variable z [13–15]. After the analysis is finished, we can then change z back to the corresponding complex variables.

3.2 The temperature field

The thermal boundary conditions on the interface in Eq. (15) can be expressed in terms of $g_1(z)$ defined in the upper half-plane and $g_2(z)$ defined in the lower half-plane as

$$\gamma_1\left[g_1'^{+}(x_1)-\bar{g}_1'^{-}(x_1)\right]=\gamma_2\left[g_2'^{-}(x_1)-\bar{g}_2'^{+}(x_1)\right],\quad x_2=0, \tag{17}$$

$$g_1'^{+}(x_1)+\bar{g}_1'^{-}(x_1)-g_2'^{-}(x_1)-\bar{g}_2'^{+}(x_1)=\mathrm{i}\alpha_0\gamma_2\left[g_2''^{-}(x_1)-\bar{g}_2''^{+}(x_1)\right],\quad x_2=0. \tag{18}$$

It follows from Eq. (17) that

$$g_1'(z)=-\Gamma\bar{g}_2'(z)+g_0'(z)+\bar{g}_0'(z), \tag{19.1}$$

$$\bar{g}_1'(z)=-\Gamma g_2'(z)+g_0'(z)+\bar{g}_0'(z), \tag{19.2}$$

where $\Gamma=\gamma_2/\gamma_1$ is a dimensionless two-phase temperature parameter and

$$g_0'(z)=X_0\ln(z-\hat{z}_0), \tag{20}$$

with $X_0=-\frac{Q_0}{4\pi\gamma_1}-\frac{T_0}{4\pi}\mathrm{i}$ and $\hat{z}_0=\hat{x}_1+p_0\hat{x}_2$ is the complex thermal potential in a homogeneous plane occupied by material 1 [9].

Substituting Eq. (19) into Eq. (18) and eliminating $g_1'^{+}(x_1)$ and $\bar{g}_1'^{-}(x_1)$, we can obtain the following:

$$-(1+\Gamma)\bar{g}_2'^{+}(x_1)+\mathrm{i}\alpha_0\gamma_2\bar{g}_2''^{+}(x_1)+2\bar{g}_0'(x_1)=(1+\Gamma)g_2'^{-}(x_1)+\mathrm{i}\alpha_0\gamma_2 g_2''^{-}(x_1)-2g_0'(x_1),\quad x_2=0. \tag{21}$$

It is apparent that the left hand side of Eq. (21) is analytic in the upper half-plane, while the right hand side of Eq. (21) is analytic in the lower half-plane. Consequently the continuity condition in Eq. (21) implies that the left and right sides of Eq. (21) are identically zero in the upper and lower half-planes, respectively. It follows that

$$-\mathrm{i}\lambda_0 g_2'(z)+g_2''(z)=-\frac{2\mathrm{i}\lambda_0 X_0}{1+\Gamma}\ln(z-\hat{z}_0),\quad \mathrm{Im}\{z\}<0, \tag{22}$$

where $\lambda_0=\frac{1+\Gamma}{\alpha_0\gamma_2}\geq 0$ with its dimension 1/length is the interface thermal contact resistance parameter for the temperature field.

The solution to Eq. (22) can be easily obtained as

$$g_2''(z)=\frac{2\mathrm{i}\lambda_0 X_0}{1+\Gamma}\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right], \tag{23}$$

where $E_1(z)$ is the exponential integral [10] defined by

$$E_1(z)=\int_z^{\infty}\frac{e^{-t}}{t}\mathrm{d}t. \tag{24}$$

Consequently $g_1''(z)$ can be obtained by substituting Eq. (23) into Eq. (19.1) as follows:

$$\begin{aligned}g_1''(z)&=-\Gamma\bar{g}_2''(z)+g_0''(z)+\bar{g}_0''(z)\\&=\frac{2\mathrm{i}\Gamma\lambda_0\bar{X}_0}{1+\Gamma}\exp\left[-\mathrm{i}\lambda_0(z-\bar{\hat{z}}_0)\right]E_1\left[-\mathrm{i}\lambda_0(z-\bar{\hat{z}}_0)\right]+\frac{X_0}{z-\hat{z}_0}+\frac{\bar{X}_0}{z-\bar{\hat{z}}_0}.\end{aligned} \tag{25}$$

In addition it is not difficult to write down the full field expressions as follows:

$$\begin{aligned}g_1''(z_0)&=\frac{2\mathrm{i}\Gamma\lambda_0\bar{X}_0}{1+\Gamma}\exp\left[-\mathrm{i}\lambda_0(z_0-\bar{\hat{z}}_0)\right]E_1\left[-\mathrm{i}\lambda_0(z_0-\bar{\hat{z}}_0)\right]+\frac{\bar{X}_0}{z_0-\bar{\hat{z}}_0}\\&\quad+\frac{X_0}{z_0-\hat{z}_0},\quad z_0=x_1+p_0x_2,\end{aligned} \tag{26}$$

$$g_2''(z_0^*)=\frac{2\mathrm{i}\lambda_0 X_0}{1+\Gamma}\exp\left[\mathrm{i}\lambda_0(z_0^*-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z_0^*-\hat{z}_0)\right],\quad z_0^*=x_1+p_0^*x_2, \tag{27}$$

where the superscript '*' is utilized to distinguish the thermal eigenvalue p_0^* associated with the lower half-plane with p_0 associated with the upper half-plane. Integration of Eqs. (26) and (27) will yield the following expressions of $g_1'(z_0)$ and $g_2'(z_0^*)$:

$$g_1'(z_0) = -\frac{2\Gamma\bar{X}_0}{1+\Gamma}\exp\left[-\mathrm{i}\lambda_0(z_0-\bar{\hat{z}}_0)\right]E_1\left[-\mathrm{i}\lambda_0(z_0-\bar{\hat{z}}_0)\right] + X_0\ln(z_0-\hat{z}_0) + \frac{1-\Gamma}{1+\Gamma}\bar{X}_0\ln(z_0-\bar{\hat{z}}_0), \quad (28)$$

$$g_2'(z_0^*) = \frac{2X_0}{1+\Gamma}\exp\left[\mathrm{i}\lambda_0(z_0^*-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z_0^*-\hat{z}_0)\right] + \frac{2X_0}{1+\Gamma}\ln(z_0^*-\hat{z}_0). \quad (29)$$

3.3 The thermoelastic field

The mechanical boundary conditions on the interface in Eq. (16) can be expressed in terms of the two analytic function vectors $\mathbf{f}_1(z)$ defined in the upper half-plane and $\mathbf{f}_2(z)$ defined in the lower half-plane as

$$\begin{aligned}&\mathbf{B}_1\mathbf{f}_1^+(x_1) + \bar{\mathbf{B}}_1\bar{\mathbf{f}}_1^-(x_1) + \mathbf{d}_1 g_1^+(x_1) + \bar{\mathbf{d}}_1\bar{g}_1^-(x_1)\\ &\quad = \mathbf{B}_2\mathbf{f}_2^-(x_1) + \bar{\mathbf{B}}_2\bar{\mathbf{f}}_2^+(x_1) + \mathbf{d}_2 g_2^-(x_1) + \bar{\mathbf{d}}_2\bar{g}_2^+(x_1), \quad x_2 = 0,\end{aligned} \quad (30)$$

$$\begin{aligned}&\mathbf{A}_1\mathbf{f}_1^+(x_1) + \bar{\mathbf{A}}_1\bar{\mathbf{f}}_1^-(x_1) + \mathbf{c}_1 g_1^+(x_1) + \bar{\mathbf{c}}_1\bar{g}_1^-(x_1) - \mathbf{A}_2\mathbf{f}_2^-(x_1) - \bar{\mathbf{A}}_2\bar{\mathbf{f}}_2^+(x_1) - \mathbf{c}_2 g_2^-(x_1) - \bar{\mathbf{c}}_2\bar{g}_2^+(x_1)\\ &\quad = \mathbf{\Lambda}\left[\mathbf{B}_2\mathbf{f}_2'^-(x_1) + \bar{\mathbf{B}}_2\bar{\mathbf{f}}_2'^+(x_1) + \mathbf{d}_2 g_2'^-(x_1) + \bar{\mathbf{d}}_2\bar{g}_2'^+(x_1)\right], \quad x_2 = 0.\end{aligned} \quad (31)$$

It follows from Eqs. (30) and (19) that

$$\begin{aligned}\mathbf{f}_1(z) &= \mathbf{B}_1^{-1}\bar{\mathbf{B}}_2\bar{\mathbf{f}}_2(z) + \mathbf{f}_0(z) - \mathbf{B}_1^{-1}\bar{\mathbf{B}}_1\bar{\mathbf{f}}_0(z) + \mathbf{B}_1^{-1}(\Gamma\mathbf{d}_1 + \bar{\mathbf{d}}_2)\bar{g}_2(z) - \mathbf{B}_1^{-1}(\mathbf{d}_1 + \bar{\mathbf{d}}_1)\bar{g}_0(z),\\ \bar{\mathbf{f}}_1(z) &= \bar{\mathbf{B}}_1^{-1}\mathbf{B}_2\mathbf{f}_2(z) - \bar{\mathbf{B}}_1^{-1}\mathbf{B}_1\mathbf{f}_0(z) + \bar{\mathbf{f}}_0(z) + \bar{\mathbf{B}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1 + \mathbf{d}_2)g_2(z) - \bar{\mathbf{B}}_1^{-1}(\mathbf{d}_1 + \bar{\mathbf{d}}_1)g_0(z),\end{aligned} \quad (32)$$

where $\mathbf{f}_0(z)$ is the analytic function vector in a homogeneous plane occupied by material 1 given by [9]

$$\mathbf{f}_0(z) = \langle(z-\hat{z}_\alpha)\ln(z-\hat{z}_\alpha)\rangle\mathbf{e}, \quad (33)$$

with $\mathbf{e} = -2\mathrm{i}\mathbf{A}_1^T\mathrm{Im}\{\mathbf{d}_1X_0\} - 2\mathrm{i}\mathbf{B}_1^T\mathrm{Im}\{\mathbf{c}_1X_0\}$, $\hat{z}_\alpha = \hat{x}_1 + p_\alpha\hat{x}_2$ and $<*>$ being a 3×3 diagonal matrix in which each component is varied according to the Greek index α (from 1 to 3).

Substituting Eq. (32) into Eq. (31), we finally obtain the following:

$$\begin{aligned}&\overline{\mathbf{N}\mathbf{B}}_2\bar{\mathbf{f}}_2^+(x_1) - \mathrm{i}\mathbf{\Lambda}\bar{\mathbf{B}}_2\bar{\mathbf{f}}_2'^+(x_1) - 2\mathbf{L}_1^{-1}\bar{\mathbf{B}}_1\bar{\mathbf{f}}_0(x_1) - \mathrm{i}\mathbf{\Lambda}\bar{\mathbf{d}}_2\bar{g}_2'(x_1)\\ &\quad + \left[-\mathrm{i}(\Gamma\mathbf{c}_1 + \bar{\mathbf{c}}_2) + \mathbf{M}_1^{-1}(\Gamma\mathbf{d}_1 + \bar{\mathbf{d}}_2)\right]\bar{g}_2(x_1) + \left[\mathrm{i}(\mathbf{c}_1 + \bar{\mathbf{c}}_1) - \mathbf{M}_1^{-1}(\mathbf{d}_1 + \bar{\mathbf{d}}_1)\right]\bar{g}_0(x_1)\\ &= \mathbf{N}\mathbf{B}_2\mathbf{f}_2^-(x_1) + \mathrm{i}\mathbf{\Lambda}\mathbf{B}_2\mathbf{f}_2'^-(x_1) - 2\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{f}_0(x_1) + \mathrm{i}\mathbf{\Lambda}\mathbf{d}_2 g_2'(x_1)\\ &\quad + \left[\mathrm{i}(\Gamma\bar{\mathbf{c}}_1 + \mathbf{c}_2) + \bar{\mathbf{M}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1 + \mathbf{d}_2)\right]g_2(x_1) - \left[\mathrm{i}(\mathbf{c}_1 + \bar{\mathbf{c}}_1) + \bar{\mathbf{M}}_1^{-1}(\mathbf{d}_1 + \bar{\mathbf{d}}_1)\right]g_0(x_1), \quad x_2 = 0,\end{aligned} \quad (34)$$

where $\mathbf{M}_k^{-1}(k = 1, 2)$ and $\mathbf{N}$ are 3×3 positive definite Hermitian matrices given by [8]

$$\begin{aligned}\mathbf{N} &= \bar{\mathbf{M}}_1^{-1} + \mathbf{M}_2^{-1} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1} + \mathrm{i}(\mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1}),\\ \mathbf{M}_k^{-1} &= \mathrm{i}\mathbf{A}_k\mathbf{B}_k^{-1} = (\mathbf{I} - \mathrm{i}\mathbf{S}_k)\mathbf{L}_k^{-1}, \quad (k = 1, 2).\end{aligned} \quad (35)$$

It is apparent that the left hand side of Eq. (34) is analytic in the upper half-plane, while the right hand side of Eq. (34) is analytic in the lower half-plane. Consequently the continuity condition in Eq. (34) implies that the left and right sides of Eq. (34) are identically zero in the upper and lower half-planes, respectively. It follows that

$$\begin{aligned}&\mathbf{N}\mathbf{B}_2\mathbf{f}_2(z) + \mathrm{i}\mathbf{\Lambda}\mathbf{B}_2\mathbf{f}_2'(z) = 2\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{f}_0(z) - \mathrm{i}\mathbf{\Lambda}\mathbf{d}_2 g_2'(z)\\ &\quad - \left[\mathrm{i}(\Gamma\bar{\mathbf{c}}_1 + \mathbf{c}_2) + \bar{\mathbf{M}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1 + \mathbf{d}_2)\right]g_2(z) + \left[\mathrm{i}(\mathbf{c}_1 + \bar{\mathbf{c}}_1) + \bar{\mathbf{M}}_1^{-1}(\mathbf{d}_1 + \bar{\mathbf{d}}_1)\right]g_0(z), \quad \mathrm{Im}\{z\} < 0,\end{aligned} \quad (36)$$

or equivalently and more explicitly

$$\begin{aligned}\mathbf{N}\mathbf{B}_2\mathbf{f}_2'(z)+\mathrm{i}\boldsymbol{\Lambda}\mathbf{B}_2\mathbf{f}_2''(z)&=2\mathbf{L}_1^{-1}\mathbf{B}_1\langle\ln(z-\hat{z}_\alpha)+1\rangle\mathbf{e}-X_0\ln(z-\hat{z}_0)\mathbf{G}_1\\&-\frac{2X_0}{1+\Gamma}\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]\left[\mathrm{i}(\Gamma\bar{\mathbf{c}}_1+\mathbf{c}_2)+\bar{\mathbf{M}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1+\mathbf{d}_2)-\lambda_0\boldsymbol{\Lambda}\mathbf{d}_2\right],\end{aligned}\tag{37}$$

where the vector $\mathbf{G}_1$ is defined as

$$\mathbf{G}_1=\frac{2}{1+\Gamma}(\mathrm{i}\mathbf{c}_2+\bar{\mathbf{M}}_1^{-1}\mathbf{d}_2)+\frac{\Gamma-1}{1+\Gamma}(\mathrm{i}\bar{\mathbf{c}}_1+\bar{\mathbf{M}}_1^{-1}\bar{\mathbf{d}}_1)-\mathrm{i}\mathbf{c}_1-\bar{\mathbf{M}}_1^{-1}\mathbf{d}_1.\tag{38}$$

If we introduce a new analytic function vector $\mathbf{h}(z)$ such that

$$\begin{aligned}\mathbf{f}_2'(z)&=\mathbf{h}(z)+2\mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\langle\ln(z-\hat{z}_\alpha)+1\rangle\mathbf{e}-X_0\ln(z-\hat{z}_0)\mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{G}_1\\&-\frac{2X_0}{1+\Gamma}\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]\mathbf{B}_2^{-1}(\mathbf{N}-\lambda_0\boldsymbol{\Lambda})^{-1}\left[\mathrm{i}(\Gamma\bar{\mathbf{c}}_1+\mathbf{c}_2)+\bar{\mathbf{M}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1+\mathbf{d}_2)-\lambda_0\boldsymbol{\Lambda}\mathbf{d}_2\right],\end{aligned}\tag{39}$$

then Eq. (37) can be expressed in terms of the new function vector $\mathbf{h}(z)$ as

$$\mathbf{N}\mathbf{B}_2\mathbf{h}(z)+\mathrm{i}\boldsymbol{\Lambda}\mathbf{B}_2\mathbf{h}'(z)=-2\mathrm{i}\boldsymbol{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\left\langle\frac{1}{z-\hat{z}_\alpha}\right\rangle\mathbf{e}-\frac{\mathrm{i}X_0\mathbf{G}}{z-\hat{z}_0},\tag{40}$$

where the vector $\mathbf{G}$ is defined by

$$\mathbf{G}=\frac{2}{1+\Gamma}\boldsymbol{\Lambda}(\mathbf{N}-\lambda_0\boldsymbol{\Lambda})^{-1}\left[\mathrm{i}(\Gamma\bar{\mathbf{c}}_1+\mathbf{c}_2)+\bar{\mathbf{M}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1+\mathbf{d}_2)-\lambda_0\boldsymbol{\Lambda}\mathbf{d}_2\right]-\boldsymbol{\Lambda}\mathbf{N}^{-1}\mathbf{G}_1.\tag{41}$$

It is observed that Eq. (40) only contains known first-order poles on its right-hand side. In order to solve the coupled set of differential equations in Eq. (40), we first consider the eigenvalue problem

$$(\mathbf{N}-\lambda\boldsymbol{\Lambda})\mathbf{v}=\mathbf{0}.\tag{42}$$

It is observed that Eq. (42) has three real and non-negative eigenvalues for λ (see Appendix A for a strict proof). Let λ_i $(i=1\text{–}3)$ be the three distinct roots and $\mathbf{v}_i$ the associated eigenvectors, then the following orthogonal relationship can be easily proved (also see Appendix A for a strict proof):

$$\bar{\boldsymbol{\Psi}}^T\mathbf{N}\boldsymbol{\Psi}=\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2,\quad\bar{\boldsymbol{\Psi}}^T\boldsymbol{\Lambda}\boldsymbol{\Psi}=\boldsymbol{\Lambda}_2,\tag{43}$$

where $\boldsymbol{\Lambda}_2$ is a 3×3 real diagonal matrix, and

$$\boldsymbol{\Psi}=\left[\mathbf{v}_1\quad\mathbf{v}_2\quad\mathbf{v}_3\right],\tag{44}$$

$$\boldsymbol{\Lambda}_1=\mathrm{diag}\left[\lambda_1\quad\lambda_2\quad\lambda_3\right].\tag{45}$$

Here λ_i $(i=1\text{–}3)$ with their unit 1/length can be considered as three interface rigidities for the elastic field (our definition of the dimensional interface rigidity is a little bit different than the dimensionless one in [6]). Note that in writing Eq. (39) we have assumed that the interface thermal contact resistance parameter for the thermal field is not equal to any one of the three interface rigidities for the elastic field, i.e., $\lambda_0\neq\lambda_1\neq\lambda_2\neq\lambda_3$. A discussion on the degenerate case in which the interface thermal contact resistance parameter for the thermal field is equal to one of the three interface rigidities for the elastic field is presented in the next section.

Next we introduce an analytic function vector $\boldsymbol{\Omega}(z)$ such that

$$\mathbf{B}_2\mathbf{h}(z)=\boldsymbol{\Psi}\boldsymbol{\Omega}(z).\tag{46}$$

Employing the orthogonal relationship in Eq. (43), then Eq. (40) can be decoupled into

$$-\mathrm{i}\boldsymbol{\Lambda}_1\boldsymbol{\Omega}(z)+\boldsymbol{\Omega}'(z)=-2\boldsymbol{\Lambda}_2^{-1}\bar{\boldsymbol{\Psi}}^T\boldsymbol{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\left\langle\frac{1}{z-\hat{z}_\alpha}\right\rangle\mathbf{e}-\frac{X_0\boldsymbol{\Lambda}_2^{-1}\bar{\boldsymbol{\Psi}}^T\mathbf{G}}{z-\hat{z}_0},\tag{47}$$

whose solution can be easily obtained as

$$\begin{aligned}\mathbf{\Omega}(z) = 2\sum_{k=1}^{3}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right] E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{I}_k\mathbf{e}\\ +\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right] E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right]\right\rangle X_0\mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{G},\end{aligned} \tag{48}$$

where

$$\mathbf{I}_1 = \mathrm{diag}\begin{bmatrix}1 & 0 & 0\end{bmatrix},\quad \mathbf{I}_2 = \mathrm{diag}\begin{bmatrix}0 & 1 & 0\end{bmatrix},\quad \mathbf{I}_3 = \mathrm{diag}\begin{bmatrix}0 & 0 & 1\end{bmatrix}. \tag{49}$$

In view of Eqs. (46) and (48), the function vector $\mathbf{h}(z)$ can then be obtained as

$$\begin{aligned}\mathbf{h}(z) = 2\mathbf{B}_2^{-1}\mathbf{\Psi}\sum_{k=1}^{3}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right] E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{I}_k\mathbf{e}\\ +X_0\mathbf{B}_2^{-1}\mathbf{\Psi}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right] E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{G}.\end{aligned} \tag{50}$$

Consequently in view of Eqs. (32) and (39), we can obtain the expressions of $\mathbf{f}_2'(z)$ and $\mathbf{f}_1'(z)$ as

$$\begin{aligned}\mathbf{f}_2'(z) = {}& 2\mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\langle\ln(z-\hat{z}_\alpha)+1\rangle\mathbf{e} - \ln(z-\hat{z}_0)X_0\mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{G}_1\\ &+2\mathbf{B}_2^{-1}\mathbf{\Psi}\sum_{k=1}^{3}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right] E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{I}_k\mathbf{e}\\ &+X_0\mathbf{B}_2^{-1}\mathbf{\Psi}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right] E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{G}\\ &-X_0\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right] E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]\mathbf{B}_2^{-1}(\mathbf{\Lambda}^{-1}\mathbf{G}+\mathbf{N}^{-1}\mathbf{G}_1),\end{aligned} \tag{51}$$

$$\begin{aligned}\mathbf{f}_1'(z) = {}& \langle\ln(z-\hat{z}_\alpha)+1\rangle\mathbf{e}+\mathbf{B}_1^{-1}\bar{\mathbf{N}}^{-1}(\bar{\mathbf{M}}_1^{-1}-\bar{\mathbf{M}}_2^{-1})\bar{\mathbf{B}}_1\langle\ln(z-\bar{\hat{z}}_\alpha)+1\rangle\bar{\mathbf{e}} - \ln(z-\bar{\hat{z}}_0)\bar{X}_0\mathbf{B}_1^{-1}\bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_2\\ &+2\mathbf{B}_1^{-1}\bar{\mathbf{\Psi}}\sum_{k=1}^{3}\left\langle \exp\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_k)\right] E_1\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_k)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\mathbf{\Lambda}\bar{\mathbf{N}}^{-1}\mathbf{L}_1^{-1}\bar{\mathbf{B}}_1\mathbf{I}_k\bar{\mathbf{e}}\\ &+\bar{X}_0\mathbf{B}_1^{-1}\bar{\mathbf{\Psi}}\left\langle \exp\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_0)\right] E_1\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_0)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\bar{\mathbf{G}}\\ &-\exp\left[-\mathrm{i}\lambda_0(z-\bar{\hat{z}}_0)\right] E_1\left[-\mathrm{i}\lambda_0(z-\bar{\hat{z}}_0)\right]\bar{X}_0\mathbf{B}_1^{-1}(\mathbf{\Lambda}^{-1}\bar{\mathbf{G}}+\bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_2-\mathbf{d}_1-\bar{\mathbf{d}}_1),\end{aligned} \tag{52}$$

where the vector $\mathbf{G}_2$ is defined as

$$\mathbf{G}_2 = \frac{2}{1+\Gamma}(\mathrm{i}\mathbf{c}_2-\mathbf{M}_2^{-1}\mathbf{d}_2)+\frac{\Gamma-1}{1+\Gamma}(\mathrm{i}\bar{\mathbf{c}}_1-\mathbf{M}_2^{-1}\bar{\mathbf{d}}_1)-\mathrm{i}\mathbf{c}_1+\mathbf{M}_2^{-1}\mathbf{d}_1. \tag{53}$$

It is not difficult to write down the full field expressions of $\mathbf{f}_1'(z)$ and $\mathbf{f}_2'(z)$ as follows:

$$\begin{aligned}\mathbf{f}_1'(z) = {}& \langle\ln(z_\alpha-\hat{z}_\alpha)+1\rangle\mathbf{e}+\sum_{k=1}^{3}\langle\ln(z_\alpha-\bar{\hat{z}}_k)+1\rangle\mathbf{B}_1^{-1}\bar{\mathbf{N}}^{-1}(\bar{\mathbf{M}}_1^{-1}-\bar{\mathbf{M}}_2^{-1})\bar{\mathbf{B}}_1\mathbf{I}_k\bar{\mathbf{e}}\\ &-\langle\ln(z_\alpha-\bar{\hat{z}}_0)\rangle\bar{X}_0\mathbf{B}_1^{-1}\bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_2\\ &+2\sum_{m=1}^{3}\sum_{k=1}^{3}\left\langle \exp\left[-\mathrm{i}\lambda_m(z_\alpha-\bar{\hat{z}}_k)\right] E_1\left[-\mathrm{i}\lambda_m(z_\alpha-\bar{\hat{z}}_k)\right]\right\rangle \mathbf{B}_1^{-1}\bar{\mathbf{\Psi}}\mathrm{I}_m\mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\mathbf{\Lambda}\bar{\mathbf{N}}^{-1}\mathbf{L}_1^{-1}\bar{\mathbf{B}}_1\mathbf{I}_k\bar{\mathbf{e}}\\ &+\sum_{k=1}^{3}\left\langle \exp\left[-\mathrm{i}\lambda_k(z_\alpha-\bar{\hat{z}}_0)\right] E_1\left[-\mathrm{i}\lambda_k(z_\alpha-\bar{\hat{z}}_0)\right]\right\rangle \bar{X}_0\mathbf{B}_1^{-1}\bar{\mathbf{\Psi}}\mathbf{I}_k\mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\bar{\mathbf{G}}\\ &-\left\langle \exp\left[-\mathrm{i}\lambda_0(z_\alpha-\bar{\hat{z}}_0)\right] E_1\left[-\mathrm{i}\lambda_0(z_\alpha-\bar{\hat{z}}_0)\right]\right\rangle \bar{X}_0\mathbf{B}_1^{-1}(\mathbf{\Lambda}^{-1}\bar{\mathbf{G}}+\bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_2-\mathbf{d}_1-\bar{\mathbf{d}}_1),\end{aligned} \tag{54}$$

$$\begin{aligned}\mathbf{f}_2'(z) = {} & 2\sum_{k=1}^{3}\left\langle \ln(z_\alpha^* - \hat{z}_k) + 1\right\rangle \mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{I}_k\mathbf{e} - \langle \ln(z_\alpha^* - \hat{z}_0)\rangle X_0\mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{G}_1 \\ & + 2\sum_{m=1}^{3}\sum_{k=1}^{3}\left\langle \exp\left[\mathrm{i}\lambda_m(z_\alpha^* - \hat{z}_k)\right] E_1\left[\mathrm{i}\lambda_m(z_\alpha^* - \hat{z}_k)\right]\right\rangle \mathbf{B}_2^{-1}\mathbf{\Psi}\mathbf{I}_m\mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{I}_k\mathbf{e} \\ & + \sum_{k=1}^{3}\left\langle \exp\left[\mathrm{i}\lambda_k(z_\alpha^* - \hat{z}_0)\right] E_1\left[\mathrm{i}\lambda_k(z_\alpha^* - \hat{z}_0)\right]\right\rangle X_0\mathbf{B}_2^{-1}\mathbf{\Psi}\mathbf{I}_k\mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{G} \\ & - \left\langle \exp\left[\mathrm{i}\lambda_0(z_\alpha^* - \hat{z}_0)\right] E_1\left[\mathrm{i}\lambda_0(z_\alpha^* - \hat{z}_0)\right]\right\rangle X_0\mathbf{B}_2^{-1}(\mathbf{\Lambda}^{-1}\mathbf{G} + \mathbf{N}^{-1}\mathbf{G}_1), \end{aligned} \tag{55}$$

where once again the superscript '*' is utilized to distinguish the Stroh eigenvalues associated with the lower half-plane (z_α^*) from those associated with the upper half-plane (z_α).

Now that $\mathbf{f}_1'(z)$ and $\mathbf{f}_2'(z)$ in Eqs. (54) and (55) together with $g_1'(z_0)$ and $g_2'(z_0^*)$ in Eqs. (28) and (29) can be utilized to uniquely determine the thermal stress field. It is observed that the first term in Eq. (50) is the logarithmic singular term in $\mathbf{f}_1'(z)$ induced by the heat source and thermal dislocation. Consequently the stresses and strains will exhibit logarithmic singular behavior at the location $\left[\hat{x}_1, \hat{x}_2\right]$ ($\hat{x}_2 > 0$) of the heat source and the thermal dislocation.

4 Special and degenerate cases

In this section two special cases for the interface and the degenerate case in which the interface thermal contact resistance parameter for the thermal field is equal to one of the three interface rigidities for the elastic field will be discussed to illustrate and also to verify the obtained solutions.

4.1 A thermally imperfect and mechanically perfect interface

When the interface is mechanically perfect, $\mathbf{\Lambda} = \mathbf{0}$. In this case Eq. (36) reduces to

$$\begin{aligned}\mathbf{B}_2\mathbf{f}_2'(z) = {} & 2\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{f}_0'(z) - \mathbf{N}^{-1}\left[\mathrm{i}(\Gamma\bar{\mathbf{c}}_1 + \mathbf{c}_2) + \bar{\mathbf{M}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1 + \mathbf{d}_2)\right] g_2'(z) \\ & + \mathbf{N}^{-1}\left[\mathrm{i}(\mathbf{c}_1 + \bar{\mathbf{c}}_1) + \bar{\mathbf{M}}_1^{-1}(\mathbf{d}_1 + \bar{\mathbf{d}}_1)\right] g_0'(z), \end{aligned} \tag{56}$$

and consequently it follows from Eq. (32) that

$$\begin{aligned}\mathbf{B}_1\mathbf{f}_1'(z) = {} & \mathbf{B}_1\mathbf{f}_0'(z) + \bar{\mathbf{N}}^{-1}(\bar{\mathbf{M}}_1^{-1} - \bar{\mathbf{M}}_2^{-1})\bar{\mathbf{B}}_1\bar{\mathbf{f}}_0'(z) \\ & + \bar{\mathbf{N}}^{-1}\left[\mathrm{i}(\Gamma\mathbf{c}_1 + \bar{\mathbf{c}}_2) + \bar{\mathbf{M}}_2^{-1}(\Gamma\mathbf{d}_1 + \bar{\mathbf{d}}_2)\right]\bar{g}_2'(z) - \bar{\mathbf{N}}^{-1}\left[\mathrm{i}(\mathbf{c}_1 + \bar{\mathbf{c}}_1) + \bar{\mathbf{M}}_2^{-1}(\mathbf{d}_1 + \bar{\mathbf{d}}_1)\right]\bar{g}_0'(z). \end{aligned} \tag{57}$$

Furthermore when the interface is also thermally perfect (i.e., the interface is perfect), we have $\alpha_0 = 0$ (or equivalently $\lambda_0 \to \infty$). Then it follows from Eq. (29) that

$$g_2'(z) = \frac{2}{1+\Gamma}g_0'(z). \tag{58}$$

As a result Eqs. (56) and (57) reduce to

$$\mathbf{B}_2\mathbf{f}_2'(z) = 2\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{f}_0'(z) - \mathbf{N}^{-1}\mathbf{G}_1 g_0'(z) \tag{59}$$

and

$$\mathbf{B}_1\mathbf{f}_1'(z) = \mathbf{B}_1\mathbf{f}_0'(z) + \bar{\mathbf{N}}^{-1}(\bar{\mathbf{M}}_1^{-1} - \bar{\mathbf{M}}_2^{-1})\bar{\mathbf{B}}_1\bar{\mathbf{f}}_0'(z) - \bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_2\bar{g}_0'(z). \tag{60}$$

It is observed that Eqs. (59) and (60) are in agreement with those derived by Kattis et al. [9] (notice that in the last term in Eq. (91) of [9], $\mathbf{G}^R$, which corresponds to $\mathbf{G}_2$ in our definition, should read $\bar{\mathbf{G}}^R$). In fact Eqs. (56) and (57) can also be directly obtained from Eqs. (51) and (52) by letting $\lambda_1, \lambda_2, \lambda_3 \to \infty$ for a mechanically perfect interface, and Eqs. (59) and (60) can also be directly obtained from Eqs. (51) and (52) by letting $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \to \infty$ for a thermally and mechanically perfect interface.

4.2 A thermally perfect and mechanically imperfect interface

When the interface is thermally perfect, we have $\alpha_0 = 0$ (or equivalently $\boldsymbol{\lambda}_0 \to \infty$). In this case Eqs. (28) and (29) reduce to

$$g_1'(z_0) = g_0'(z_0) + \frac{1-\Gamma}{1+\Gamma}\bar{g}_0'(z_0), \tag{61}$$

$$g_2'(z_0^*) = \frac{2}{1+\Gamma} g_0'(z_0^*), \tag{62}$$

which are just those derived in [9].

As a result Eq. (36) will reduce to

$$\mathbf{N}\mathbf{B}_2\mathbf{f}_2(z) + \mathrm{i}\boldsymbol{\Lambda}\mathbf{B}_2\mathbf{f}_2'(z) = 2\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{f}_0(z) - \frac{2\mathrm{i}\boldsymbol{\Lambda}\mathbf{d}_2}{1+\Gamma} g_0'(z) - \mathbf{G}_1 g_0(z), \quad \mathrm{Im}\{z\} < 0, \tag{63}$$

or equivalently and more explicitly

$$\mathbf{N}\mathbf{B}_2\mathbf{f}_2'(z) + \mathrm{i}\boldsymbol{\Lambda}\mathbf{B}_2\mathbf{f}_2''(z) = 2\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{e} + 2\mathbf{L}_1^{-1}\mathbf{B}_1\left\langle \ln(z-\hat{z}_\alpha)\right\rangle\mathbf{e} - \ln(z-\hat{z}_0)X_0\mathbf{G}_1 - \frac{2\mathrm{i}X_0\boldsymbol{\Lambda}\mathbf{d}_2}{1+\Gamma}\frac{1}{z-\hat{z}_0},$$
$$\mathrm{Im}\{z\} < 0, \tag{64}$$

whose solution can be similarly given by

$$\begin{aligned}\mathbf{B}_2\mathbf{f}_2'(z) = {} & 2\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\left\langle \ln(z-\hat{z}_\alpha)+1\right\rangle\mathbf{e} - X_0\ln(z-\hat{z}_0)\mathbf{N}^{-1}\mathbf{G}_1 \\ & + 2\Psi\sum_{k=1}^{3}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right]E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right]\right\rangle\boldsymbol{\Lambda}_2^{-1}\bar{\boldsymbol{\Psi}}^T\boldsymbol{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{I}_k\mathbf{e} \\ & + X_0\boldsymbol{\Psi}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right]\right\rangle\boldsymbol{\Lambda}_2^{-1}\bar{\boldsymbol{\Psi}}^T\boldsymbol{\Lambda}\left(\frac{2\mathbf{d}_2}{1+\Gamma} - \mathbf{N}^{-1}\mathbf{G}_1\right),\end{aligned} \tag{65}$$

and consequently we obtain

$$\begin{aligned}\mathbf{B}_1\mathbf{f}_1'(z) = {} & \mathbf{B}_1\left\langle \ln(z-\hat{z}_\alpha)+1\right\rangle\mathbf{e} + \bar{\mathbf{N}}^{-1}(\bar{\mathbf{M}}_1^{-1} - \bar{\mathbf{M}}_2^{-1})\bar{\mathbf{B}}_1\left\langle \ln(z-\bar{\hat{z}}_\alpha)+1\right\rangle\bar{\mathbf{e}} - \ln(z-\bar{\hat{z}}_0)\bar{X}_0\bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_2 \\ & + 2\bar{\boldsymbol{\Psi}}\sum_{k=1}^{3}\left\langle \exp\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_k)\right]E_1\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_k)\right]\right\rangle\boldsymbol{\Lambda}_2^{-1}\boldsymbol{\Psi}^T\boldsymbol{\Lambda}\bar{\mathbf{N}}^{-1}\mathbf{L}_1^{-1}\bar{\mathbf{B}}_1\mathbf{I}_k\bar{\mathbf{e}} \\ & + \bar{X}_0\bar{\boldsymbol{\Psi}}\left\langle \exp\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_0)\right]E_1\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_0)\right]\right\rangle\boldsymbol{\Lambda}_2^{-1}\boldsymbol{\Psi}^T\boldsymbol{\Lambda}\left(\frac{2\bar{\mathbf{d}}_2}{1+\Gamma} - \bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_1\right).\end{aligned} \tag{66}$$

Equations (65) and (66) can also be directly obtained from Eqs. (51) and (52) by letting $\lambda_0 \to \infty$ for a thermally perfect interface. It is also observed from the above discussions in Sects. 4.1 and 4.2 that our solution procedure is consistent in itself and that the derived solutions Eqs. (51) and (52) can be reduced to known ones [9].

4.3 The degenerate case: $\lambda_0 = \lambda_1 \neq \lambda_2 \neq \lambda_3$

Equations (51) and (52) can be equivalently written into the following forms:

$$\begin{aligned}\mathbf{f}_2'(z) = {} & 2\mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\langle \ln(z-\hat{z}_\alpha)+1\rangle\mathbf{e} - \ln(z-\hat{z}_0)X_0\mathbf{B}_2^{-1}\mathbf{N}^{-1}\mathbf{G}_1 \\ & + 2\mathbf{B}_2^{-1}\boldsymbol{\Psi}\sum_{k=1}^{3}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right]E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_k)\right]\right\rangle\boldsymbol{\Lambda}_2^{-1}\bar{\boldsymbol{\Psi}}^T\boldsymbol{\Lambda}\mathbf{N}^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{I}_k\mathbf{e} \\ & + \frac{2X_0}{1+\Gamma}\mathbf{B}_2^{-1}\boldsymbol{\Psi}\langle Y_\alpha(z)\rangle\boldsymbol{\Lambda}_2^{-1}\bar{\boldsymbol{\Psi}}^T\left[\mathrm{i}(\Gamma\bar{\mathbf{c}}_1 + \mathbf{c}_2) + \bar{\mathbf{M}}_1^{-1}(\Gamma\bar{\mathbf{d}}_1 + \mathbf{d}_2) - \lambda_0\boldsymbol{\Lambda}\mathbf{d}_2\right] \\ & - X_0\mathbf{B}_2^{-1}\boldsymbol{\Psi}\left\langle \exp\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_\alpha(z-\hat{z}_0)\right]\right\rangle\boldsymbol{\Lambda}_2^{-1}\bar{\boldsymbol{\Psi}}^T\boldsymbol{\Lambda}\mathbf{N}^{-1}\mathbf{G}_1,\end{aligned} \tag{67}$$

$$
\begin{aligned}
\mathbf{f}_1'(z) = &\left\langle \ln(z-\hat{z}_\alpha)+1\right\rangle \mathbf{e} + \mathbf{B}_1^{-1}\bar{\mathbf{N}}^{-1}(\bar{\mathbf{M}}_1^{-1}-\bar{\mathbf{M}}_2^{-1})\bar{\mathbf{B}}_1\left\langle \ln(z-\bar{\hat{z}}_\alpha)+1\right\rangle \bar{\mathbf{e}} - \ln(z-\bar{\hat{z}}_0)\bar{X}_0\mathbf{B}_1^{-1}\bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_2 \\
&+ 2\mathbf{B}_1^{-1}\bar{\mathbf{\Psi}}\sum_{k=1}^{3}\left\langle \exp\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_k)\right]E_1\left[-\mathrm{i}\boldsymbol{\lambda}_\alpha(z-\bar{\hat{z}}_k)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\mathbf{\Lambda}\bar{\mathbf{N}}^{-1}\mathbf{L}_1^{-1}\bar{\mathbf{B}}_1\mathbf{I}_k\bar{\mathbf{e}} \\
&+ \frac{2\bar{X}_0}{1+\Gamma}\mathbf{B}_1^{-1}\bar{\mathbf{\Psi}}\left\langle \bar{Y}_\alpha(z)\right\rangle \mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\left[-\mathrm{i}(\Gamma\mathbf{c}_1+\bar{\mathbf{c}}_2)+\mathbf{M}_1^{-1}(\Gamma\mathbf{d}_1+\bar{\mathbf{d}}_2)-\lambda_0\mathbf{\Lambda}\bar{\mathbf{d}}_2\right] \\
&- \bar{X}_0\mathbf{B}_1^{-1}\bar{\mathbf{\Psi}}\left\langle \exp\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_0)\right]E_1\left[-\mathrm{i}\lambda_\alpha(z-\bar{\hat{z}}_0)\right]\right\rangle \mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\mathbf{\Lambda}\bar{\mathbf{N}}^{-1}\bar{\mathbf{G}}_1 \\
&+ \frac{2\bar{X}_0}{1+\Gamma}\exp\left[-\mathrm{i}\lambda_0(z-\bar{\hat{z}}_0)\right]E_1\left[-\mathrm{i}\lambda_0(z-\bar{\hat{z}}_0)\right]\mathbf{B}_1^{-1}(\Gamma\mathbf{d}_1+\bar{\mathbf{d}}_2),
\end{aligned}
\tag{68}
$$

where the analytic functions $Y_i(z)$, (i=1–3) are defined as

$$
Y_1(z) = \frac{\exp\left[\mathrm{i}\lambda_1(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_1(z-\hat{z}_0)\right]-\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]}{\lambda_1-\lambda_0}, \tag{69.1}
$$

$$
Y_2(z) = \frac{\exp\left[\mathrm{i}\lambda_2(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_2(z-\hat{z}_0)\right]-\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]}{\lambda_2-\lambda_0}, \tag{69.2}
$$

$$
Y_3(z) = \frac{\exp\left[\mathrm{i}\lambda_3(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_3(z-\hat{z}_0)\right]-\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]}{\lambda_3-\lambda_0}. \tag{69.3}
$$

When $\lambda_0 = \lambda_1 \neq \lambda_2 \neq \lambda_3$, the right hand side of Eq. (69.1) will become a 0/0 type. Applying the L'Hôpital's rule to the right hand side of Eq. (69.1) when $\lambda_1 \to \lambda_0$ yields

$$
Y_1(z) = \frac{\mathrm{i}\lambda_0(z-\hat{z}_0)\exp\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]E_1\left[\mathrm{i}\lambda_0(z-\hat{z}_0)\right]-1}{\lambda_0}, \tag{70}
$$

which approaches zero as $z \to \infty$. The other degenerate cases $\lambda_0 = \lambda_2 \neq \lambda_1 \neq \lambda_3$ and $\lambda_0 = \lambda_3 \neq \lambda_1 \neq \lambda_2$ can be similarly discussed.

5 Conclusions

We have derived the two-dimensional thermal Green's functions for a steady line heat source and a temperature dislocation located in the upper half-plane of an anisotropic bimaterial with a weakly conducting and compliant planar imperfect interface. The full-field solutions of $\mathbf{f}_1'(z)$ and $\mathbf{f}_2'(z)$ in Eqs. (54) and (55) are expressed in terms of the exponential integral $E_1(z)$. The solutions for the non-degenerate case in which $\lambda_0 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3$ and for the degenerate case in which $\lambda_0 = \lambda_1 \neq \lambda_2 \neq \lambda_3$ were all derived. Two special cases for the interface; (1) a thermally imperfect and mechanically perfect interface, and (2) a thermally perfect and mechanically imperfect interface, were also discussed to illustrate and also to verify the obtained solutions. The corresponding thermal Green's functions for an anisotropic bimaterial with a highly conducting and compliant (or even stiff) interface can be derived similarly. In Appendix B we present the isothermal Green's functions in anisotropic bimaterials with an elastically stiff interface to demonstrate the basic ingredients in the analyses of a stiff interface.

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Appendix A: Proof of the properties of the eigenvalue problem Eq. (42)

We consider the eigenvalue problem

$$
\mathbf{N}\mathbf{v} = \lambda\mathbf{\Lambda}\mathbf{v}. \tag{A.1}
$$

Premultiplication of Eq. (A.1) by $\bar{\mathbf{v}}^T$ yields

$$
\bar{\mathbf{v}}^T\mathbf{N}\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{\Lambda}\mathbf{v}, \tag{A.2}
$$

or equivalently

$$\frac{1}{\lambda} = \frac{\bar{\mathbf{v}}^T \mathbf{\Lambda} \mathbf{v}}{\bar{\mathbf{v}}^T \mathbf{N} \mathbf{v}}, \tag{A.3}$$

which indicates that the eigenvalue λ must be real and non-negative because $\bar{\mathbf{v}}^T \mathbf{N} \mathbf{v} > 0$ ($\mathbf{N}$ is positive definite) and $\bar{\mathbf{v}}^T \mathbf{\Lambda} \mathbf{v} \geq 0$ ($\mathbf{\Lambda}$ is positive semidefinite).

Let λ_1 and λ_2 be the two distinct eigenvalues of Eq. (A.1) and $\mathbf{v}_1$ and $\mathbf{v}_2$ be the associated eigenvectors of Eq. (A.1), we then have

$$\mathbf{N}\mathbf{v}_1 = \lambda_1 \mathbf{\Lambda} \mathbf{v}_1. \tag{A.4}$$

Pre-multiplying Eq. (A.4) by $\bar{\mathbf{v}}_2^T$, we obtain

$$\bar{\mathbf{v}}_2^T \mathbf{N} \mathbf{v}_1 = \lambda_1 \bar{\mathbf{v}}_2^T \mathbf{\Lambda} \mathbf{v}_1. \tag{A.5}$$

Meanwhile the following relation is also valid

$$\mathbf{N}\mathbf{v}_2 = \lambda_2 \mathbf{\Lambda} \mathbf{v}_2. \tag{A.6}$$

Taking the conjugate transpose of Eq. (A.6), and then post-multiplying both sides by $\mathbf{v}_1$, we finally obtain

$$\bar{\mathbf{v}}_2^T \mathbf{N} \mathbf{v}_1 = \lambda_2 \bar{\mathbf{v}}_2^T \mathbf{\Lambda} \mathbf{v}_1. \tag{A.7}$$

Subtracting Eq. (A.5) from Eq. (A.7) we have

$$(\lambda_1 - \lambda_2) \bar{\mathbf{v}}_2^T \mathbf{\Lambda} \mathbf{v}_1 = 0. \tag{A.8}$$

Due to the fact that $\lambda_1 \neq \lambda_2$, then the following orthogonal relation with respect to the real and symmetric matrix $\mathbf{\Lambda}$ holds:

$$\bar{\mathbf{v}}_2^T \mathbf{\Lambda} \mathbf{v}_1 = 0. \tag{A.9}$$

In view of Eqs. (A.5) and (A.9), the following orthogonal relation with respect to the Hermitian matrix $\mathbf{N}$ is also established:

$$\bar{\mathbf{v}}_2^T \mathbf{N} \mathbf{v}_1 = 0. \tag{A.10}$$

Appendix B: Isothermal Green's functions in an anisotropic bimaterial with a stiff interface

In order to illustrate the discussion of the stiff interface more clearly, we will ignore the thermal effect in the following discussions. More specifically we consider a line force $\hat{\mathbf{f}}$ and a line dislocation of Burgers vector $\hat{\mathbf{b}}$ located at $\left[\hat{x}_1, \hat{x}_2\right]$, ($\hat{x}_2 > 0$) in the upper half-plane of the anisotropic bimaterial. We will first develop the boundary conditions on the stiff interface.

The constitutive equations for an interphase of constant thickness h between the upper semi-infinite anisotropic solid 1 and the lower semi-infinite anisotropic solid 2 can be equivalently written into

$$\boldsymbol{\sigma}_1 = \mathbf{Q}_0 \mathbf{u}_{,1} + \mathbf{R}_0 \mathbf{u}_{,2}, \tag{B.1.1}$$

$$\boldsymbol{\sigma}_2 = \mathbf{R}_0^T \mathbf{u}_{,1} + \mathbf{T}_0 \mathbf{u}_{,2}, \tag{B.1.2}$$

where $\boldsymbol{\sigma}_1 = \left[\sigma_{11}\ \sigma_{21}\ \sigma_{31}\right]^T$, $\boldsymbol{\sigma}_2 = \left[\sigma_{12}\ \sigma_{22}\ \sigma_{32}\right]^T$, and the subscript 0 is used to identify the quantities associated with the interphase.

Taking the derivative of the two sides of Eq. (B.1.1) with respect to x_1 and making use of the equilibrium equations $\sigma_{1,1} + \sigma_{2,2} = \mathbf{0}$, we obtain

$$\boldsymbol{\sigma}_{2,2} = -\mathbf{Q}_0 \mathbf{u}_{,11} - \mathbf{R}_0 \mathbf{u}_{,12}. \tag{B.2}$$

If we assume that the elastic constants $(C_{ijkl})_0$ of the interphase are of the same order of magnitude and that $(C_{ijkl})_0 >> (C_{ijkl})_1, (C_{ijkl})_2$ (the stiff interphase) [1], then we can obtain from Eq. (B.1.2) the following connection

$$\mathbf{u}_{,2} = -\mathbf{T}_0^{-1}\mathbf{R}_0^T\mathbf{u}_{,1}. \tag{B.3}$$

Substitution of the above into Eq. (B.2) yields

$$\boldsymbol{\sigma}_{2,2} = -(\mathbf{Q}_0 - \mathbf{R}_0\mathbf{T}_0^{-1}\mathbf{R}_0^T)\mathbf{u}_{,11}, \tag{B.4}$$

When the interphase is also very thin, then the above can be approximated by

$$(\boldsymbol{\sigma}_2)_1 - (\boldsymbol{\sigma}_2)_2 = -\mathbf{E}\mathbf{u}_{,11}, \quad x_2 = 0, \tag{B.5}$$

where $\mathbf{E}$ is a positive semi-definite symmetric matrix given by

$$\mathbf{E} = h(\mathbf{Q}_0 - \mathbf{R}_0\mathbf{T}_0^{-1}\mathbf{R}_0^T). \tag{B.6}$$

It is easily checked that Eq. (B.5) is in agreement with Eq. (6.5) in [1]. According to Ting [8], we can further write $\mathbf{E}$ into the form

$$\mathbf{E} = \begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & 0 & 0 \\ E_{13} & 0 & E_{33} \end{bmatrix}, \tag{B.7}$$

where $E_{11} > 0$, $E_{33} > 0$, $E_{11}E_{33} - E_{13}^2 > 0$.

The boundary conditions in Eq. (B.5) can also be written in terms of $\mathbf{u}$ and $\boldsymbol{\Phi}$ as

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2 = -\mathbf{E}\mathbf{u}_{2,1}, \quad x_2 = 0, \tag{B.8}$$

or in terms of the two analytic function vectors $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ as

$$\mathbf{A}_1\mathbf{f}_1^+(x_1) + \bar{\mathbf{A}}_1\bar{\mathbf{f}}_1^-(x_1) = \mathbf{A}_2\mathbf{f}_2^-(x_1) + \bar{\mathbf{A}}_2\bar{\mathbf{f}}_2^+(x_1), \quad x_2 = 0, \tag{B.9}$$

$$\mathbf{B}_1\mathbf{f}_1^+(x_1) + \bar{\mathbf{B}}_1\bar{\mathbf{f}}_1^-(x_1) - \mathbf{B}_2\mathbf{f}_2^-(x_1) - \bar{\mathbf{B}}_2\bar{\mathbf{f}}_2^+(x_1) = -\mathbf{E}\left[\mathbf{A}_2\mathbf{f}_2'^-(x_1) + \bar{\mathbf{A}}_2\bar{\mathbf{f}}_2'^+(x_1)\right], \quad x_2 = 0. \tag{B.10}$$

It follows from Eq. (B.9) that

$$\mathbf{f}_1(z) = \mathbf{A}_1^{-1}\bar{\mathbf{A}}_2\bar{\mathbf{f}}_2(z) + \mathbf{f}_0(z) - \mathbf{A}_1^{-1}\bar{\mathbf{A}}_1\bar{\mathbf{f}}_0(z), \tag{B.11.1}$$

$$\bar{\mathbf{f}}_1(z) = \bar{\mathbf{A}}_1^{-1}\mathbf{A}_2\mathbf{f}_2(z) - \bar{\mathbf{A}}_1^{-1}\mathbf{A}_1\mathbf{f}_0(z) + \bar{\mathbf{f}}_0(z), \tag{B.11.2}$$

where $\mathbf{f}_0(z)$ is the analytic function vector for a line force $\hat{\mathbf{f}}$ and a line dislocation of Burgers vector $\hat{\mathbf{b}}$ in a homogeneous plane occupied by material 1:

$$\mathbf{f}_0(z) = \frac{1}{2\pi \mathrm{i}}\left\langle \ln(z - \hat{z}_\alpha)\right\rangle (\mathbf{A}_1^T\hat{\mathbf{f}} + \mathbf{B}_1^T\hat{\mathbf{b}}). \tag{B.12}$$

Substituting Eq. (B.11) into Eq. (B.10) and applying the Liouville's theorem, we can finally arrive at the following set of differential equations

$$\boldsymbol{\Pi}\mathbf{A}_2\mathbf{f}_2(z) + \mathrm{i}\mathbf{E}\mathbf{A}_2\mathbf{f}_2'(z) = 2\mathbf{H}_1^{-1}\mathbf{A}_1\mathbf{f}_0(z), \quad \mathrm{Im}\{z\} < 0, \tag{B.13}$$

where Π is a positive definite Hermitian matrix defined by [8]

$$\boldsymbol{\Pi} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \bar{\Pi}_{12} & \Pi_{22} & \Pi_{23} \\ \bar{\Pi}_{13} & \bar{\Pi}_{23} & \Pi_{33} \end{bmatrix} = \bar{\mathbf{M}}_1 + \mathbf{M}_2, \tag{B.14}$$

$$\mathbf{M}_k = -\mathrm{i}\mathbf{B}_k\mathbf{A}_k^{-1} = \mathbf{H}_k^{-1}(\mathbf{I} + \mathrm{i}\mathbf{S}_k), \quad (k = 1, 2). \tag{B.15}$$

In order to solve Eq. (B.13), we consider the eigenvalue problem

$$(\mathbf{E} - \lambda\boldsymbol{\Pi})\mathbf{v} = \mathbf{0}. \tag{B.16}$$

The three real eigenvalues λ_i ($i = 1$–3) to Eq. (B.16) can be explicitly determined as

$$\begin{aligned}
\lambda_1 &= \frac{a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_2} > 0, \\
\lambda_2 &= \frac{a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2} > 0, \\
\lambda_3 &= 0,
\end{aligned} \tag{B.17}$$

where

$$a_2 = |\mathbf{\Pi}|, \quad a_1 = E_{11}\widehat{\Pi}_{11} + E_{33}\widehat{\Pi}_{33} + 2E_{13}\mathrm{Re}\left\{\widehat{\Pi}_{13}\right\}, \quad a_0 = \Pi_{22}\widehat{E}_{22}, \tag{B.18}$$

with $\widehat{\Pi}_{ij}$ and $\widehat{E}_{ij}$ denoting the cofactors of the matrices $\mathbf{\Pi}$ and $\mathbf{E}$, respectively. In addition the following orthogonal relationship can be easily proved:

$$\bar{\mathbf{\Psi}}^T\mathbf{E}\mathbf{\Psi} = \mathbf{\Lambda}_1\mathbf{\Lambda}_2, \quad \bar{\mathbf{\Psi}}^T\mathbf{\Pi}\mathbf{\Psi} = \mathbf{\Lambda}_2, \tag{B.19}$$

where $\mathbf{\Lambda}_2$ is a 3×3 positive real diagonal matrix and

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}, \tag{B.20}$$

$$\mathbf{\Lambda}_1 = \mathrm{diag}\begin{bmatrix} \lambda_1 & \lambda_2 & 0 \end{bmatrix}. \tag{B.21}$$

Next we introduce an analytic function vector $\mathbf{\Omega}(z)$ such that

$$\mathbf{A}_2\mathbf{f}_2(z) = \mathbf{\Psi}\mathbf{\Omega}(z). \tag{B.22}$$

Employing the orthogonal relationship in Eq. (B.19), then Eq. (B.13) can be decoupled into

$$\mathbf{\Omega}(z) + \mathrm{i}\mathbf{\Lambda}_1\mathbf{\Omega}'(z) = 2\mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{H}_1^{-1}\mathbf{A}_1\mathbf{f}_0(z), \quad \mathrm{Im}\{z\} < 0, \tag{B.23}$$

whose solution can be easily obtained as

$$\mathbf{\Omega}'(z) = \frac{1}{\pi}\sum_{k=1}^{3}\left\langle F_\alpha(z, \hat{z}_k)\right\rangle\mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{H}_1^{-1}\mathbf{A}_1\mathbf{I}_k(\mathbf{A}_1^T\hat{\mathbf{f}} + \mathbf{B}_1^T\hat{\mathbf{b}}), \tag{B.24}$$

where

$$\begin{aligned}
F_1(z, \hat{z}_k) &= \lambda_1^{-1}\exp\left[\mathrm{i}\lambda_1^{-1}(z - \hat{z}_k)\right]E_1\left[\mathrm{i}\lambda_1^{-1}(z - \hat{z}_k)\right], \\
F_2(z, \hat{z}_k) &= \lambda_2^{-1}\exp\left[\mathrm{i}\lambda_2^{-1}(z - \hat{z}_k)\right]E_1\left[\mathrm{i}\lambda_2^{-1}(z - \hat{z}_k)\right], \\
F_3(z, \hat{z}_k) &= \frac{1}{\mathrm{i}(z - \hat{z}_k)}.
\end{aligned} \tag{B.25}$$

Consequently it follows from Eqs. (B.22), (B.24) and (B.11.1) that

$$\mathbf{f}_2'(z) = \frac{1}{\pi}\mathbf{A}_2^{-1}\mathbf{\Psi}\sum_{k=1}^{3}\left\langle F_\alpha(z, \hat{z}_k)\right\rangle\mathbf{\Lambda}_2^{-1}\bar{\mathbf{\Psi}}^T\mathbf{H}_1^{-1}\mathbf{A}_1\mathbf{I}_k(\mathbf{A}_1^T\hat{\mathbf{f}} + \mathbf{B}_1^T\hat{\mathbf{b}}), \tag{B.26}$$

$$\begin{aligned}
\mathbf{f}_1'(z) = {} & \frac{1}{\pi}\mathbf{A}_1^{-1}\bar{\mathbf{\Psi}}\sum_{k=1}^{3}\left\langle \bar{F}_\alpha(z, \hat{z}_k)\right\rangle\mathbf{\Lambda}_2^{-1}\mathbf{\Psi}^T\mathbf{H}_1^{-1}\bar{\mathbf{A}}_1\mathbf{I}_k(\bar{\mathbf{A}}_1^T\hat{\mathbf{f}} + \bar{\mathbf{B}}_1^T\hat{\mathbf{b}}) \\
& + \frac{1}{2\pi\mathrm{i}}\left\langle\frac{1}{z - \hat{z}_\alpha}\right\rangle(\mathbf{A}_1^T\hat{\mathbf{f}} + \mathbf{B}_1^T\hat{\mathbf{b}}) + \frac{1}{2\pi\mathrm{i}}\mathbf{A}_1^{-1}\bar{\mathbf{A}}_1\left\langle\frac{1}{z - \bar{\hat{z}}_\alpha}\right\rangle(\bar{\mathbf{A}}_1^T\hat{\mathbf{f}} + \bar{\mathbf{B}}_1^T\hat{\mathbf{b}}).
\end{aligned} \tag{B.27}$$

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