## Provided for non-commercial research and education use. Not for reproduction, distribution or commercial use.



This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:
http://www.elsevier.com/copyright

# Two-dimensional Eshelby's problem for two imperfectly bonded piezoelectric half-planes 

Xu Wang ${ }^{\text {a,* }}$, Ernian Pan ${ }^{\text {b }}$<br>${ }^{\text {a }}$ Center for Composite Materials, University of Delaware, Newark, DE 19716, USA<br>${ }^{\mathrm{b}}$ Department of Civil Engineering, University of Akron, Akron, OH 44325-3905, USA

## A R T I C L E I N F O

## Article history:

Received 13 July 2009
Received in revised form 9 September 2009
Available online 23 September 2009

## Keywords:

Stroh formalism
Eshelby inclusion
Piezoelectric materials
Imperfect interface
Reduced generalized compliance


#### Abstract

General solutions are derived to the two-dimensional Eshelby's problem of an inclusion of arbitrary shape embedded in one of two imperfectly bonded anisotropic piezoelectric half-planes. The inclusion undergoes uniform eigenstrains and eigenelectric fields. In this work four different kinds of imperfect interface models with vanishing thickness are considered: (i) a compliant and weakly conducting interface, (ii) a stiff and highly conducting interface, (iii) a compliant and highly conducting interface, and (iv) a stiff and weakly conducting interface. Furthermore the obtained general solutions are illustrated in detail through an example of an elliptical inclusion near the imperfect interface. It is observed that the full-field expressions of the three analytic function vectors characterizing the electroelastic field in the two piezoelectric half-planes including the elliptical inclusion can be elegantly and concisely presented through the introduction of an integral function. We also present the tractions and normal electric displacement along a compliant and weakly conducting imperfect interface induced by the elliptical inclusion. It is found that the imperfection of the interface has no influence on the leading term in the far-field asymptotic expansion of the tractions and normal electric displacement along the compliant and weakly conducting interface induced by an arbitrary shaped inclusion. The far-field expansions of the analytic function vectors in the two imperfectly bonded half-planes for an arbitrary shaped inclusion are also derived. Some new identities and structures of the matrices $\mathbf{N}_{i}$ and $\mathbf{N}_{i}^{(-1)}$ for anisotropic piezoelectric materials are obtained.


© 2009 Elsevier Ltd. All rights reserved.

## 1. Introduction

The Eshelby's problem of an inclusion with eigenstrains (or transformation strains) has been a topic in micromechanics for more than fifty years (Eshelby, 1957; Mura, 1987). When addressing the three-dimensional Eshelby's problem, the Green's function approach is prevalent (Eshelby, 1957; Mura, 1987; Nozaki and Taya, 2001). However when discussing two-dimensional (2D) Eshelby's problem in isotropic or anisotropic solids, the complex variable method is more effective (see for example Jaswon and Bhargava, 1961; Bhargava and Radhakrishna, 1964; Willis, 1964; Yang and Chou, 1976, 1977; Ru, 2000, 2001; Pan, 2004; Jiang and Pan, 2004; Wang et al., 2007). It has been found in recent years that studies on Eshelby's problem are essential in understanding the behaviors of quantum dots and quantum wires in nanocomposite solids (see recent reviews by Ovid'ko and Sheinerman, 2005 and Malanganti and Sharma, 2005).

When addressing the inclusion problems in a two-phase infinite medium (say with a flat interface), it is found that the perfect inter-

[^0]face assumption was adopted in the majority of the previous studies (see for example, Zhang and Chou, 1985. Yu and Sanday, 1991; Jiang and Pan, 2004). In a recent study, Wang et al. (2007) considered aD thermal inclusion of arbitrary shape embedded in one of two imperfectly bonded isotropic elastic half-planes by using Muskhelishvili's complex variable method (Muskhelishvili, 1963). The imperfect interface in that study was simulated by using the linear spring layer with vanishing thickness. However, the corresponding Eshelby's problem for two imperfectly bonded dissimilar anisotropic piezoelectric half-planes still remains a challenging problem.

It is of interest to point out also that so far various interface models have been proposed to simulate an interphase layer with finite thickness (Needleman, 1990; Benveniste and Miloh, 2001; Benveniste and Baum, 2007; Bertoldi et al., 2007a,b; Benveniste, 2006, 2009), to account for damage (for example, micro-cracks and mi-cro-voids) occurring on the interface (Fan and Sze, 2001), and to study their influence on the effective properties of the composites (Lu and Lin, 2003; Wang and Pan, 2007) and on the interfacial wave propagation (Melkumyan and Mai, 2008). Nondistructive evaluation methods were also proposed to detect and characterize the interface imperfection (Nagy, 1992; Hu and Nagy, 1998). It was reported that the effect of interfacial stress, defects, impurities,
and electrodes on the variation of polarization in ferroelectric thin films could be significant (Lu and Cao, 2002). However, as expected that if the piezoelectricity of an interphase layer is taken into consideration (Benveniste, 2009), the scenarios of the imperfect interface will become more complex in view of the fact that now the interface has imperfection in both elasticity and dielectricity.

In this work we consider the 2D problem of an Eshelby inclusion of arbitrary shape with uniform eigenstrains and eigenelectric fields embedded in one of two bonded anisotropic piezoelectric half-planes by means of the Stroh formalism (Suo et al., 1992; Suo, 1993; Wang, 1994; Chung and Ting, 1996; Ru, 2000, 2001). In extending previous works (Ru, 2001; Pan, 2004; Jiang and Pan, 2004; Wang et al., 2008), the two anisotropic piezoelectric halfplanes are now bonded through a thin anisotropic piezoelectric layer. It is found that closed-form solutions can be derived when the middle piezoelectric layer is replaced by an imperfect interface with vanishing thickness. The imperfect interface models discussed in this work can be classified into the following four different kinds:
(i) Compliant and weakly conducting interface. This imperfect interface is based on the assumption that tractions and normal electric displacement are continuous across the interface, whereas the elastic displacements and electric potential undergo jumps on the interface which are proportional to the interface tractions and normal electric displacement.
(ii) Stiff and highly conducting interface. This imperfect interface is based on the assumption that displacements and electric potential are continuous across the interface, whereas tractions and normal electric displacement undergo jumps on the interface which are proportional to certain surface differential operators of the interface displacements and electric potential.
(iii) Compliant and highly conducting interface. This imperfect interface is based on the assumption that tractions and tangential electric field are continuous across the interface, whereas the elastic displacements and charge potential undergo jumps on the interface which are proportional to the interface tractions and tangential electric field.
(iv) Stiff and weakly conducting interface. This imperfect interface is based on the assumption that displacements and charge potential are continuous across the interface, whereas tractions and tangential electric field undergo jumps on the interface which are proportional to certain surface differential operators of the interface displacements and charge potential.

Our theoretical development demonstrates that the parameters in all the four kinds of imperfect interface models can be explicitly expressed in terms of the electroelastic moduli and the thickness of the piezoelectric layer.

## 2. The Stroh formalism for anisotropic piezoelectric materials

In the following we will present two different schemes of the Stroh formalism. Scheme 1 of the Stroh formalism will be adopted in the analyses of a compliant and weakly conducting interface (Section 3), and a stiff and highly conducting interface (Section 4). Scheme 2 will be adopted in the analyses of a compliant and highly conducting interface (Section 5), and a stiff and weakly conducting interface (Section 6).

### 2.1. Scheme 1 of the Stroh formalism

The basic equations for an anisotropic piezoelectric material can be expressed in a fixed rectangular coordinate system $x_{i}(i=1,2,3)$ as

$$
\begin{align*}
& \sigma_{i j}=C_{i j k l} u_{k, l}+e_{k i j} \phi_{k}, \quad D_{k}=e_{k j} u_{i, j}-\epsilon_{k l} \phi_{, l},  \tag{1}\\
& \sigma_{i j, j}=0, \quad D_{i, i}=0,
\end{align*}
$$

where repeated indices mean summation, a comma follows by $i(i=1,2,3)$ stands for the derivative with respect to the $i$ th spatial coordinate; $u_{i}$ and $\phi$ are the elastic displacement and electric potential; $\sigma_{i j}$ and $D_{i}$ are the stress and electric displacement; $C_{i j k l}, \epsilon_{i j}$ and $e_{i j k}$ are the elastic, dielectric and piezoelectric coefficients, respectively.

For 2D problems in which all quantities depend only on $x_{1}$ and $x_{2}$, the general solutions can be expressed as (Suo et al., 1992; Wang, 1994; Ting, 1996)

$$
\begin{align*}
& \mathbf{u}=\left[\begin{array}{llll}
u_{1} & u_{2} & u_{3} & \phi
\end{array}\right]^{T}=\mathbf{A f}(z)+\overline{\mathbf{A f}(z)},  \tag{2}\\
& \boldsymbol{\Phi}=\left[\begin{array}{llll}
\Phi_{1} & \Phi_{2} & \Phi_{3} & \varphi
\end{array}\right]^{T}=\mathbf{B f}(z)+\overline{\mathbf{B}(z)},
\end{align*}
$$

where

$$
\begin{align*}
& \mathbf{A}=\left[\begin{array}{llll}
\mathbf{a}_{1} & \mathbf{a}_{2} & \mathbf{a}_{3} & \mathbf{a}_{4}
\end{array}\right], \quad \mathbf{B}=\left[\begin{array}{llll}
\mathbf{b}_{1} & \mathbf{b}_{2} & \mathbf{b}_{3} & \mathbf{b}_{4}
\end{array}\right], \\
& \mathbf{f}(z)=\left[\begin{array}{llll}
f_{1}\left(z_{1}\right) & f_{2}\left(z_{2}\right) & f_{3}\left(z_{3}\right) & f_{4}\left(z_{4}\right)
\end{array}\right]^{T},  \tag{3}\\
& z_{i}=x_{1}+p_{i} x_{2}, \\
& \operatorname{Im}\left\{p_{i}\right\}>0, \\
& (i=1-4),
\end{align*}
$$

with
$\left[\begin{array}{ll}\mathbf{N}_{1} & \mathbf{N}_{2} \\ \mathbf{N}_{3} & \mathbf{N}_{1}^{T}\end{array}\right]\left[\begin{array}{l}\mathbf{a}_{i} \\ \mathbf{b}_{i}\end{array}\right]=p_{i}\left[\begin{array}{l}\mathbf{a}_{i} \\ \mathbf{b}_{i}\end{array}\right], \quad(i=1-4)$
$\mathbf{N}_{1}=-\mathbf{T}^{-1} \mathbf{R}^{T}, \quad \mathbf{N}_{2}=\mathbf{T}^{-1}, \quad \mathbf{N}_{3}=\mathbf{R T}^{-1} \mathbf{R}^{T}-\mathbf{Q}$,
and
$\mathbf{Q}=\left[\begin{array}{ll}\mathbf{Q}^{E} & \mathbf{e}_{11} \\ \mathbf{e}_{11}^{T} & -\epsilon_{11}\end{array}\right], \quad \mathbf{R}=\left[\begin{array}{ll}\mathbf{R}^{E} & \mathbf{e}_{21} \\ \mathbf{e}_{12}^{T} & -\epsilon_{12}\end{array}\right], \quad \mathbf{T}=\left[\begin{array}{ll}\mathbf{T}^{E} & \mathbf{e}_{22} \\ \mathbf{e}_{22}^{T} & -\epsilon_{22}\end{array}\right]$,
$\left(\mathbf{Q}^{E}\right)_{i k}=C_{i 1 k 1}, \quad\left(\mathbf{R}^{E}\right)_{i k}=C_{i 1 k 2}, \quad\left(\mathbf{T}^{E}\right)_{i k}=C_{i 2 k 2}, \quad\left(\mathbf{e}_{i j}\right)_{m}=e_{i j m}$.
In addition the extended stress function vector $\boldsymbol{\Phi}$ is defined, in terms of the stresses and electric displacements, as follows:

$$
\begin{array}{lll}
\sigma_{i 1}=-\Phi_{i, 2}, & \sigma_{i 2}=\Phi_{i, 1}, \quad(i=1-3)  \tag{8}\\
D_{1}=-\varphi_{, 2}, & D_{2}=\varphi_{, 1} .
\end{array}
$$

Here we can call $\varphi$ a charge potential (Suo, 1993). Due to the fact that the two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following normalized orthogonal relationship:
$\left[\begin{array}{ll}\mathbf{B}^{T} & \mathbf{A}^{T} \\ \overline{\mathbf{B}}^{T} & \overline{\mathbf{A}}^{T}\end{array}\right]\left[\begin{array}{cc}\mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}}\end{array}\right]=\mathbf{I}$,
then three real Barnett-Lothe tensors $\mathbf{S}, \mathbf{H}$ and $\mathbf{L}$ can be introduced
$\mathbf{S}=\mathrm{i}\left(2 \mathbf{A B}^{T}-\mathbf{I}\right), \quad \mathbf{H}=2 \mathrm{i} \mathbf{A A}^{T}, \quad \mathbf{L}=-2 \mathbf{i} \mathbf{B B}^{T}$.
During this investigation, the following identities will also be utilized:

$$
\begin{align*}
& 2 \mathbf{A}\left\langle p_{\alpha}\right\rangle \mathbf{A}^{T}=\mathbf{N}_{2}-\mathrm{i}\left(\mathbf{N}_{1} \mathbf{H}+\mathbf{N}_{2} \mathbf{S}^{T}\right), \\
& 2 \mathbf{A}\left\langle p_{\alpha}\right\rangle \mathbf{B}^{T}=\mathbf{N}_{1}+\mathrm{i}\left(\mathbf{N}_{2} \mathbf{L}-\mathbf{N}_{1} \mathbf{S}\right),  \tag{11}\\
& 2 \mathbf{Z}\left\langle p_{\alpha}\right\rangle \mathbf{B}^{T}=\mathbf{N}_{3}+\mathrm{i}\left(\mathbf{N}_{1}^{T} \mathbf{L}-\mathbf{N}_{3} \mathbf{S}\right),
\end{align*}
$$

where $\langle *\rangle$ is a $4 \times 4$ diagonal matrix in which each component is varied according to the Greek index $\alpha$ (from 1 to 4).

It can also be easily checked that
$\left[\begin{array}{ll}\mathbf{N}_{1}^{(-1)} & \mathbf{N}_{2}^{(-1)} \\ \mathbf{N}_{3}^{(-1)} & \mathbf{N}_{1}^{(-1)^{T}}\end{array}\right]\left[\begin{array}{l}\mathbf{a}_{i} \\ \mathbf{b}_{i}\end{array}\right]=\frac{1}{p_{i}}\left[\begin{array}{l}\mathbf{a}_{i} \\ \mathbf{b}_{i}\end{array}\right], \quad(i=1-4)$
where
$\mathbf{N}_{1}^{(-1)}=-\mathbf{Q}^{-1} \mathbf{R}, \quad \mathbf{N}_{2}^{(-1)}=-\mathbf{Q}^{-1}, \quad \mathbf{N}_{3}^{(-1)}=\mathbf{T}-\mathbf{R}^{T} \mathbf{Q}^{-1} \mathbf{R}$.
The detailed structures and identities of $\mathbf{N}_{i}$ and $\mathbf{N}_{i}^{(-1)}(i=1,2,3)$ for Scheme 1 can be found in Appendix A.

### 2.2. Scheme 2 of the Stroh formalism

In this scheme, the constitutive equations can be written into (Suo, 1993)
$\sigma_{i j}=C_{i j k l} u_{k, l}+h_{k i j} D_{k}, \quad E_{k}=h_{k i j} u_{i, j}+\beta_{k l} D_{l}$,
where $E_{i}$ is the electric field; $C_{i j k l}, \beta_{i j}$ and $h_{i j k}$ are the elastic, dielectric and piezoelectric coefficients.

For 2D problems in which all quantities depend only on $x_{1}$ and $x_{2}$, the general solutions can be expressed as (Suo, 1993; Wang, 1994)

$$
\begin{align*}
& \mathbf{u}=\left[\begin{array}{llll}
u_{1} & u_{2} & u_{3} & \varphi
\end{array}\right]^{T}=\mathbf{A f}(z)+\overline{\mathbf{A f}(z)}  \tag{15}\\
& \mathbf{\Phi}=\left[\begin{array}{llll}
\Phi_{1} & \Phi_{2} & \Phi_{3} & \phi
\end{array}\right]^{T}=\mathbf{B}(z)+\overline{\mathbf{B}(z)}
\end{align*}
$$

where the functions $\Phi_{i}(i=1-3)$ and $\varphi$ have been defined in Eq. (8), and

$$
\begin{align*}
& \mathbf{A}=\left[\begin{array}{llll}
\mathbf{a}_{1} & \mathbf{a}_{2} & \mathbf{a}_{3} & \mathbf{a}_{4}
\end{array}\right], \quad \mathbf{B}=\left[\begin{array}{llll}
\mathbf{b}_{1} & \mathbf{b}_{2} & \mathbf{b}_{3} & \mathbf{b}_{4}
\end{array}\right], \\
& \mathbf{f}(z)=\left[\begin{array}{lll}
f_{1}\left(z_{1}\right) & f_{2}\left(z_{2}\right) & f_{3}\left(z_{3}\right) \\
f_{4}\left(z_{4}\right)
\end{array}\right]^{T},  \tag{16}\\
& z_{i}=x_{1}+p_{i} x_{2}, \quad \operatorname{Im}\left\{p_{i}\right\}>0, \\
& (i=1-4)
\end{align*}
$$

with
$\left[\begin{array}{ll}\mathbf{N}_{1} & \mathbf{N}_{2} \\ \mathbf{N}_{3} & \mathbf{N}_{1}^{T}\end{array}\right]\left[\begin{array}{l}\mathbf{a}_{i} \\ \mathbf{b}_{i}\end{array}\right]=p_{i}\left[\begin{array}{l}\mathbf{a}_{i} \\ \mathbf{b}_{i}\end{array}\right], \quad(i=1-4)$
$\mathbf{N}_{1}=-\mathbf{T}^{-1} \mathbf{R}^{T}, \quad \mathbf{N}_{2}=\mathbf{T}^{-1}, \quad \mathbf{N}_{3}=\mathbf{R T}^{-1} \mathbf{R}^{T}-\mathbf{Q}$,
and
$\mathbf{Q}=\left[\begin{array}{ll}\mathbf{Q}^{E} & \mathbf{h}_{21} \\ \mathbf{h}_{21}^{T} & \beta_{22}\end{array}\right], \quad \mathbf{R}=\left[\begin{array}{cc}\mathbf{R}^{E} & -\mathbf{h}_{11} \\ \mathbf{h}_{22}^{T} & -\beta_{12}\end{array}\right], \quad \mathbf{T}=\left[\begin{array}{cc}\mathbf{T}^{E} & -\mathbf{h}_{12} \\ -\mathbf{h}_{12}^{T} & \beta_{11}\end{array}\right]$,
$\left(\mathbf{Q}^{E}\right)_{i k}=C_{i 1 k 1}, \quad\left(\mathbf{R}^{E}\right)_{i k}=C_{i 1 k 2}, \quad\left(\mathbf{T}^{E}\right)_{i k}=C_{i 2 k 2}, \quad\left(\mathbf{h}_{i j}\right)_{m}=h_{i j m}$.
The identities in Eqs. (9)-(13) are also valid in this scheme. It is stressed that in this scheme, both the two $4 \times 4$ symmetric matrices $\mathbf{Q}$ and $\mathbf{T}$ are positive definite. The structures and identities of $\mathbf{N}_{i}$ and $\mathbf{N}_{i}^{(-1)}(i=1,2,3)$ for Scheme 2 can be found in Appendix B. We add that the formulations for Scheme 2 presented here are somewhat different than those presented by Suo (1993) in view of the fact that $\varphi=-\xi$ with $\xi$ defined in Eq. (B1) by Suo (1993).

## 3. The Eshelby's problem for two bonded piezoelectric halfplanes with a compliant and weakly conducting interface

### 3.1. The general solution

Now we consider two dissimilar anisotropic piezoelectric halfplanes imperfectly bonded along the real axis $x_{2}=0$, as shown in Fig. 1. Here we assume that the upper half-plane contains a subdomain of arbitrary shape which has the same elastic, piezoelectric and dielectric constants as the upper half-plane and undergoes uniform eigenstrains $\left(\varepsilon_{11}^{*}, \varepsilon_{12}^{*}, \varepsilon_{22}^{*}, \varepsilon_{31}^{*}, \varepsilon_{32}^{*}\right)$ and eigenelectric fields $\left(E_{1}^{*}, E_{2}^{*}\right)$. Let $S_{0}$ and $S_{1}$ denote the subdomain and its supplement to the upper half-plane, $\Gamma$ the perfect interface separating $S_{0}$ and $S_{1}, S_{2}$ the lower half-plane. In this research all quantities in $S_{0}, S_{1}$ and $S_{2}$ will be attached with the subscripts 0,1 and 2 or the superscripts (0), (1) and (2). For example the three analytic functions $\mathbf{f}_{0}(z), \mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ are defined respectively in $S_{0}, S_{1}$ and $S_{2}$. In the analysis carried out in this section, we will adopt Scheme 1 of the Stroh formalism.

The interface conditions along the perfect interface $\Gamma$ can be expressed as (Ru, 2001)
$\mathbf{u}_{1}=\mathbf{u}_{0}+\mathbf{u}^{*}, \quad \boldsymbol{\Phi}_{1}=\boldsymbol{\Phi}_{0} \quad$ on $\Gamma$,
where $\mathbf{u}^{*}$ is the additional displacements and electric potential within the Eshelby's inclusion $S_{0}$ due to uniform eigenstrains and eigenelectric fields
$\mathbf{u}^{*}=\left(\begin{array}{c}\varepsilon_{11}^{*} x_{1}+\varepsilon_{12}^{*} x_{2} \\ \varepsilon_{12}^{*} x_{1}+\varepsilon_{22}^{*} x_{2} \\ 2\left(\varepsilon_{31}^{*} x_{1}+\varepsilon_{32}^{*} x_{2}\right) \\ -\left(E_{1}^{*} x_{1}+E_{2}^{*} x_{2}\right)\end{array}\right) \quad$ within $S_{0}$
In view of Eq. (21), we introduce the following auxiliary function vector $\mathbf{g}(z)$ :
$\mathbf{g}(z)=\left\{\begin{array}{c}\mathbf{f}_{0}(z)+\left\langle z_{\alpha}\right\rangle \mathbf{c}+<P_{\alpha}\left(z_{\alpha}\right)>\mathbf{d} \\ \mathbf{f}_{1}(z)-\left\langle D_{\alpha}\left(z_{\alpha}\right)-P_{\alpha}\left(z_{\alpha}\right)\right\rangle \mathbf{d}\end{array}\right.$,
where $\bar{z}_{\alpha}=D_{\alpha}\left(z_{\alpha}\right)$ along the interface $\Gamma$ (Ru, 2001). In addition $D_{\alpha}\left(z_{\alpha}\right)=P_{\alpha}\left(z_{\alpha}\right)+o(1)$ as $\left|z_{\alpha}\right| \rightarrow \infty$. The two complex vectors $\mathbf{c}$ and $\mathbf{d}$ appearing in Eq. (23) are related to the uniform eigenstrains and eigenelectric fields as
$\mathbf{c}=\left\langle\frac{\bar{p}_{\alpha}}{\bar{p}_{\alpha}-p_{\alpha}}\right\rangle \mathbf{B}_{1}^{T}\left[\begin{array}{c}\varepsilon_{11}^{*} \\ \varepsilon_{12}^{*} \\ 2 \varepsilon_{31}^{*} \\ -E_{1}^{*}\end{array}\right]-\left\langle\frac{1}{\bar{p}_{\alpha}-p_{\alpha}}\right\rangle \mathbf{B}_{1}^{T}\left[\begin{array}{c}\varepsilon_{12}^{*} \\ \varepsilon_{22}^{*} \\ 2 \varepsilon_{32}^{*} \\ -E_{2}^{*}\end{array}\right]$,
$\mathbf{d}=\left\langle\frac{1}{\bar{p}_{\alpha}-p_{\alpha}}\right\rangle \mathbf{B}_{1}^{T}\left[\begin{array}{c}\varepsilon_{12}^{*} \\ \varepsilon_{22}^{*} \\ 2 \varepsilon_{32}^{*} \\ -E_{2}^{*}\end{array}\right]-\left\langle\frac{p_{\alpha}}{p_{\alpha}-p_{\alpha}}\right\rangle \mathbf{B}_{1}^{T}\left[\begin{array}{c}\varepsilon_{11}^{*} \\ \varepsilon_{12}^{*} \\ 2 \varepsilon_{31}^{*} \\ -E_{1}^{*}\end{array}\right]$,
where the Stroh eigenvalues $p_{k}(k=1-4)$ are those pertaining to the upper half-plane within which the Eshelby inclusion is embedded. Our analysis (suppressed here) demonstrates that $\mathbf{g}(z)$ is analytic, continuous and single-valued everywhere in the whole upper half-plane $S_{0}+S_{1}$ including the point at infinity. We observe from Eq. (23) that $\mathbf{f}_{1}(z) \equiv \mathbf{g}(z)+\mathbf{h}(z)$ where $\mathbf{h}(z)=\left\langle D_{\alpha}\left(z_{\alpha}\right)-P_{\alpha}\left(z_{\alpha}\right)\right\rangle \mathbf{d}$ is the singular part while $\mathbf{g}(z)$ is the regular part of $\mathbf{f}_{1}(z)$ if we extend the definition region of $\mathbf{f}_{1}(z)$ to the whole upper half-plane including the domain $S_{0}$.

In addition the boundary conditions on the compliant and weakly conducting imperfect interface $x_{2}=0$ separating the two piezoelectric half-planes can be expressed as
$\sigma_{12}^{(1)}=\sigma_{12}^{(2)}, \quad \sigma_{22}^{(1)}=\sigma_{22}^{(2)}, \quad \sigma_{32}^{(1)}=\sigma_{32}^{(2)}, \quad D_{2}^{(1)}=D_{2}^{(2)}$,
$\left[\begin{array}{l}u_{1}^{(1)}-u_{1}^{(2)} \\ u_{2}^{(1)}-u_{2}^{(2)} \\ u_{3}^{(1)}-u_{3}^{(2)} \\ \phi^{(1)}-\phi^{(2)}\end{array}\right]=\Lambda\left[\begin{array}{c}\sigma_{12}^{(2)} \\ \sigma_{22}^{(2)} \\ \sigma_{32}^{(2)} \\ D_{2}^{(2)}\end{array}\right], \quad x_{2}=0$,
where the $4 \times 4$ real and symmetric matrix $\boldsymbol{\Lambda}$ is explicitly given by
$\boldsymbol{\Lambda}=\boldsymbol{\Lambda}^{T}=\left[\begin{array}{llll}\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{34} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44}\end{array}\right]$,
where $\left[\begin{array}{lll}\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33}\end{array}\right]$ is positive definite whereas $\alpha_{44}<0$. Eq. (25) states that tractions and normal electric displacement are continuous across the interface, whereas the elastic displacements and electric potential undergo jumps on the interface which are proportional to the interface tractions and normal electric displacement. A detailed derivation of the above imperfect interface model in Eqs. (25) and (26) can be found in Appendix C.

The above imperfect boundary conditions in Eq. (25) can also be conveniently expressed in terms of $\mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ as


Fig. 1. Eshelby's problem for two imperfectly bonded anisotropic piezoelectric half-planes with an inclusion of arbitrary shape.

$$
\begin{align*}
& \mathbf{B}_{1} \mathbf{f}_{1}^{+}\left(x_{1}\right)+\overline{\mathbf{B}}_{1} \overline{\mathbf{f}}_{1}^{-}\left(x_{1}\right)=\mathbf{B}_{2} \mathbf{f}_{2}^{-}\left(x_{1}\right)+\overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}\right), \\
& \mathbf{A}_{1} \mathbf{f}_{1}^{+}\left(x_{1}\right)+\overline{\mathbf{A}}_{\mathbf{1}} \mathbf{f}_{1}^{( }\left(x_{1}\right)-\mathbf{A}_{2} \mathbf{f}_{2}^{-}\left(x_{1}\right)-\overline{\mathbf{A}}_{2} \mathbf{f}_{2}^{+}\left(x_{1}\right) \quad x_{2}=0  \tag{27}\\
& \quad=\boldsymbol{\Lambda}\left[\mathbf{B}_{2} \mathbf{f}_{2}^{\prime}\left(x_{1}\right)+\overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}^{\prime+}\left(x_{1}\right)\right],
\end{align*}
$$

It follows from Eq. (27) that
$\mathbf{f}_{1}(z)=\mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}(z)+\mathbf{h}(z)-\mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1} \overline{\mathbf{h}}(z)$,
$\overline{\mathbf{f}}_{1}(z)=\overline{\mathbf{B}}_{1}^{-1} \mathbf{B}_{2} \mathbf{f}_{2}(z)-\overline{\mathbf{B}}_{1}^{-1} \mathbf{B}_{1} \mathbf{h}(z)+\overline{\mathbf{h}}(z)$.
In writing Eq. (28), we have implicitly replaced the complex variables $z_{k},(k=1-4)$ by the common complex variable $z=x_{1}+\mathrm{i} x_{2}$ in view of the fact that $z_{1}=z_{2}=z_{3}=z_{4}=z$ on the interface $x_{2}=0$. After the analysis is finished, we will change $z$ back to the corresponding complex variables. Substituting Eq. (28) into Eq. (27) ${ }_{2}$, we obtain

$$
\begin{align*}
& \overline{\mathbf{N B}}_{2} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}\right)-\mathrm{i} \Lambda \overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}^{\prime+}\left(x_{1}\right)-2 \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \overline{\mathbf{h}}\left(x_{1}\right) \\
& \quad=\mathbf{N B}_{2} \mathbf{f}_{2}^{-}\left(x_{1}\right)+\mathrm{i} \boldsymbol{\Lambda} \mathbf{B}_{2} \mathbf{f}_{2}^{\prime-}\left(x_{1}\right)-2 \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{h}\left(x_{1}\right), \quad x_{2}=0, \tag{29}
\end{align*}
$$

where $\mathbf{M}_{k}^{-1},(k=1,2)$ and $\mathbf{N}$ are $4 \times 4$ Hermitian matrices given by (Suo et al., 1992; Wang, 1994)

$$
\begin{align*}
& \mathbf{N}=\overline{\mathbf{M}}_{1}^{-1}+\mathbf{M}_{2}^{-1}=\mathbf{L}_{1}^{-1}+\mathbf{L}_{2}^{-1}+\mathrm{i}\left(\mathbf{S}_{1} \mathbf{L}_{1}^{-1}-\mathbf{S}_{2} \mathbf{L}_{2}^{-1}\right) \\
& \mathbf{M}_{k}^{-1}=\mathrm{i} \mathbf{A}_{k} \mathbf{B}_{k}^{-1}=\left(\mathbf{I}-\mathrm{i} \mathbf{S}_{k}\right) \mathbf{L}_{k}^{-1}, \quad(k=1,2) \tag{30}
\end{align*}
$$

We add that $\mathbf{M}_{k}^{-1},(k=1,2)$ and $\mathbf{N}$ are not positive definite (Lothe and Barnett, 1975; Suo et al., 1992). It is apparent that the left hand side of Eq. (29) is analytic in the upper half-plane, while the right hand side of Eq. (29) is analytic in the lower half-plane. Consequently the continuity condition in Eq. (29) implies that the left and right sides of Eq. (29) are identically zero in the upper and lower half-planes, respectively. It follows that:
$\mathbf{N B}_{2} \mathbf{f}_{2}(z)+\mathbf{i} \Lambda \mathbf{B}_{2} \mathbf{f}_{2}^{\prime}(z)=2 \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{h}(z), \quad \operatorname{Im}\{z\}<0$

In order to solve the coupled set of first-order differential equations in Eq. (31), we first consider the following eigenvalue problem:
$(\mathbf{N}-\lambda \boldsymbol{\Lambda}) \mathbf{v}=\mathbf{0}$.
There exist four eigenvalues to the above eigenvalue problem (the four eigenvalues are not necessarily real in view of the fact that both $\mathbf{N}$ and $\mathbf{\Lambda}$ are not positive definite). If $\lambda$ is an eigenvalue, then its conjugate $\bar{\lambda}$ is also an eigenvalue. In addition $\operatorname{Re}\{\lambda\}>0$. Let that $\lambda_{i},(i=1-4)$ be the four distinct roots and $\mathbf{v}_{i}$ the associated eigenvectors, then the following orthogonal relationship can be easily proved:
$\mathbf{J} \overline{\mathbf{\Psi}}^{\boldsymbol{T}} \mathbf{N} \boldsymbol{\Psi}=\boldsymbol{\Lambda}_{1} \boldsymbol{\Lambda}_{2}, \quad \mathbf{J} \overline{\boldsymbol{\Psi}}^{\boldsymbol{T}} \boldsymbol{\Lambda} \boldsymbol{\Psi}=\boldsymbol{\Lambda}_{2}$,
where $\Lambda_{2}$ is a $4 \times 4$ diagonal matrix, and
$\boldsymbol{\Psi}=\left[\begin{array}{llll}\mathbf{v}_{1} & \mathbf{v}_{2} & \mathbf{v}_{3} & \mathbf{v}_{4}\end{array}\right]$,
$\boldsymbol{\Lambda}_{1}=\operatorname{diag}\left[\begin{array}{llll}\lambda_{1} & \lambda_{2} & \lambda_{3} & \lambda_{4}\end{array}\right]$.
In addition the $4 \times 4$ real and symmetric matrix $J$ appearing in Eq. (33) is dependent on the nature of the four eigenvalues $\lambda_{i}, \quad(i=1-4)$. A detailed classification is given below.
(i) Four real eigenvalues (i.e., $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}>0$ ):
$\mathbf{J}=\operatorname{diag}\left[\begin{array}{llll}1 & 1 & 1 & 1\end{array}\right]$,
(ii) Two real and two complex conjugate eigenvalues (i.e., $\left.\lambda_{1}, \lambda_{2}>0, \lambda_{3}=\bar{\lambda}_{4}\right):$
$\mathbf{J}=\left[\begin{array}{llll}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0\end{array}\right]$,
(iii) Four complex eigenvalues (i.e., $\lambda_{1}=\bar{\lambda}_{2}, \lambda_{3}=\bar{\lambda}_{4}$ ):
$\mathbf{J}=\left[\begin{array}{llll}0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0\end{array}\right]$.
Next we introduce an analytic function vector $\boldsymbol{\Omega}(z)$ such that
$\mathbf{B}_{2} \mathbf{f}_{2}(z)=\boldsymbol{\Psi} \boldsymbol{\Omega}(z)$.
Employing the orthogonal relationship in Eq. (33), then Eq. (31) can be decoupled into
$-\mathrm{i} \boldsymbol{\Lambda}_{1} \boldsymbol{\Omega}(z)+\boldsymbol{\Omega}^{\prime}(z)=-2 \mathrm{i} \boldsymbol{\Lambda}_{2}^{-1} \mathbf{J} \bar{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1}\left\langle D_{\alpha}(z)-P_{\alpha}(z)\right\rangle \mathbf{d}, \quad \operatorname{Im}\{z\}<0$

The general solution to the above set of decoupled differential equations can be conveniently expressed as (Yoon et al., 2006; Wang et al., 2007)
$\boldsymbol{\Omega}(z)=-2 \mathrm{i} \sum_{k=1}^{4}\left\langle\exp \left(\mathrm{i} \lambda_{\alpha} z\right)\right\rangle \int_{-\infty \mathrm{i}}^{z}\left[D_{k}(\xi)-P_{k}(\xi)\right]$

$$
\begin{equation*}
\times\left\langle\exp \left(-\mathrm{i} \lambda_{\alpha} \xi\right)\right\rangle d \xi \boldsymbol{\Lambda}_{2}^{-1} \mathbf{J} \bar{\Psi}^{\mathrm{T}} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k} \mathbf{d}, \quad \operatorname{Im}\{z\}<0 \tag{41}
\end{equation*}
$$

where
$\mathbf{I}_{1}=\operatorname{diag}\left[\begin{array}{llll}1 & 0 & 0 & 0\end{array}\right], \quad \mathbf{I}_{2}=\operatorname{diag}\left[\begin{array}{llll}0 & 1 & 0 & 0\end{array}\right]$,
$\mathbf{I}_{3}=\operatorname{diag}\left[\begin{array}{llll}0 & 0 & 1 & 0\end{array}\right]$,
$\mathbf{I}_{4}=\operatorname{diag}\left[\begin{array}{llll}0 & 0 & 0 & 1\end{array}\right]$.
Once $\boldsymbol{\Omega}(z)$ has been obtained, it is easy to arrived at $\mathbf{f}_{0}(z), \mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ by using Eqs. (23), (28) and (39). Before ending this subsection, it is of interest to look into in more detail the four eigenvalues $\lambda_{i},(i=1-4)$ determined by Eq. (32) through a specific case. Here we assume that the two piezoelectric half-planes and the middle piezoelectric interphase layer are orthotropic (Pan, 2001). In addition the two half-planes have the same material property except that the poling direction of the upper half-plane is in the positive $x_{2}$-direction while that of the lower one is in the negative $x_{2}$-direction, and the interphase is poled in the $x_{1}$ direction. Consequently the complex Hermitian matrix $\mathbf{N}$ (Suo et al., 1992; Ru, 1999) and the real and symmetric matrix $\boldsymbol{\Lambda}$ can be explicitly given by
$\mathbf{N}=2\left[\begin{array}{cccc}\frac{1}{C_{L}} & 0 & 0 & -\mathrm{i} \beta \\ 0 & \frac{1}{C_{T}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{C_{44} C_{55}}} & 0 \\ i \beta & 0 & 0 & -\frac{1}{\epsilon}\end{array}\right], \quad\left(C_{L}, C_{T} \in>0, \beta<0\right)$
and
$\boldsymbol{\Lambda}=\left[\begin{array}{cccc}\rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ \rho_{14} & 0 & 0 & -\rho_{44}\end{array}\right]$,
where

$$
\begin{align*}
& \rho_{11}=\frac{h t c_{22}^{(c)}}{C_{66}^{(c)} \epsilon_{22}^{(c)}+e_{62}^{(c) 2}}>0, \quad \rho_{22}=\frac{h}{C_{22}^{(c)}}>0, \quad \rho_{33}=\frac{h}{C_{44}^{(c)}}>0,  \tag{45}\\
& \rho_{44}=\frac{h c_{66}^{(c)}}{C_{66}^{(c)} \epsilon_{22}^{(c)}+e_{26}^{(c)}}>0, \quad \rho_{14}=\frac{h_{26}^{(c)}}{C_{66}^{(c)}\left(c_{22}^{(c)}+e_{26}^{(c)}\right.} .
\end{align*}
$$

Now that the four eigenvalues to Eq. (32) can be given by
$\lambda_{1}=\frac{2}{\rho_{22} C_{T}}>0, \quad \lambda_{2}=\frac{2}{\rho_{33} \sqrt{C_{44} C_{55}}}>0$,
and

$$
\begin{align*}
& \lambda_{3,4}=\frac{\frac{\rho_{11}}{\epsilon}+\frac{\rho_{44}}{C_{L}} \pm \sqrt{\left(\frac{\rho_{11}}{\epsilon}+\frac{\rho_{44}}{C_{L}}\right)^{2}-4\left(\frac{1}{\epsilon C_{L}}+\beta^{2}\right)\left(\rho_{11} \rho_{44}+\rho_{14}^{2}\right)}}{\rho_{11} \rho_{44}+\rho_{14}^{2}}>0, \\
& \text { when } \left\lvert\, \frac{1}{\epsilon} \sqrt{\frac{\rho_{11}}{\rho_{44}}-\frac{1}{C_{L}} \sqrt{\frac{\rho_{44}}{\rho_{14}}} \left\lvert\, \geqslant 2 \sqrt{\beta^{2}+\frac{\rho_{14}^{2}}{\rho_{11} \rho_{44}}\left(\frac{1}{\epsilon C_{L}}+\beta^{2}\right)}\right. ; \text { or }}\right.  \tag{47}\\
& \lambda_{3,4}=\frac{\frac{\rho_{11}}{\epsilon}+\frac{\rho_{44}}{C_{L}} \pm i \sqrt{4\left(\frac{1}{\epsilon C_{L}}+\beta^{2}\right)\left(\rho_{11} \rho_{44}+\rho_{14}^{2}\right)-\left(\frac{\rho_{11}}{\epsilon}+\frac{\rho_{44}}{C_{L}}\right)^{2}}}{\rho_{11} \rho_{44}+\rho_{14}^{2}}, \\
& \operatorname{Re}\left\{\lambda_{3}\right\}=\operatorname{Re}\left\{\lambda_{4}\right\}>0 \tag{48}
\end{align*}
$$

when $\left|\frac{1}{\epsilon} \sqrt{\frac{\rho_{11}}{\rho_{44}}}-\frac{1}{C_{L}} \sqrt{\frac{\rho_{44}}{\rho_{11}}}\right|<2 \sqrt{\beta^{2}+\frac{\rho_{14}^{2}}{\rho_{11} \rho_{44}}\left(\frac{1}{\epsilon C_{L}}+\beta^{2}\right)}$. Among the above four eigenvalues, $\lambda_{2}$ belongs to the decoupled anti-plane deformation.

### 3.2. An example of elliptical inclusion

In the following we illustrate the obtained general solution through an example of an elliptical inclusion with semi-major and semi-minor axes $a$ and $b$. We further assume that the major axis is parallel to the $x_{1}$-axis and the center of the ellipse is located at $x_{1}=0$ and $x_{2}=\delta(\delta>b)$. In this case $D_{k}(z), P_{k}(z)$ and $D_{k}(z)-P_{k}(z)$ can be explicitly determined as!
$D_{k}(z)=\frac{a^{2}+\left|p_{k}\right|^{2} b^{2}}{a^{2}+p_{k}^{2} b^{2}}\left(z-p_{k} \delta\right)+\bar{p}_{k} \delta+\frac{\mathrm{i}\left(p_{k}-\bar{p}_{k}\right) a b}{a^{2}+p_{k}^{2} b^{2}}$

$$
\begin{equation*}
\times \sqrt{\left(z-p_{k} \delta\right)^{2}-\left(a^{2}+p_{k}^{2} b^{2}\right)} \tag{49}
\end{equation*}
$$

$P_{k}(z)=\frac{a-\mathrm{i} \bar{p}_{k} b}{a-\mathrm{i} p_{k} b} z-\frac{a\left(p_{k}-\bar{p}_{k}\right)}{a-\mathrm{i} p_{k} b} \delta$,
$D_{k}(z)-P_{k}(z)=\frac{\mathrm{i}\left(\bar{p}_{k}-p_{k}\right) a b}{z-p_{k} \delta+\sqrt{\left(z-p_{k} \delta\right)^{2}-\left(a^{2}+p_{k}^{2} b^{2}\right)}}$.
Consequently the analytic function vector $\boldsymbol{\Omega}(z)$ can be explicitly determined as

$$
\begin{array}{r}
\boldsymbol{\Omega}(z)=a b \sum_{k=1}^{4}\left(p_{k}-\bar{p}_{k}\right)\left\langle\operatorname { e x p } [ \mathrm { i } \lambda _ { \alpha } ( z - p _ { k } \delta ) ] Y \left[\mathbf{i} \lambda_{\alpha}\left(z-p_{k} \delta\right),\right.\right. \\
 \tag{52}\\
\left.\left.\lambda_{\alpha} \sqrt{a^{2}+p_{k}^{2} b^{2}}\right]\right\rangle \boldsymbol{\Lambda}_{2}^{-1} \mathbf{J}^{T} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k} \mathbf{d}, \quad \operatorname{Im}\{z\}<0,
\end{array}
$$

where $Y(z, \beta)$ is an introduced integral function defined by
$Y(z, \beta)=\int_{z}^{\infty} \frac{2 \exp (-\xi)}{\xi+\sqrt{\xi^{2}+\beta^{2}}} \mathrm{~d} \xi$.
Apparently $Y(z, 0)=E_{1}(z)$, the exponential integral (Abramovitz and Stegun, 1972). In view of Eqs. (23), (28), (39) and (52), the three analytic function vectors $\mathbf{f}_{0}(z)$ within the inclusion, $\mathbf{f}_{1}(z)$ in the upper half-plane but outside the inclusion and $\mathbf{f}_{2}(z)$ in the lower half-plane can now be explicitly given by

$$
\begin{align*}
& \mathbf{f}_{0}(z)= a b \\
& \sum_{k=1}^{4}\left(\bar{p}_{k}-p_{k}\right) \mathbf{B}_{1}^{-1} \bar{\Psi}\left\langle\exp \left[-\mathrm{i} \bar{\lambda}_{\alpha}\left(z-\bar{p}_{k} \delta\right)\right] Y\right. \\
&\left.\times\left[-\mathrm{i} \overline{\mathrm{\lambda}}_{\alpha}\left(z-\bar{p}_{k} \delta\right), \bar{\lambda}_{\alpha} \sqrt{a^{2}+\bar{p}_{k}^{2} b^{2}}\right]\right\rangle \overline{\boldsymbol{\Lambda}}_{2}^{-1} \mathbf{J}^{\top} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}} \\
&-\mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1}\left\langle\frac{\operatorname{iab}\left(\bar{p}_{\alpha}-p_{\alpha}\right)}{z-\bar{p}_{\alpha} \delta+\sqrt{\left(z-\bar{p}_{\alpha} \delta\right)^{2}-\left(a^{2}+\bar{p}_{\alpha}^{2} b^{2}\right)}}\right\rangle \overline{\mathbf{d}}  \tag{54}\\
&-z\left[\mathbf{c}+\left\langle\frac{a-\mathrm{i} \overline{\mathrm{p}}_{\alpha} b}{a-\mathrm{i} p_{\alpha} b}\right\rangle \mathbf{d}\right], \quad z \in S_{0}
\end{align*}
$$

$$
\begin{align*}
\mathbf{f}_{1}(z)= & a b \sum_{k=1}^{4}\left(\bar{p}_{k}-p_{k}\right) \mathbf{B}_{1}^{-1} \overline{\boldsymbol{\Psi}}\left\langle\exp \left[-\mathrm{i} \bar{\lambda}_{\alpha}\left(z-\bar{p}_{k} \delta\right)\right] Y\right. \\
& \left.\times\left[-\mathrm{i} \bar{\lambda}_{\alpha}\left(z-\bar{p}_{k} \delta\right), \bar{\lambda}_{\alpha} \sqrt{a^{2}+\bar{p}_{k}^{2} b^{2}}\right]\right\rangle \overline{\boldsymbol{\Lambda}}_{2}^{-1} \mathbf{J}^{T} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}} \\
& -\mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1}\left\langle\frac{\mathrm{i} a b\left(\bar{p}_{\alpha}-p_{\alpha}\right)}{z-\bar{p}_{\alpha} \delta+\sqrt{\left(z-\bar{p}_{\alpha} \delta\right)^{2}-\left(a^{2}+\bar{p}_{\alpha}^{2} b^{2}\right)}}\right\rangle \overline{\mathbf{d}} \\
& +\left\langle\frac{\mathrm{i} a b\left(\bar{p}_{\alpha}-p_{\alpha}\right)}{z-p_{\alpha} \delta+\sqrt{\left(z-p_{\alpha} \delta\right)^{2}-\left(a^{2}+p_{\alpha}^{2} b^{2}\right)}}\right\rangle \mathbf{d}, \quad z \in S_{1} \tag{55}
\end{align*}
$$

$$
\begin{align*}
& \mathbf{f}_{2}(z)=a b \sum_{k=1}^{4}\left(p_{k}-\bar{p}_{k}\right) \mathbf{B}_{2}^{-1} \boldsymbol{\Psi}\left\langle\exp \left[\mathrm{i} \lambda_{\alpha}\left(z-p_{k} \delta\right)\right] Y\right. \\
& \left.\times\left[\mathrm{i} \lambda_{\alpha}\left(z-p_{k} \delta\right), \lambda_{\alpha} \sqrt{a^{2}+p_{k}^{2} b^{2}}\right]\right\rangle \boldsymbol{\Lambda}_{2}^{-1} \mathbf{J} \bar{\Psi}^{T} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k} \mathbf{d}, \quad z \in S_{2} \tag{56}
\end{align*}
$$

It is not difficult to write down the full-field expressions of $\mathbf{f}_{0}(z), \mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ as follows:

$$
\begin{align*}
\mathbf{f}_{0}(z)= & a b \sum_{m=1}^{4} \sum_{k=1}^{4}\left(\bar{p}_{k}-p_{k}\right)\left\langle\exp \left[-\mathrm{i} \bar{\lambda}_{m}\left(z_{\alpha}-\bar{p}_{k} \delta\right)\right] Y\right. \\
& \left.\times\left[-\mathrm{i} \bar{\lambda}_{m}\left(z_{\alpha}-\bar{p}_{k} \delta\right), \bar{\lambda}_{m} \sqrt{a^{2}+\bar{p}_{k}^{2} b^{2}}\right]\right\rangle \mathbf{B}_{1}^{-1} \bar{\Psi}_{m} \overline{\boldsymbol{\Lambda}}_{2}^{-1} \mathbf{J}^{T} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}} \\
& -\sum_{k=1}^{4}\left\langle\frac{\mathrm{i} a b\left(\bar{p}_{\alpha}-p_{\alpha}\right)}{z_{\alpha}-\bar{p}_{\alpha} \delta+\sqrt{\left(z_{\alpha}-\bar{p}_{\alpha} \delta\right)^{2}-\left(a^{2}+\bar{p}_{\alpha}^{2} b^{2}\right)}}\right\rangle \mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}} \\
& -\left\langle z_{\alpha}\right\rangle\left[\mathbf{c}+\left\langle\frac{a-\mathrm{i} \overline{\mathrm{p}}_{\alpha} b}{a-\mathrm{i} p_{\alpha} b}\right\rangle \mathbf{d}\right] \tag{57}
\end{align*}
$$

$$
\begin{align*}
\mathbf{f}_{1}(z)= & a b \sum_{m=1}^{4} \sum_{k=1}^{4}\left(\bar{p}_{k}-p_{k}\right)\left\langle\exp \left[-\mathrm{i} \bar{\lambda}_{m}\left(z_{\alpha}-\bar{p}_{k} \delta\right)\right] Y\right. \\
& \left.\times\left[-\mathrm{i} \bar{\lambda}_{m}\left(z_{\alpha}-\bar{p}_{k} \delta\right), \bar{\lambda}_{m} \sqrt{a^{2}+\bar{p}_{k}^{2} b^{2}}\right]\right\rangle \mathbf{B}_{1}^{-1} \bar{\Psi} \mathbf{I}_{m} \overline{\boldsymbol{\Lambda}}_{2}^{-1} \mathbf{J} \Psi^{T} \mathbf{L}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}} \\
& -\sum_{k=1}^{4}\left\langle\frac{\mathrm{i} a b\left(\bar{p}_{\alpha}-p_{\alpha}\right)}{z_{\alpha}-\bar{p}_{\alpha} \delta+\sqrt{\left(z_{\alpha}-\bar{p}_{\alpha} \delta\right)^{2}-\left(a^{2}+\bar{p}_{\alpha}^{2} b^{2}\right)}}\right\rangle \mathbf{B}_{1}^{-1} \overline{\mathbf{B}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}} \\
& +\left\langle\frac{\mathrm{i} a b\left(\bar{p}_{\alpha}-p_{\alpha}\right)}{z_{\alpha}-p_{\alpha} \delta+\sqrt{\left(z_{\alpha}-p_{\alpha} \delta\right)^{2}-\left(a^{2}+p_{\alpha}^{2} b^{2}\right)}}\right\rangle \mathbf{d} \tag{58}
\end{align*}
$$

$$
\begin{align*}
\mathbf{f}_{2}(z)= & a b \sum_{m=1}^{4} \sum_{k=1}^{4}\left(p_{k}-\bar{p}_{k}\right)\left\langle\exp \left[\mathrm{i} \lambda_{m}\left(z_{\alpha}^{*}-p_{k} \delta\right)\right] Y\right. \\
& \left.\times\left[\mathrm{i} \lambda_{m}\left(z_{\alpha}^{*}-p_{k} \delta\right), \lambda_{m} \sqrt{a^{2}+p_{k}^{2} b^{2}}\right]\right\rangle \mathbf{B}_{2}^{-1} \Psi \mathbf{I}_{m} \mathbf{\Lambda}_{2}^{-1} \mathbf{J} \bar{\Psi}^{T} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k} \mathbf{d} \tag{59}
\end{align*}
$$

where the superscript '*' is utilized to distinguish the Stroh eignvalues associated with the lower half-plane $\left(z_{\alpha}^{*}\right)$ from those associated with the upper half-plane $\left(z_{\alpha}\right)$.

It is clearly observed from Eq. (57) that the electroelastic field inside the elliptical inclusion is intrinsically non-uniform even when the material properties of the two piezoelectric half-planes are exactly the same. The tractions and normal electric displacement distributed along the whole imperfect interface $x_{2}=0$ can also be simply given by

$$
\begin{align*}
{\left[\begin{array}{l}
\sigma_{12} \\
\sigma_{22} \\
\sigma_{32} \\
D_{2}
\end{array}\right]=} & 4 a b R e\left\{\sum_{k=1}^{4}\left(p_{k}-\bar{p}_{k}\right) \boldsymbol{\Psi}\left\langle\frac{\omega\left[\mathrm{i} \lambda_{\alpha}\left(x_{1}-p_{k} \delta\right), \lambda_{\alpha} \sqrt{a^{2}+p_{k}^{2} b^{2}}\right]-1}{x_{1}-p_{k} \delta+\sqrt{\left(x_{1}-p_{k} \delta\right)^{2}-\left(a^{2}+p_{k}^{2} b^{2}\right)}}\right\rangle\right. \\
& \left.\times \boldsymbol{\Lambda}_{2}^{-1} \mathbf{J} \bar{\Psi}^{T} \mathbf{L}_{1}^{-1} \mathbf{B}_{1} \mathbf{I}_{k} \mathbf{d}\right\}, x_{2}=0 \tag{60}
\end{align*}
$$

where the function $\omega(z, \beta)$ is defined by (Wang et al., 2007)
$\omega(z, \beta)=\exp (z)\left[z+\sqrt{z^{2}+\beta^{2}}\right] \int_{z}^{\infty} \frac{\exp (-\xi)}{\xi+\sqrt{\xi^{2}+\beta^{2}}} \mathrm{~d} \xi$.
The jumps in elastic displacements and electric potential along the imperfect interface can then be easily obtained by using Eqs. (25) and (60). By noticing the following far field asymptotic behavior of $\omega(z, \beta)$
$\omega(z, \beta) \cong 1-\frac{1}{z}+o\left(\frac{1}{z^{2}}\right), \quad|z| \rightarrow \infty$
then the tractions and normal electric displacement along $x_{2}=0$ at far field when $\left|x_{1}\right| \rightarrow \infty$ are
$\left[\begin{array}{c}\sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \\ D_{2}\end{array}\right] \cong \frac{2 a b}{x_{1}^{2}} \operatorname{Im}\left\{\mathbf{N}^{-1} \mathbf{L}_{1}^{-1} \mathbf{B}_{1}\left\langle\bar{p}_{\alpha}-p_{\alpha}\right\rangle \mathbf{d}\right\}+o\left(\frac{1}{x_{1}^{3}}\right)$,
$\left|x_{1}\right| \rightarrow \infty, \quad$ and $\quad x_{2}=0$
which clearly indicates that the imperfection of the interface has no influence on the leading $1 / x_{1}^{2}$ term in the far-field asymptotic expansion! Expression (63) can be easily generalized to an arbitrary shaped inclusion of area $A$ embedded in the upper half-plane such that

$$
\begin{align*}
{\left[\begin{array}{c}
\sigma_{12} \\
\sigma_{22} \\
\sigma_{32} \\
D_{2}
\end{array}\right] } & \cong \frac{A}{\pi x_{1}^{2}}\left(\widetilde{\mathbf{L}}\left[\begin{array}{c}
\varepsilon_{12}^{*} \\
\varepsilon_{22}^{*} \\
2 \varepsilon_{32}^{*} \\
-E_{2}^{*}
\end{array}\right]-\left[\widetilde{\mathbf{L}} \mathbf{L}_{1}^{-1}\left(\mathbf{N}_{1}^{(1) T} \mathbf{L}_{1}-\mathbf{N}_{3}^{(1)} \mathbf{S}_{1}\right)\right.\right. \\
& \left.\left.-\widetilde{W} \mathbf{L}_{1}^{-1} \mathbf{N}_{3}^{(1)}\right]\left[\begin{array}{c}
\varepsilon_{11}^{*} \\
\varepsilon_{12}^{*} \\
2 \varepsilon_{31}^{*} \\
-E_{1}^{*}
\end{array}\right]\right)+o\left(\frac{1}{x_{1}^{3}}\right), \quad\left|x_{1}\right| \rightarrow \infty, \text { and } x_{2}=0 \tag{64}
\end{align*}
$$

where
$\widetilde{\mathbf{L}}=\left(\mathbf{D}-\mathbf{W}^{T} \mathbf{D}^{-1} \mathbf{W}\right)^{-1}, \quad \widetilde{\mathbf{W}}=\mathbf{D}^{-1} \mathbf{W} \widetilde{\mathbf{L}}$,
and
$\mathbf{D}=\mathbf{L}_{1}^{-1}+\mathbf{L}_{2}^{-1}, \quad \mathbf{W}=\mathbf{S}_{1} \mathbf{L}_{1}^{-1}-\mathbf{S}_{2} \mathbf{L}_{2}^{-1}$.
During the derivation of the above real form solution, we have adopted the identities in Eqs. (10) and (11).
3.3. Far-field expansions of the analytic function vectors for an arbitrary shaped inclusion

It follows from Eqs. (58) and (59) that the far-field asymptotic expansions of $\mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ for an arbitrary shaped inclusion of area $A$ embedded in the upper half-plane can be simply derived as

$$
\begin{align*}
\mathbf{f}_{1}(z)= & \frac{A}{2 \pi}\left\langle\left(z_{\alpha}\right)^{-1}\right\rangle \mathbf{B}_{1}^{-1}\left\{\overline { \mathbf { N } } ^ { - 1 } \mathbf { L } _ { 1 } ^ { - 1 } \left[\left(\mathbf{N}_{1}^{(1) T} \mathbf{L}_{1}-\mathbf{N}_{3}^{(1)} \mathbf{S}_{1}\right.\right.\right. \\
& \left.\left.\left.+\mathrm{i} \mathbf{N}_{3}^{(1)}\right) \varepsilon_{1}^{*}-\mathbf{L}_{1} \boldsymbol{\varepsilon}_{2}^{*}\right]-\mathrm{i} \mathbf{N}_{3}^{(1)} \boldsymbol{\varepsilon}_{1}^{*}\right\}+o\left(\left\langle\left(z_{\alpha}\right)^{-2}\right\rangle\right), \tag{67}
\end{align*}
$$

$$
\begin{align*}
\mathbf{f}_{2}(z)= & \frac{A}{2 \pi}\left\langle\left(z_{\alpha}^{*}\right)^{-1}\right\rangle \mathbf{B}_{2}^{-1} \mathbf{N}^{-1} \mathbf{L}_{1}^{-1}\left[\left(\mathbf{N}_{1}^{(1) T} \mathbf{L}_{1}-\mathbf{N}_{3}^{(1)} \mathbf{S}_{1}-\mathrm{i} \mathbf{N}_{3}^{(1)}\right) \boldsymbol{\varepsilon}_{1}^{*}\right. \\
& \left.-\mathbf{L}_{1} \boldsymbol{\varepsilon}_{2}^{*}\right]+o\left(\left\langle\left(z_{\alpha}^{*}\right)^{-2}\right\rangle\right), \tag{68}
\end{align*}
$$

where
$\boldsymbol{\varepsilon}_{1}^{*}=\left[\begin{array}{llll}\varepsilon_{11}^{*} & \varepsilon_{12}^{*} & 2 \varepsilon_{31}^{*} & -E_{1}^{*}\end{array}\right]^{T}, \quad \boldsymbol{\varepsilon}_{2}^{*}=\left[\begin{array}{llll}\varepsilon_{12}^{*} & \varepsilon_{22}^{*} & 2 \varepsilon_{32}^{*} & -E_{2}^{*}\end{array}\right]^{T}$.

Interestingly the leading terms in the far-field asymptotic behaviors of $\mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ are independent of the imperfection of the interface, and they satisfy the perfect boundary conditions on $x_{2}=0$. The result in Eq. (64) can also be obtained from Eq. (68) by taking differentiation. When the two piezoelectric halfplanes are exactly the same, it can be easily deduced from Eqs.
(67) and (68) that

$$
\begin{align*}
\mathbf{f}_{1}(z)=\mathbf{f}_{2}(z)= & \frac{A}{4 \pi}\left\langle\left(z_{\alpha}\right)^{-1}\right\rangle \mathbf{B}^{-1} \\
& \times\left[\left(\mathbf{N}_{1}^{T} \mathbf{L}-\mathbf{N}_{3} \mathbf{S}-\mathrm{i} \mathbf{N}_{3}\right) \boldsymbol{\varepsilon}_{1}^{*}-\mathbf{L} \boldsymbol{\varepsilon}_{2}^{*}\right]+o\left(\left\langle\left(z_{\alpha}\right)^{-2}\right\rangle\right) . \tag{70}
\end{align*}
$$

## 4. The Eshelby's problem for two bonded piezoelectric halfplanes with a stiff and highly conducting interface

### 4.1. The general solution

In this section, we discuss the case in which the interface between the two piezoelectric half-planes is stiff and highly conducting. The boundary conditions on the stiff and highly conducting imperfect interface $x_{2}=0$ can be expressed as
$u_{1}^{(1)}=u_{1}^{(2)}, \quad u_{2}^{(1)}=u_{2}^{(2)}, \quad u_{3}^{(1)}=u_{3}^{(2)}, \quad \phi^{(1)}=\phi^{(2)}$,
$\left[\begin{array}{l}\sigma_{12}^{(1)} \\ \sigma_{22}^{(1)} \\ \sigma_{32}^{(1)} \\ D_{2}^{(1)}\end{array}\right]-\left[\begin{array}{l}\sigma_{12}^{(2)} \\ \sigma_{22}^{(2)} \\ \sigma_{32}^{(2)} \\ D_{2}^{(2)}\end{array}\right]=-\mathbf{E} \frac{\partial^{2}}{\partial x_{1}^{2}}\left[\begin{array}{l}u_{1}^{(2)} \\ u_{2}^{(2)} \\ u_{3}^{(2)} \\ \phi^{(2)}\end{array}\right], \quad x_{2}=0$,
where
$\mathbf{E}=\mathbf{E}^{T}=\left[\begin{array}{cccc}E_{11} & 0 & E_{13} & E_{14} \\ 0 & 0 & 0 & 0 \\ E_{13} & 0 & E_{33} & E_{34} \\ E_{14} & 0 & E_{34} & E_{44}\end{array}\right]$,
with $E_{11}>0, E_{33}>0, E_{11} E_{33}-E_{13}^{2}>0$ and $E_{44}<0$. Eq. (71), which can be termed a generalized "membrane type interface" (Benveniste and Miloh, 2001; Benveniste, 2006; Erdogan and Ozturk, 2008; Guler, 2008), states that displacements and electric potential are continuous across the interface, whereas tractions and normal electric displacement undergo jumps on the interface which are proportional to certain surface differential operators of the interface displacements and electric potential. It is clearly observed from Eq. (71) that the normal stress $\sigma_{22}$ is still continuous across the imperfect interface. A detailed derivation of the above imperfect interface model in Eqs. (71) and (72) can be found in Appendix C. In the analysis carried out in this section, we will also adopt Scheme 1 of the Stroh formalism.

The boundary conditions in Eq. (71) can also be concisely written in terms of $\mathbf{u}$ and $\boldsymbol{\Phi}$ as
$\mathbf{u}_{1}=\mathbf{u}_{2}, \quad \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{2}=-\mathbf{E} \mathbf{u}_{2,1}, \quad \chi_{2}=0$
or in terms of the two analytic function vectors $\mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ as

$$
\begin{align*}
& \mathbf{A}_{\mathbf{1}} \mathbf{f}_{1}^{+}\left(x_{1}\right)+\overline{\mathbf{A}}_{1} \overline{\mathbf{f}}_{1}^{-}\left(x_{1}\right)=\mathbf{A}_{2} \mathbf{f}_{2}^{-}\left(x_{1}\right)+\overline{\mathbf{A}}_{2} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}\right), \\
& \mathbf{B}_{1} \mathbf{f}_{1}^{+}\left(x_{1}\right)+\overline{\mathbf{B}}_{1} \overline{\mathbf{f}}_{1}^{-}\left(x_{1}\right)-\mathbf{B}_{2} \mathbf{f}_{2}^{-}\left(x_{1}\right)-\overline{\mathbf{B}}_{2} \overline{\mathbf{f}}_{2}^{+}\left(x_{1}\right) \quad x_{2}=0  \tag{74}\\
& \quad=-\mathbf{E}\left[\mathbf{A}_{2} \mathbf{f}_{2}^{-}\left(x_{1}\right)+\overline{\mathbf{A}}_{2} \mathbf{f}_{2}^{+}\left(x_{1}\right)\right],
\end{align*}
$$

It follows from Eq. (74) that:

$$
\begin{align*}
& \mathbf{f}_{1}(z)=\mathbf{A}_{1}^{-1} \overline{\mathbf{A}}_{2} \overline{\mathbf{f}}_{2}(z)+\mathbf{h}(z)-\mathbf{A}_{1}^{-1} \overline{\mathbf{A}}_{1} \overline{\mathbf{h}}(z), \\
& \overline{\mathbf{f}}_{1}(z)=\overline{\mathbf{A}}_{1}^{-1} \mathbf{A}_{2} \mathbf{f}_{2}(z)-\overline{\mathbf{A}}_{1}^{-1} \mathbf{A}_{1} \mathbf{h}(z)+\overline{\mathbf{h}}(z) . \tag{75}
\end{align*}
$$

Substituting Eq. (75) into Eq. (74) 2 , we can finally arrive at the following set of coupled differential equations:
$\boldsymbol{\Pi} \mathbf{A}_{2} \mathbf{f}_{2}(z)+\mathbf{i} \mathbf{A}_{2} \mathbf{f}_{2}^{\prime}(z)=2 \mathbf{H}_{1}^{-1} \mathbf{A}_{1} \mathbf{h}(z), \quad \operatorname{Im}\{z\}<0$,
where $\Pi$ is a $4 \times 4$ Hermitian matrix defined by
$\boldsymbol{\Pi}=\overline{\mathbf{M}}_{1}+\mathbf{M}_{2}$,
$\mathbf{M}_{k}=-\mathrm{i} \mathbf{B}_{k} \mathbf{A}_{k}^{-1}=\mathbf{H}_{k}^{-1}\left(\mathbf{I}+\mathrm{i} \mathbf{S}_{k}\right), \quad(k=1,2)$.
In order to solve Eq. (76), we consider the following eigenvalue problem:
$(\mathbf{E}-\lambda \boldsymbol{\Pi}) \mathbf{v}=\mathbf{0}$.
It is apparent that: (i) if $\lambda$ is an eigenvalue, then its conjugate $\bar{\lambda}$ is also an eigenvalue; (ii) $\lambda=0$ is an eigenvalue, and the real parts of all the other three non-zero eigenvalues are positive. Let that $\lambda_{1}=0$ and $\lambda_{i},(i=2,3,4)$ be the four distinct roots and $\mathbf{v}_{i}$ the associated eigenvectors, then the following orthogonal relationship can be easily proved:
$\boldsymbol{J} \bar{\Psi}^{\top} \mathbf{E} \boldsymbol{\Psi}=\boldsymbol{\Lambda}_{1} \boldsymbol{\Lambda}_{2}, \quad \boldsymbol{J} \overline{\boldsymbol{\Psi}}^{\mathrm{T}} \boldsymbol{\Pi} \boldsymbol{\Psi}=\boldsymbol{\Lambda}_{2}$,
where $\boldsymbol{\Lambda}_{2}$ is a $4 \times 4$ diagonal matrix, and
$\boldsymbol{\Psi}=\left[\begin{array}{llll}\mathbf{v}_{1} & \mathbf{v}_{2} & \mathbf{v}_{3} & \mathbf{v}_{4}\end{array}\right]$,
$\boldsymbol{\Lambda}_{1}=\operatorname{diag}\left[\begin{array}{llll}0 & \lambda_{2} & \lambda_{3} & \lambda_{4}\end{array}\right]$.
In addition the $4 \times 4$ real and symmetric matrix $\mathbf{J}$ is dependent on the nature of the four eigenvalues $\lambda_{i},(i=1-4)$. A detailed classification is given below.
(i) Four real eigenvalues (i.e., $\lambda_{2}, \lambda_{3}, \lambda_{4}>0$ ):
$\mathbf{J}=\operatorname{diag}\left[\begin{array}{llll}1 & 1 & 1 & 1\end{array}\right]$,
(ii) Two real and two complex conjugate eigenvalues (i.e., $\left.\lambda_{2}>0, \lambda_{3}=\bar{\lambda}_{4}\right):$
$\mathbf{J}=\left[\begin{array}{llll}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0\end{array}\right]$.
Next we introduce an analytic function vector $\boldsymbol{\Omega}(z)=\left[\Omega_{1}(z)\right.$ $\left.\Omega_{2}(z) \Omega_{3}(z) \Omega_{4}(z)\right]^{T}$ such that
$\mathbf{A}_{\mathbf{2}} \mathbf{f}_{2}(z)=\boldsymbol{\Psi} \boldsymbol{\Omega}(z)$.
Employing the orthogonal relationship in Eq. (80), then Eq. (76) can be decoupled into
$\boldsymbol{\Omega}(z)+\mathbf{i} \boldsymbol{\Lambda}_{1} \boldsymbol{\Omega}^{\prime}(z)=2 \boldsymbol{\Lambda}_{2}^{-1} \overline{\mathbf{J}}^{\mathrm{T}} \mathbf{H}_{1}^{-1} \mathbf{A}_{1}\left\langle D_{\alpha}(z)-P_{\alpha}(z)\right\rangle \mathbf{d}, \quad \operatorname{Im}\{z\}<0$.

The general solution to the above set of decoupled differential equations can be conveniently expressed as
$\boldsymbol{\Omega}(z)=\sum_{k=1}^{4}\left\langle F_{\alpha}^{k}(z)\right\rangle \boldsymbol{\Lambda}_{2}^{-1} \bar{J}^{\top} \mathbf{H}_{1}^{-1} \mathbf{A}_{1} \mathbf{I}_{k} \mathbf{d}$,
where the analytic functions $F_{m}^{k}(z)$ are defined by
$F_{m}^{k}(z)=\left\{\begin{array}{lr}2\left[D_{k}(z)-P_{k}(z)\right], & m=1 \\ -2 \mathrm{i} \lambda_{m}^{-1} \exp \left(\mathrm{i} \lambda_{m}^{-1} z\right) \int_{-\infty \mathrm{i}}^{z}\left[D_{k}(\xi)-P_{k}(\xi)\right] \exp \left(-\mathrm{i} \lambda_{m}^{-1} \xi\right) \mathrm{d} \xi, & m=2,3,4\end{array}\right.$

Consequently it is not difficult to write down the full-field solutions of $\mathbf{f}_{0}(z), \mathbf{f}_{1}(z)$ and $\mathbf{f}_{2}(z)$ as follows:

$$
\begin{align*}
\mathbf{f}_{0}(z)= & \sum_{m=1}^{4} \sum_{k=1}^{4}\left\langle\bar{F}_{m}^{k}\left(z_{\alpha}\right)\right\rangle \mathbf{A}_{1}^{-1} \overline{\boldsymbol{\Psi}} \mathbf{I}_{m} \overline{\boldsymbol{\Lambda}}_{2}^{-1} \mathbf{J} \boldsymbol{\Psi}^{T} \mathbf{H}_{1}^{-1} \overline{\mathbf{A}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}} \\
& -\sum_{k=1}^{4}\left\langle\bar{D}_{k}\left(z_{\alpha}\right)-\bar{P}_{k}\left(z_{\alpha}\right)\right\rangle \mathbf{A}_{1}^{-1} \overline{\mathbf{A}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}}-\left\langle z_{\alpha}\right\rangle \mathbf{c}-\left\langle P_{\alpha}\left(z_{\alpha}\right)\right\rangle \mathbf{d},  \tag{89}\\
\mathbf{f}_{1}(z)= & \sum_{m=1}^{4} \sum_{k=1}^{4}\left\langle\bar{F}_{m}^{k}\left(z_{\alpha}\right)\right\rangle \mathbf{A}_{1}^{-1} \overline{\boldsymbol{\Psi}} \mathbf{I}_{m} \overline{\boldsymbol{\Lambda}}_{2}^{-1} \mathbf{J} \boldsymbol{\Psi}^{T} \mathbf{H}_{1}^{-1} \overline{\mathbf{A}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}}-\sum_{k=1}^{4}\left\langle\bar{D}_{k}\left(z_{\alpha}\right)\right. \\
& \left.-\bar{P}_{k}\left(z_{\alpha}\right)\right\rangle \mathbf{A}_{1}^{-1} \overline{\mathbf{A}}_{1} \mathbf{I}_{k} \overline{\mathbf{d}}+\left\langle D_{\alpha}\left(z_{\alpha}\right)-P_{\alpha}\left(z_{\alpha}\right)\right\rangle \mathbf{d},  \tag{90}\\
\mathbf{f}_{2}(z)= & \sum_{m=1}^{4} \sum_{k=1}^{4}\left\langle F_{m}^{k}\left(z_{\alpha}^{*}\right)\right\rangle \mathbf{A}_{2}^{-1} \boldsymbol{\Psi} \mathbf{I}_{m} \mathbf{\Lambda}_{2}^{-1} \mathbf{J} \bar{\Psi}^{T} \mathbf{H}_{1}^{-1} \mathbf{A}_{1} \mathbf{I}_{k} \mathbf{d} . \tag{91}
\end{align*}
$$

### 4.2. An example of elliptical inclusion

We illustrate the obtained general solution in Section 4.1 through an example of an elliptical inclusion with semi-major and semi-minor axes $a$ and $b$. We further assume that the major axis is parallel to the $x_{1}$-axis and the center of the ellipse is located at $x_{1}=0$ and $x_{2}=\delta(\delta>b)$. As a result the explicit expressions of $F_{m}^{k}(z)$ can be easily given by
$F_{m}^{k}(z)= \begin{cases}\frac{2 \mathrm{i}\left(p_{k}-p_{k}\right) a b}{z-p_{k} \delta+\sqrt{\left(z-p_{k} \delta\right)^{2}-\left(a^{2}+p_{k}^{2} b^{2}\right)}}, & m=1, \\ \lambda_{m}^{-1}\left(p_{k}-\bar{p}_{k}\right) a b \exp \left[\mathrm{i} \lambda_{m}^{-1}\left(z-p_{k} \delta\right)\right] Y\left[\mathrm{i} \lambda_{m}^{-1}\left(z-p_{k} \delta\right), \lambda_{m}^{-1} \sqrt{a^{2}+p_{k}^{2} b^{2}}\right], & m=2,3,4\end{cases}$
where the integral function $Y(z, \beta)$ has been defined in Eq. (53). Apparently $F_{m}^{k}(z)=F_{1}^{k}(z)$ as $\lambda_{m} \rightarrow 0,(m=2,3,4)$.
5. The Eshelby's problem for two bonded piezoelectric halfplanes with a compliant and highly conducting interface

In this section we consider the case in which the interface between the two piezoelectric half-planes is compliant and highly conducting. When we adopt Scheme 2 of the Stroh formalism, the interface conditions along the perfect interface $\Gamma$ can be expressed as (Ru, 2001)
$\mathbf{u}_{1}=\mathbf{u}_{0}+\mathbf{u}^{*}, \quad \boldsymbol{\Phi}_{1}=\boldsymbol{\Phi}_{0}+\boldsymbol{\Phi}^{*} \quad$ on $\Gamma$,
where $\mathbf{u}^{*}$ and $\boldsymbol{\Phi}^{*}$ are the additional displacements and electric potential within the Eshelby's inclusion $S_{0}$ due to uniform eigenstrains and eigenelectric fields
$\mathbf{u}^{*}=\left(\begin{array}{c}\varepsilon_{11}^{*} x_{1}+\varepsilon_{12}^{*} x_{2} \\ \varepsilon_{12}^{*} x_{1}+\varepsilon_{22}^{*} x_{2} \\ 2\left(\varepsilon_{31}^{*} x_{1}+\varepsilon_{32}^{*} x_{2}\right) \\ 0\end{array}\right)$ and $\boldsymbol{\Phi}^{*}=\left(\begin{array}{c}0 \\ 0 \\ 0 \\ -\left(E_{1}^{*} x_{1}+E_{2}^{*} x_{2}\right)\end{array}\right)$ within $S_{0}$

In this case we can still introduce the auxiliary function vector $\mathbf{g}(z)$ defined in Eq. (23). However now the vectors $\mathbf{c}$ and $\mathbf{d}$ are re-defined by

$$
\begin{align*}
& \mathbf{c}=\left\langle\frac{\bar{p}_{\alpha}}{p_{\alpha}-p_{\alpha}}\right\rangle\left(\mathbf{B}_{1}^{T}\left[\begin{array}{c}
\varepsilon_{11}^{*} \\
\varepsilon_{12}^{*} \\
2 \varepsilon_{31}^{*} \\
0
\end{array}\right]-\mathbf{A}_{1}^{T}\left[\begin{array}{c}
0 \\
0 \\
0 \\
E_{1}^{*}
\end{array}\right]\right)-\left\langle\frac{1}{p_{\alpha}-p_{\alpha}}\right\rangle\left(\mathbf{B}_{1}^{T}\left[\begin{array}{c}
\varepsilon_{12}^{*} \\
\varepsilon_{22}^{*} \\
2 \varepsilon_{32}^{*} \\
0
\end{array}\right]-\mathbf{A}_{1}^{T}\left[\begin{array}{c}
0 \\
0 \\
0 \\
E_{2}^{*}
\end{array}\right]\right), \\
& \mathbf{d}=\left\langle\frac{1}{p_{\alpha}-p_{\alpha}}\right\rangle\left(\mathbf{B}_{1}^{T}\left[\begin{array}{c}
\varepsilon_{11}^{*} \\
\varepsilon_{12}^{*} \\
2 \varepsilon_{31}^{*} \\
0
\end{array}\right]-\mathbf{A}_{1}^{T}\left[\begin{array}{c}
0 \\
0 \\
0 \\
E_{1}^{*}
\end{array}\right]\right)-\left\langle\frac{p_{\alpha}}{p_{\alpha-}-p_{\alpha}}\right\rangle\left(\mathbf{B}_{1}^{T}\left[\begin{array}{c}
\varepsilon_{12}^{*} \\
\varepsilon_{22}^{*} \\
2 \varepsilon_{32}^{*} \\
0
\end{array}\right]-\mathbf{A}_{1}^{T}\left[\begin{array}{c}
0 \\
0 \\
0 \\
E_{2}^{*}
\end{array}\right]\right) . \tag{95}
\end{align*}
$$

During the above derivation, we have adopted the normalized orthogonal relationship in Eq. (9). Similar to the situation in Section 3, the introduced $\mathbf{g}(z)$ is still analytic, continuous and single-valued everywhere in the whole upper half-plane $S_{0}+S_{1}$ including the point at infinity.

In addition the boundary conditions on the compliant and highly conducting imperfect interface $x_{2}=0$ separating the two piezoelectric half-planes can be expressed as
$\sigma_{12}^{(1)}=\sigma_{12}^{(2)}, \quad \sigma_{22}^{(1)}=\sigma_{22}^{(2)}, \quad \sigma_{32}^{(1)}=\sigma_{32}^{(2)}, \quad E_{1}^{(1)}=E_{1}^{(2)}$,
$\left[\begin{array}{c}u_{1}^{(1)}-u_{1}^{(2)} \\ u_{2}^{(1)}-u_{2}^{(2)} \\ u_{3}^{(1)}-u_{3}^{(2)} \\ \varphi^{(1)}-\varphi^{(2)}\end{array}\right]=\Lambda\left[\begin{array}{c}\sigma_{12}^{(2)} \\ \sigma_{22}^{(2)} \\ \sigma_{32}^{(2)} \\ -E_{1}^{(2)}\end{array}\right], \quad x_{2}=0$,
where $\Lambda$ is a $4 \times 4$ positive definite real and symmetric matrix. Eq. (96) states that tractions and tangential electric field are continuous across the interface, whereas the elastic displacements and charge potential undergo jumps on the interface which are proportional to the interface tractions and tangential electric field. A detailed derivation of the above imperfect interface model in Eq. (96) can be found in Appendix D.

Once we have introduced the above, all the rest analysis is similar to that in Section 3. In fact the analysis becomes much simpler because in this case both $\mathbf{N}$ defined in Eq. (30) and $\mathbf{\Lambda}$ are positive definite (Suo, 1993). Thus we observe that: (i) all the four eigenvalues of Eq. (32) are positive real; and (ii) it is sufficient to treat J in Eq. (33) as an identity matrix and $\boldsymbol{\Lambda}_{2}$ in Eq. (33) as a $4 \times 4$ positive real diagonal matrix.

## 6. The Eshelby's problem for bonded two piezoelectric halfplanes with a stiff and weakly conducting interface

In this section we consider the case in which the interface between the two piezoelectric half-planes is stiff and weakly conducting. We will adopt Scheme 2 of the Stroh formalism in the following analysis. The boundary conditions on the stiff and weakly conducting imperfect interface $x_{2}=0$ can be expressed as
$u_{1}^{(1)}=u_{1}^{(2)}, \quad u_{2}^{(1)}=u_{2}^{(2)}, \quad u_{3}^{(1)}=u_{3}^{(2)}, \quad \varphi^{(1)}=\varphi^{(2)}$,
$\left[\begin{array}{c}\sigma_{12}^{(1)} \\ \sigma_{22}^{(1)} \\ \sigma_{32}^{(1)} \\ -E_{1}^{(1)}\end{array}\right]-\left[\begin{array}{c}\sigma_{12}^{(2)} \\ \sigma_{22}^{(2)} \\ \sigma_{32}^{(2)} \\ -E_{1}^{(2)}\end{array}\right]=-\mathbf{E} \frac{\partial^{2}}{\partial x_{1}^{2}}\left[\begin{array}{c}u_{1}^{(2)} \\ u_{2}^{(2)} \\ u_{3}^{(2)} \\ \varphi^{(2)}\end{array}\right], \quad x_{2}=0$,
where $\mathbf{E}$ is a positive semidefinite matrix given by
$\mathbf{E}=\mathbf{E}^{T}=\left[\begin{array}{cccc}E_{11} & 0 & E_{13} & E_{14} \\ 0 & 0 & 0 & 0 \\ E_{13} & 0 & E_{33} & E_{34} \\ E_{14} & 0 & E_{34} & E_{44}\end{array}\right]$

Eq. (97) states that displacements and charge potential are continuous across the interface, whereas tractions and tangential electric field undergo jumps on the interface which are proportional to certain surface differential operators of the interface displacements and charge potential. It is also clearly observed from Eq. (97) that the normal stress $\sigma_{22}$ is still continuous across the imperfect interface. A detailed derivation of the above imperfect interface model in Eqs. (97) and (98) can be found in Appendix D.

Once we have introduced the above, all the rest analysis is similar to that in Section 4. Keep in mind that now the two vectors $\mathbf{c}$ and $\mathbf{d}$ have been re-defined in Eq. (95). In fact the analysis becomes simpler because in this case $\Pi$ defined in Eq. (77) is positive definite whilst $\mathbf{E}$ is positive semidefinite. Thus we observe that: (i) The nature of the eigenvalues of Eq. (79) is: $\lambda_{1}=0$ and $\lambda_{2}, \lambda_{3}, \lambda_{4}>0$; and (ii) it is sufficient to treat $\mathbf{J}$ in Eq. (80) as an identity matrix and $\Lambda_{2}$ in Eq. (80) as a $4 \times 4$ positive real diagonal matrix.

## 7. Conclusions

In this research we derived closed-form solutions to the 2D problem of an Eshelby's inclusion of arbitrary shape embedded in one of two imperfectly bonded piezoelectric half-planes. Full-field solutions for an elliptical inclusion embedded in the upper halfplane were presented in Eqs. (57)-(59) in terms of the introduced integral function $Y(z, \beta)$. A concise expression of the tractions and normal electric displacement along the interface was given by Eq. (60) through the introduction of the function $\omega(z, \beta)$. The far-field asymptotic expansions of the tractions and normal electric displacement along the imperfect interface as well as those of the analytic function vectors in the two half-planes due to an arbitrary shaped inclusion were also presented. It was observed that the leading terms in these expansions are in fact independent of the imperfection of the interface. We then presented in Eqs. (89)(91) the full-field general solutions of the Esheby's problem for two bonded piezoelectric half-planes with a stiff and highly conducting interface. The obtained general solutions were demonstrated through the example of an elliptical inclusion. We discussed in Sections 5 and 6 the Eshelby's problem in piezoelectric bimaterials with a compliant and highly conducting interface and with a stiff and weakly conducting interface. We observed that the discussions on a compliant and highly conducting interface or a stiff and weakly conducting interface become simpler in view of the fact that in these two cases the complex Hermitian matrices for the piezoelectric bimaterial are positive definite (Suo, 1993) whilst the real and symmetric matrices for the imperfect interface are positive definite [see Eq. (96)] or positive semi-definite [see Eq. (98)]. During the theoretical development, we also derived explicit expressions of $\mathbf{N}_{i}$ and $\mathbf{N}_{i}^{(-1)}$ for anisotropic piezoelectric materials in terms of the introduced 28 reduced generalized compliances $S_{i j}$ (see Appendices A and B).

In this work we only considered the case in which the imperfect interface is infinitely long. When the imperfection is finite along the interface, the problem basically reduces to interface bridged cracks or interface bridged anti-cracks with the imperfect boundary conditions being used as the "bridging force" for compliant interface (Ni and Nemat-Nasser, 2000) or "bridging strain" for stiff interface (Erdogan and Ozturk, 2008). In this case in principle we can resort to the interfacial Green's functions for an extended line dislocation and an extended line force (Ting, 1996) to construct a system of Cauchy singular integral equations for the distributed dislocation density and the distributed line force density whose explicit solutions can be given in terms of Chebyshev polynomials or Jacobi polynomials (Erdogan and Gupta, 1972; Ni and Nemat-Nasser, 2000). Particularly when the bridged cracks or anti-cracks are
located in homogeneous materials, a decoupling methodology similar to that proposed in this research can still be conveniently adopted to arrive at a decoupled set of singular integral equations, with each equation in a form similar to Eq. (5.1) by Erdogan and Gupta (1972).

## Acknowledgements

Part of the work in this article was started while the first author (XW) was at University of Akron, supported by AFOSR FA9550-06-1-0317. The reviewers' comments are highly appreciated.

## Appendix $A$. The structures of $N_{i}$ and $N_{i}^{(-1)}$ for Scheme 1

First we discuss the structures of $\mathbf{N}_{i}(i=1,2,3)$. It has been proved that (Ting, 1996)

$$
\mathbf{N}_{1}=\left[\begin{array}{cccc}
* & -1 & * & *  \tag{A1}\\
* & 0 & * & * \\
* & 0 & * & * \\
* & 0 & * & *
\end{array}\right], \quad \mathbf{N}_{3}=\left[\begin{array}{cccc}
* & 0 & * & * \\
0 & 0 & 0 & 0 \\
* & 0 & * & * \\
* & 0 & * & *
\end{array}\right],
$$

where $*$ denotes a possibly nonzero element.
We then introduce $S_{i j},(i, j=1-8$ and $i, j \neq 3)$ such that

$$
\begin{align*}
& {\left[\begin{array}{lllllll}
C_{11} & C_{12} & C_{14} & C_{15} & C_{16} & e_{11} & e_{21} \\
C_{12} & C_{22} & C_{24} & C_{25} & C_{26} & e_{12} & e_{22} \\
C_{14} & C_{24} & C_{44} & C_{45} & C_{46} & e_{14} & e_{24} \\
C_{15} & C_{25} & C_{45} & C_{55} & C_{56} & e_{15} & e_{25} \\
C_{16} & C_{26} & C_{46} & C_{56} & C_{66} & e_{16} & e_{26} \\
e_{11} & e_{12} & e_{14} & e_{15} & e_{16} & -\epsilon_{11} & -\epsilon_{12} \\
e_{21} & e_{22} & e_{24} & e_{25} & e_{26} & -\epsilon_{12} & -\epsilon_{22}
\end{array}\right]}  \tag{A2}\\
& \quad\left[\begin{array}{lllllll}
S_{11} & S_{12} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\
S_{12} & S_{22} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} \\
S_{14} & S_{24} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\
S_{15} & S_{25} & S_{45} & S_{55} & S_{56} & S_{57} & S_{58} \\
S_{16} & S_{26} & S_{46} & S_{56} & S_{66} & S_{67} & S_{68} \\
S_{17} & S_{27} & S_{47} & S_{57} & S_{67} & S_{77} & S_{78} \\
S_{18} & S_{28} & S_{48} & S_{58} & S_{68} & S_{78} & S_{88}
\end{array}\right]=\mathbf{I} .
\end{align*}
$$

Our task below is to present the expressions of $\mathbf{N}_{1}, \mathbf{N}_{2}$ and $\mathbf{N}_{3}$ in terms of $S_{i j}$. After arranging the columns and rows, Eq. (A2) can be equivalently written into the following:

$$
\begin{align*}
& {\left[\begin{array}{ccccccc}
C_{11} & C_{15} & e_{11} & C_{16} & C_{12} & C_{14} & e_{21} \\
C_{15} & C_{55} & e_{15} & C_{56} & C_{25} & C_{45} & e_{25} \\
e_{11} & e_{15} & -\epsilon_{11} & e_{16} & e_{12} & e_{14} & -\epsilon_{12} \\
C_{16} & C_{56} & e_{16} & C_{66} & C_{26} & C_{46} & e_{26} \\
C_{12} & C_{25} & e_{12} & C_{26} & C_{22} & C_{24} & e_{22} \\
C_{14} & C_{45} & e_{14} & C_{46} & C_{24} & C_{44} & e_{24} \\
e_{21} & e_{25} & -\epsilon_{12} & e_{26} & e_{22} & e_{24} & -\epsilon_{22}
\end{array}\right]}  \tag{A3}\\
& \quad\left[\begin{array}{lllllll}
S_{11} & S_{15} & S_{17} & S_{16} & S_{12} & S_{14} & S_{18} \\
S_{15} & S_{55} & S_{57} & S_{56} & S_{25} & S_{45} & S_{58} \\
S_{17} & S_{57} & S_{77} & S_{67} & S_{27} & S_{47} & S_{78} \\
S_{16} & S_{56} & S_{67} & S_{66} & S_{26} & S_{46} & S_{68} \\
S_{12} & S_{25} & S_{27} & S_{26} & S_{22} & S_{24} & S_{28} \\
S_{14} & S_{45} & S_{47} & S_{46} & S_{24} & S_{44} & S_{48} \\
S_{18} & S_{58} & S_{78} & S_{68} & S_{28} & S_{48} & S_{88}
\end{array}\right]=\mathbf{I},
\end{align*}
$$

or
$\left[\begin{array}{cccccccc}C_{11} & C_{16} & C_{15} & e_{11} & C_{16} & C_{12} & C_{14} & e_{21} \\ C_{16} & C_{66} & C_{56} & e_{16} & C_{66} & C_{26} & C_{46} & e_{26} \\ C_{15} & C_{56} & C_{55} & e_{15} & C_{56} & C_{25} & C_{45} & e_{25} \\ e_{11} & e_{16} & e_{15} & -\epsilon_{11} & e_{16} & e_{12} & e_{14} & -\epsilon_{12} \\ C_{16} & C_{66} & C_{56} & e_{16} & C_{66} & C_{26} & C_{46} & e_{26} \\ C_{12} & C_{26} & C_{25} & e_{12} & C_{26} & C_{22} & C_{24} & e_{22} \\ C_{14} & C_{46} & C_{45} & e_{14} & C_{46} & C_{24} & C_{44} & e_{24} \\ e_{21} & e_{26} & e_{25} & -\epsilon_{12} & e_{26} & e_{22} & e_{24} & -\epsilon_{22}\end{array}\right]$

$$
\begin{align*}
& \quad\left[\begin{array}{cccccccc}
S_{11} & 0 & S_{15} & S_{17} & S_{16} & S_{12} & S_{14} & S_{18} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_{15} & 0 & S_{55} & S_{57} & S_{56} & S_{25} & S_{45} & S_{58} \\
S_{17} & 0 & S_{57} & S_{77} & S_{67} & S_{27} & S_{47} & S_{78} \\
S_{16} & 0 & S_{56} & S_{67} & S_{66} & S_{26} & S_{46} & S_{68} \\
S_{12} & 0 & S_{25} & S_{27} & S_{26} & S_{22} & S_{24} & S_{28} \\
S_{14} & 0 & S_{45} & S_{47} & S_{46} & S_{24} & S_{44} & S_{48} \\
S_{18} & 0 & S_{58} & S_{78} & S_{68} & S_{28} & S_{48} & S_{88}
\end{array}\right] \\
& =\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] . \tag{A4}
\end{align*}
$$

Eq. (A4) can be more concisely written into

$$
\left[\begin{array}{cc}
\mathbf{Q} & \mathbf{R}  \tag{A5}\\
\mathbf{R}^{T} & \mathbf{T}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{q}_{2} & \mathbf{r}_{2} \\
\mathbf{r}_{2}^{T} & \mathbf{t}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{I}-\mathbf{I}_{2} & \mathbf{I}_{12}^{T} \\
\mathbf{0} & \mathbf{I}
\end{array}\right],
$$

where
$\mathbf{q}_{2}=\left[\begin{array}{cccc}S_{11} & 0 & S_{15} & S_{17} \\ 0 & 0 & 0 & 0 \\ S_{15} & 0 & S_{55} & S_{57} \\ S_{17} & 0 & S_{57} & S_{77}\end{array}\right], \quad \mathbf{r}_{2}=\left[\begin{array}{cccc}S_{16} & S_{12} & S_{14} & S_{18} \\ 0 & 0 & 0 & 0 \\ S_{56} & S_{25} & S_{45} & S_{58} \\ S_{67} & S_{27} & S_{47} & S_{78}\end{array}\right]$,
$\mathbf{t}=\left[\begin{array}{llll}S_{66} & S_{26} & S_{46} & S_{68} \\ S_{26} & S_{22} & S_{24} & S_{28} \\ S_{46} & S_{24} & S_{44} & S_{48} \\ S_{68} & S_{28} & S_{48} & S_{88}\end{array}\right]$,

$$
\text { and } \mathbf{I}_{12}=\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0  \tag{A7}\\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \quad \mathbf{I}_{2}=\operatorname{diag}\left[\begin{array}{llll}
0 & 1 & 0 & 0
\end{array}\right] .
$$

After carrying out a procedure similar to that by Ting (1996), we finally obtain:
$\mathbf{N}_{3}=-\mathbf{q}_{2}^{-1}=\frac{-1}{\Delta}\left[\begin{array}{cccc}S_{55} S_{77}-S_{57}^{2} & 0 & S_{17} S_{57}-S_{15} S_{77} & S_{15} S_{57}-S_{17} S_{55} \\ 0 & 0 & 0 & 0 \\ S_{17} S_{57}-S_{15} S_{77} & 0 & S_{11} S_{77}-S_{17}^{2} & S_{15} S_{17}-S_{11} S_{57} \\ S_{15} S_{57}-S_{17} S_{55} & 0 & S_{15} S_{17}-S_{11} S_{57} & S_{11} S_{55}-S_{15}^{2}\end{array}\right]$,
$\mathbf{N}_{1}=\mathbf{r}_{2}^{T} \mathbf{q}_{2}^{-1}-\mathbf{I}_{12}=\left[\begin{array}{cccc}r_{6} & -1 & s_{6} & t_{6} \\ r_{2} & 0 & s_{2} & t_{2} \\ r_{4} & 0 & s_{4} & t_{4} \\ r_{8} & 0 & s_{8} & t_{8}\end{array}\right]$,
$\mathbf{N}_{2}=\mathbf{t}-\mathbf{r}_{2}^{T} \mathbf{q}_{2}^{-1} \mathbf{r}_{2}=\left[\begin{array}{llll}\kappa_{66} & \kappa_{26} & \kappa_{46} & \kappa_{68} \\ \kappa_{26} & \kappa_{22} & \kappa_{24} & \kappa_{28} \\ \kappa_{46} & \kappa_{24} & \kappa_{44} & \kappa_{48} \\ \kappa_{68} & \kappa_{28} & \kappa_{48} & \kappa_{88}\end{array}\right]$,
where $\mathbf{q}_{2}^{-1}$ is the pseudo inverse of $\mathbf{q}_{2}$, and
$\Delta=\left|\begin{array}{lll}S_{11} & S_{15} & S_{17} \\ S_{15} & S_{55} & S_{57} \\ S_{17} & S_{57} & S_{77}\end{array}\right|$,
$r_{\alpha}=\frac{1}{\Delta}\left|\begin{array}{lll}S_{1 \alpha} & S_{5 \alpha} & S_{7 \alpha} \\ S_{15} & S_{55} & S_{57} \\ S_{17} & S_{57} & S_{77}\end{array}\right|, \quad s_{\alpha}=\frac{1}{\Delta}\left|\begin{array}{lll}S_{11} & S_{15} & S_{17} \\ S_{1 \alpha} & S_{52} & S_{7 \alpha} \\ S_{17} & S_{57} & S_{77}\end{array}\right|$,
$t_{\alpha}=\frac{1}{\Delta}\left|\begin{array}{lll}S_{11} & S_{15} & S_{17} \\ S_{15} & S_{55} & S_{57} \\ S_{1 \alpha} & S_{5 \alpha} & S_{7 \alpha}\end{array}\right|, \quad(\alpha=6,2,4,8)$
$\kappa_{\alpha \beta}=\frac{1}{\Delta}\left|\begin{array}{llll}S_{11} & S_{1 \alpha} & S_{15} & S_{17} \\ S_{1 \beta} & S_{\alpha \beta} & S_{5 \beta} & S_{7 \beta} \\ S_{15} & S_{5 \alpha} & S_{55} & S_{57} \\ S_{17} & S_{7 \alpha} & S_{57} & S_{77}\end{array}\right|, \quad(\alpha, \beta=6,2,4,8)$
In view of Eq. (A2), we have $S_{11}>0, S_{55}>0, S_{11} S_{55}-S_{15}^{2}>0$ and $S_{77}<0$. Consequently, if we write $\mathbf{N}_{3}$ into the following form:
$\mathbf{N}_{3}=-\left[\begin{array}{cccc}\Xi_{11} & 0 & \Xi_{13} & \Xi_{14} \\ 0 & 0 & 0 & 0 \\ \Xi_{13} & 0 & \Xi_{33} & \Xi_{34} \\ \Xi_{14} & 0 & \Xi_{34} & \Xi_{44}\end{array}\right]$,
then $\Xi_{11}>0, \Xi_{33}>0, \Xi_{11} \Xi_{33}-\Xi_{13}^{2}>0$ and $\Xi_{44}<0$.
In the following we discuss the structures of $\mathbf{N}_{i}^{(-1)}(i=1,2,3)$. It can be easily checked that:
$\mathbf{N}_{1}^{(-1)}=\left[\begin{array}{cccc}0 & * & * & * \\ -1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & *\end{array}\right], \quad \mathbf{N}_{3}^{(-1)}=\left[\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & *\end{array}\right]$.
In addition the following identity establishes:

$$
\left[\begin{array}{cc}
\mathbf{Q} & \mathbf{R}  \tag{A16}\\
\mathbf{R}^{T} & \mathbf{T}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{q} & \mathbf{r}_{1} \\
\mathbf{r}_{1}^{T} & \mathbf{t}_{1}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{I} & \mathbf{0} \\
\mathbf{I}_{12} & \mathbf{I}-\mathbf{I}_{1}
\end{array}\right],
$$

where
$\mathbf{q}=\left[\begin{array}{llll}S_{11} & S_{16} & S_{15} & S_{17} \\ S_{16} & S_{66} & S_{56} & S_{67} \\ S_{15} & S_{56} & S_{55} & S_{57} \\ S_{17} & S_{67} & S_{57} & S_{77}\end{array}\right], \quad \mathbf{r}_{1}=\left[\begin{array}{llll}0 & S_{12} & S_{14} & S_{18} \\ 0 & S_{26} & S_{46} & S_{68} \\ 0 & S_{25} & S_{45} & S_{58} \\ 0 & S_{27} & S_{47} & S_{78}\end{array}\right]$,
$\mathbf{t}_{1}=\left[\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & S_{22} & S_{24} & S_{28} \\ 0 & S_{24} & S_{44} & S_{48} \\ 0 & S_{28} & S_{48} & S_{88}\end{array}\right]$,
and
$\mathbf{I}_{1}=\operatorname{diag}\left[\begin{array}{llll}1 & 0 & 0 & 0\end{array}\right]$.
(A18)

We can finally arrive at
$\mathbf{N}_{3}^{(-1)}=\mathbf{t}_{1}^{-1}=\frac{1}{\Delta^{\prime}}\left[\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & S_{44} S_{88}-S_{48}^{2} & S_{28} S_{48}-S_{24} S_{88} & S_{24} S_{48}-S_{28} S_{44} \\ 0 & S_{28} S_{48}-S_{24} S_{88} & S_{22} S_{88}-S_{28}^{2} & S_{24} S_{28}-S_{22} S_{48} \\ 0 & S_{24} S_{48}-S_{28} S_{44} & S_{24} S_{28}-S_{22} S_{48} & S_{22} S_{44}-S_{24}^{2}\end{array}\right]$,
$\mathbf{N}_{1}^{(-1)}=\mathbf{r}_{1} \mathbf{t}_{1}^{-1}-\mathbf{I}_{12}^{T}=\left[\begin{array}{cccc}0 & r_{1}^{\prime} & s_{1}^{\prime} & t_{1}^{\prime} \\ -1 & r_{6}^{\prime} & s_{6}^{\prime} & t_{6}^{\prime} \\ 0 & r_{5}^{\prime} & s_{5}^{\prime} & t_{5}^{\prime} \\ 0 & r_{7}^{\prime} & s_{7}^{\prime} & t_{7}^{\prime}\end{array}\right]$,
$\mathbf{N}_{2}^{(-1)}=-\mathbf{q}+\mathbf{r}_{1} \mathbf{t}_{1}^{-1} \mathbf{r}_{1}^{T}=-\left[\begin{array}{llll}\kappa_{11}^{\prime} & \kappa_{16}^{\prime} & \kappa_{15}^{\prime} & \kappa_{17}^{\prime} \\ \kappa_{16}^{\prime} & \kappa_{66}^{\prime} & \kappa_{56}^{\prime} & \kappa_{67}^{\prime} \\ \kappa_{15}^{\prime} & \kappa_{56}^{\prime} & \kappa_{55}^{\prime} & \kappa_{57}^{\prime} \\ \kappa_{17}^{\prime} & \kappa_{67}^{\prime} & \kappa_{57}^{\prime} & \kappa_{77}^{\prime}\end{array}\right]$,
where
$\Delta^{\prime}=\left|\begin{array}{lll}S_{22} & S_{24} & S_{28} \\ S_{24} & S_{44} & S_{48} \\ S_{28} & S_{48} & S_{88}\end{array}\right|$,
$r_{\alpha}^{\prime}=\frac{1^{\prime}}{\Delta}\left|\begin{array}{lll}S_{2 \alpha} & S_{4 \alpha} & S_{8 \alpha} \\ S_{24} & S_{44} & S_{48} \\ S_{28} & S_{48} & S_{88}\end{array}\right|, \quad S_{\alpha}^{\prime}=\frac{1}{\Delta^{\prime}}\left|\begin{array}{lll}S_{22} & S_{24} & S_{28} \\ S_{2 \alpha} & S_{4 \alpha} & S_{8 \alpha} \\ S_{28} & S_{48} & S_{88}\end{array}\right|$,
$t_{\alpha}^{\prime}=\frac{1}{\Delta^{\prime}}\left|\begin{array}{lll}S_{22} & S_{24} & S_{28} \\ S_{24} & S_{44} & S_{48} \\ S_{2 \alpha} & S_{4 \alpha} & S_{8 \alpha}\end{array}\right|, \quad(\alpha=1,6,5,7)$
$\kappa_{\alpha \beta}^{\prime}=\frac{1}{\Delta^{\prime}}\left|\begin{array}{llll}S_{\alpha \beta} & S_{2 \beta} & S_{4 \beta} & S_{8 \beta} \\ S_{2 \alpha} & S_{22} & S_{24} & S_{28} \\ S_{4 \alpha} & S_{24} & S_{44} & S_{48} \\ S_{8 \alpha} & S_{28} & S_{48} & S_{88}\end{array}\right|, \quad(\alpha, \beta=1,6,5,7)$
In view of Eq. (A2), we have $S_{22}>0, S_{44}>0, S_{22} S_{44}-S_{24}^{2}>0$ and $S_{88}<0$. Consequently, if we write $\mathbf{N}_{3}^{(-1)}$ into the following form:
$\mathbf{N}_{3}^{(-1)}=\left[\begin{array}{cccc}0 & 0 & 0 & 0 \\ 0 & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ 0 & \Theta_{23} & \Theta_{33} & \Theta_{34} \\ 0 & \Theta_{24} & \Theta_{34} & \Theta_{44}\end{array}\right]$,
then $\Theta_{22}>0, \Theta_{33}>0, \Theta_{22} \Theta_{33}-\Theta_{23}^{2}>0$ and $\Theta_{44}<0$.
It is of interest to point out that the expressions of $\mathbf{N}_{3}$ and $\mathbf{N}_{3}^{(-1)}$ for anisotropic elastic materials in terms of the reduced elastic compliances $s_{\alpha \beta}^{\prime}$ were first obtained by Stroh (1958) and those of $\mathbf{N}_{1}$ and $\mathbf{N}_{1}^{(-1)}$ were first obtained by Ting (1988). Here we present the explicit expressions of $\mathbf{N}_{i}$ and $\mathbf{N}_{i}^{(-1)}$ for anisotropic piezoelectric materials in terms of the introduced $S_{i j}$. Thus $S_{i j}$ can be considered as the reduced generalized compliances for piezoelectric materials.

## Appendix $B$. The structures of $N_{i}$ and $N_{i}^{(-1)}$ for Scheme 2

In Scheme 2 of the Stroh formalism, the following identity is still valid:
$\mathbf{Q}_{12}=\mathbf{R}_{11}, \quad \mathbf{R}_{2 K}=\mathbf{T}_{1 K}, \quad(I, K=1,2,3,4)$
Thus the structures in Eq. (A1) for Scheme 1 are still valid for Scheme 2.

Next we introduce $S_{i j},(i, j=1-8$ and $i, j \neq 3)$ such that

$$
\begin{gather*}
{\left[\begin{array}{ccccccc}
C_{11} & C_{12} & C_{14} & C_{15} & C_{16} & h_{21} & -h_{11} \\
C_{12} & C_{22} & C_{24} & C_{25} & C_{26} & h_{22} & -h_{12} \\
C_{14} & C_{24} & C_{44} & C_{45} & C_{46} & h_{24} & -h_{14} \\
C_{15} & C_{25} & C_{45} & C_{55} & C_{56} & h_{25} & -h_{15} \\
C_{16} & C_{26} & C_{46} & C_{56} & C_{66} & h_{26} & -h_{16} \\
h_{21} & h_{22} & h_{24} & h_{25} & h_{26} & \beta_{22} & -\beta_{12} \\
-h_{11} & -h_{12} & -h_{14} & -h_{15} & -h_{16} & -\beta_{12} & \beta_{11}
\end{array}\right]} \\
\quad\left[\begin{array}{cccccccc}
S_{11} & S_{12} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\
S_{12} & S_{22} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} \\
S_{14} & S_{24} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\
S_{15} & S_{25} & S_{45} & S_{55} & S_{56} & S_{57} & S_{58} \\
S_{16} & S_{26} & S_{46} & S_{56} & S_{66} & S_{67} & S_{68} \\
S_{17} & S_{27} & S_{47} & S_{57} & S_{67} & S_{77} & S_{78} \\
S_{18} & S_{28} & S_{48} & S_{58} & S_{68} & S_{78} & S_{88}
\end{array}\right]=\mathbf{I}, \tag{B2}
\end{gather*}
$$

which can also be obtained from Eq. (A2) by using the following substitutions:

$$
\begin{align*}
& \mathbf{e}_{1 j} \rightarrow \mathbf{h}_{2 j}, \quad \mathbf{e}_{2 j} \rightarrow-\mathbf{h}_{1 j}, \quad(j=1,2)  \tag{B3}\\
& \epsilon_{11} \rightarrow-\beta_{22}, \quad \epsilon_{12} \rightarrow \beta_{12}, \quad \epsilon_{22} \rightarrow-\beta_{11}
\end{align*}
$$

It is observed from Eq. (B2) that the $7 \times 7$ real and symmetric matrix formed by $S_{i j}$ is positive definite in view of the fact that the introduced energy density function $\psi$ is convex (Suo, 1993). Once we introduce Eq. (B2), all the rest development is very similar to that in Appendix A. The only difference lies in that in scheme 2 both the two $4 \times 4$ matrices $-\mathbf{N}_{3}$ and $\mathbf{N}_{3}^{(-1)}$ are positive semidefinite, and both the two $4 \times 4$ matrices $\mathbf{N}_{2}$ and $-\mathbf{N}_{2}^{(-1)}$ are positive definite. This situation is similar to that for anisotropic elastic materials.

## Appendix C. The imperfect interface models used in Sections 3 and 4

The constitutive equations for a piezoelectric interphase of constant thickness $h$ between the upper semi-infinite anisotropic piezoelectric solid 1 and the lower semi-infinite anisotropic piezoelectric solid 2 can be equivalently written into

$$
\begin{align*}
& \boldsymbol{\sigma}_{1}=\mathbf{Q}_{c} \mathbf{u}_{11}+\mathbf{R}_{c} \mathbf{u}_{, 2} \\
& \boldsymbol{\sigma}_{2}=\mathbf{R}_{c}^{T} \mathbf{u}_{11}+\mathbf{T}_{c} \mathbf{u}_{2,} \tag{C1}
\end{align*}
$$

where $\quad \sigma_{1}=\left[\begin{array}{llll}\sigma_{11} & \sigma_{21} & \sigma_{31} & D_{1}\end{array}\right]^{T}, \quad \sigma_{2}=\left[\begin{array}{llll}\sigma_{12} & \sigma_{22} & \sigma_{32} & D_{2}\end{array}\right]^{T}$, and the subscript $c$ is used to identify the quantities associated with the interphase. All the rest notations in Eq. (C1) are the same as those adopted in Section 2.1 for Scheme 1 of the Stroh formalism.
(i) If we assume that $C_{i j k l}^{(c)} \ll C_{i j k l}^{(1)}, C_{i j k l}^{(2)} ; e_{i j k}^{(c)} \ll e_{i j k}^{(1)}, e_{i j k}^{(2)}$ and $\epsilon_{i j}^{(c)} \ll \epsilon_{i j}^{(1)}, \epsilon_{i j}^{(2)}$ (or the so-called compliant and weakly conducting interphase) and that the interphase is also very thin, then it follows from $(\mathrm{C} 1)_{2}$ that
$\boldsymbol{\sigma}_{2}^{(1)}=\boldsymbol{\sigma}_{2}^{(2)}=\frac{\mathbf{T}_{c}}{h}\left(\mathbf{u}_{1}-\mathbf{u}_{2}\right), \quad x_{2}=0$
which is equivalent to Eq. (25). As a result $\boldsymbol{\Lambda}$ in Eq. (25) is related to the electroelastic properties and the thickness of the interphase through the following:
$\boldsymbol{\Lambda}=h \mathbf{T}_{c}^{-1}=h \mathbf{N}_{2}^{(c)}$,
which clearly indicates that the property of $\boldsymbol{\Lambda}$ is exactly the same as that of $\mathbf{T}_{c}^{-1}=\mathbf{N}_{2}^{(c)}$ [or equivalently that of $\mathbf{T}_{c}$ defined in Eq. (6)]. In view of the fact that $\mathbf{T}^{E}$ is positive definite and $\epsilon_{22}>0$, then we
arrive at the conclusion that $\left[\begin{array}{lll}\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33}\end{array}\right]$ is positive definite whereas $\alpha_{44}<0$.
(ii) If we assume that $C_{i j k l}^{(c)} \gg C_{i j k l}^{(1)}, C_{i j k l}^{(2)}$ and $\epsilon_{i j}^{(c)} \gg \epsilon_{i j}^{(1)}, \epsilon_{i j}^{(2)}$ (or the so-called stiff and highly conducting interphase, it is of interest to notice that here there is no restriction on the piezoelectric constants of the interphase), then it follows from Eq. $(\mathrm{C} 1)_{2}$ that
$\mathbf{u}_{, 2}=-\mathbf{T}_{c}^{-1} \mathbf{R}_{c}^{T} \mathbf{u}_{1,}$.

Substituting the above into Eq. (C1) , we arrive at the following expression:
$\boldsymbol{\sigma}_{1}=\left(\mathbf{Q}_{c}-\mathbf{R}_{c} \mathbf{T}_{c}^{-1} \mathbf{R}_{c}^{T}\right) \mathbf{u}_{, 1}$.
By taking the derivative of both sides of Eq. (C5) with respect to $x_{1}$, and by noticing the 2 D equilibrium equations $\boldsymbol{\sigma}_{1,1}+\boldsymbol{\sigma}_{2,2}=\mathbf{0}$, we can finally obtain
$\boldsymbol{\sigma}_{2,2}=-\left(\mathbf{Q}_{c}-\mathbf{R}_{c} \mathbf{T}_{c}^{-1} \mathbf{R}_{c}^{T}\right) \mathbf{u}_{, 11}$.
If we further assume that the interphase is very thin, then we arrive at
$\boldsymbol{\sigma}_{2}^{(1)}-\boldsymbol{\sigma}_{2}^{(2)}=-\mathbf{E} \mathbf{u}_{11}, \quad \boldsymbol{x}_{2}=0$,
where the real and symmetric matrix $\mathbf{E}$ is related to the electroelastic properties and the thickness of the interphase through the following:
$\mathbf{E}=h\left(\mathbf{Q}_{c}-\mathbf{R}_{c} \mathbf{T}_{c}^{-1} \mathbf{R}_{c}^{T}\right)=-h \mathbf{N}_{3}^{(c)}$.
In view of Eq. (A14), it is then apparent that $\mathbf{E}$ can be expressed into Eq. (72) and that $E_{11}>0, E_{33}>0, E_{11} E_{33}-E_{13}^{2}>0$ and $E_{44}<0$.

## Appendix D. The imperfect interface models used in Sections 5 and 6

The constitutive equations for a piezoelectric interphase of constant thickness $h$ between the upper semi-infinite anisotropic piezoelectric solid 1 and the lower semi-infinite anisotropic piezoelectric solid 2 can be equivalently written into
$\boldsymbol{\sigma}_{1}=\mathbf{Q}_{c} \mathbf{u}_{11}+\mathbf{R}_{c} \mathbf{u}_{, 2}$,
$\boldsymbol{\sigma}_{2}=\mathbf{R}_{c}^{T} \mathbf{u}_{11}+\mathbf{T}_{c} \mathbf{u}_{, 2},{ }^{\prime}$
where $\quad \boldsymbol{\sigma}_{1}=\left[\begin{array}{llll}\sigma_{11} & \sigma_{21} & \sigma_{31} & E_{2}\end{array}\right]^{T}, \quad \boldsymbol{\sigma}_{2}=\left[\begin{array}{llll}\sigma_{12} & \sigma_{22} & \sigma_{32} & -E_{1}\end{array}\right]^{T}$, and the subscript $c$ is used to identify the quantities associated with the interphase. All the rest notations in Eq. (D1) are the same as those adopted in Section 2.2 for Scheme 2 of the Stroh formalism.
(i) If we assume that $C_{i j k l}^{(c)} \ll C_{i j k l}^{(1)}, C_{i j k l}^{(2)} ; h_{i j k}^{(c)} \ll h_{i j k}^{(1)}, h_{i j k}^{(2)}$ and $\beta_{i j}^{(c)} \ll \beta_{i j}^{(1)}, \beta_{i j}^{(2)}$ (or the so-called compliant and highly conducting interphase) and that the interphase is also very thin, then it follows that:
$\mathbf{u}_{1}-\mathbf{u}_{2}=\boldsymbol{\Lambda} \boldsymbol{\sigma}_{2}^{(1)}=\boldsymbol{\Lambda} \boldsymbol{\sigma}_{2}^{(2)}, \quad x_{2}=0$,
where $\boldsymbol{\Lambda}=h \mathbf{T}_{c}^{-1}=h \mathbf{N}_{2}^{(c)}$ is positive definite in view of the fact that $\mathbf{T}$ is positive definite.
(ii) If we assume that $C_{i j k l}^{(c)} \gg C_{i j k l}^{(1)}, C_{i j k l}^{(2)}$ and $\beta_{i j}^{(c)} \gg \beta_{i j}^{(1)}, \beta_{i j}^{(2)}$ (or the so-called stiff and weakly conducting interphase), then the following interface model establishes:
$\boldsymbol{\sigma}_{2}^{(1)}-\boldsymbol{\sigma}_{2}^{(2)}=-\mathbf{E} \mathbf{u}_{11}, \quad x_{2}=0$,
where
$\mathbf{E}=h\left(\mathbf{Q}_{c}-\mathbf{R}_{c} \mathbf{T}_{c}^{-1} \mathbf{R}_{c}^{T}\right)=-h \mathbf{N}_{3}^{(c)}$.

It is apparent that $\mathbf{E}$ is positive semidefinite in view of the fact that $-\boldsymbol{N}_{3}$ is positive semidefinite.

## References

Abramovitz, M., Stegun, I.A., 1972. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York.
Benveniste, Y., 2006. A general interface model for a three-dimensional curved thin anisotropic interphase between two anisotropic media. J. Mech. Phys. Solids 54, 708-734.
Benveniste, Y., 2009. An interface model for a three-dimensional curved thin piezoelectric interphase between two piezoelectric media. Math. Mech. Solids 14, 102-122.
Benveniste, Y., Baum, G., 2007. An interface model of a graded three-dimensional anisotropic curved interphase. Proc. R. Soc. A 463, 419-434.
Benveniste, Y., Miloh, T., 2001. Imperfect soft and stiff interfaces in twodimensional elasticity. Mech. Mater. 33, 309-323.
Bertoldi, K., Bigoni, D., Drugan, W.J., 2007a. Structural interfaces in linear elasticity. Part I: nonlocality and gradient approximations. J. Mech. Phys. Solids 55, 1-34.
Bertoldi, K., Bigoni, D., Drugan, W.J., 2007b. Structural interfaces in linear elasticity. Part II: effective properties and neutrality. J. Mech. Phys. Solids 55, 35-63.
Bhargava, R.D., Radhakrishna, H.C., 1964. Elliptic inclusion in orthotropic medium. J. Phys. Soc. Japan 19, 396-405.
Chung, M.Y., Ting, T.C.T., 1996. Piezoelectric solid with an elliptic inclusion or hole. Int. J. Solids Struct. 33, 3343-3361.
Erdogan, F., Gupta, G.D., 1972. On the numerical solution of singular integral equations. Q. Appl. Math. 30, 525-534.
Erdogan, F., Ozturk, M., 2008. On the singularities in fracture and contact mechanics. ASME J. Appl. Mech. 75, 051111. 12p.
Eshelby, J.D., 1957. The determination of the elastic fields of an ellipsoidal inclusion, and related problems. Proc. R. Soc. Lond. A 241, 376-396.
Fan, H., Sze, K.Y., 2001. A micro-mechanics model for imperfect interface in dielectric materials. Mech. Mater. 33, 363-370.
Guler, M.A., 2008. Mechanical modeling of thin films and cover plates bonded to graded substrate. ASME J. Appl. Mech. 75, 051105. 8p.
Hu, J., Nagy, P.B., 1998. On the role of interface imperfections in thermoelectric nondestructive materials characterization. Appl. Phys. Lett. 73, 127-139.
Jaswon, M.A., Bhargava, R.D., 1961. Two-dimensional elastic inclusion problems. Proc. Camb. Phil. Soc. 57, 669-690.
Jiang, X., Pan, E., 2004. Exact solution for 2D polygonal inclusion problem in anisotropic magnetoelectroelastic full-, half-, and bimaterials. Int. J. Solids Struct. 41, 4361-4382.
Lothe, J., Barnett, D.M., 1976. Integral formalism for surface waves in piezoelectric crystals. J. Appl. Phys. 47, 1799-1807.
Lu, S.Y., Lin, C.Y., 2003. Determination of effective conductivities of imperfect contact composites with first-passage simulation. Phys. Rev. B68, 056705.
Lu, T., Cao, W., 2002. Generalized continuum theory for ferroelectric thin films. Phys. Rev. B66, 024102.
Malanganti, R., Sharma, P., 2005. A review of strain field calculations in embedded quantum dots and wires. In: Rieth, M., Schommers, W. (Eds.), Handbook of Theoretical and Computational Nanotechnology, pp. 1-44.
Mura, T., 1987. Micromechanics of Defects in Solids, second revised ed. Kluwer Academic Publishers, Dordrecht.
Muskhelishvili, N.I., 1963. Some Basic Problems of the Mathematical Theory of Elasticity. P. Noordhoff Ltd., The Netherlands.
Nagy, P.B., 1992. Ultrasonic classification of imperfect interfaces. J. Nondestructive Eval. 11, 127-139.
Needleman, A., 1990. An analysis of decohesion along an imperfect interface. Int. J. Fract 42, 21-40.
Ni, L., Nemat-Nasser, S., 2000. Bridged interface cracks in anisotropic bimaterials. Philos. Mag. 80, 2675-2693.
Melkumyan, A., Mai, Y.W., 2008. Influence of imperfect bonding on interface waves guided by piezoelectric/piezomagnetic composites. Philos. Mag. 88, 467-469.
Nozaki, H., Taya, M., 2001. Elastic fields in a polyhedral inclusion with uniform eigenstrains and related problems. ASME J. Appl. Mech. 68, 441-452.
Ovid'ko, I.A., Sheinerman, A.G., 2005. Elastic fields of inclusions in nanocomposite solids. Rev. Adv. Mater. Sci. 9, 17-33.
Pan, E., 2001. Exact solution for simply supported and multilayered magneto-electro-elastic plates. ASME J. Appl. Mech. 68, 608-618.
Pan, E., 2004. Eshelby problem of polygonal inclusions in anisotropic piezoelectric full- and half-planes. J. Mech. Phys. Solids 52, 567-589.
Ru, C.Q., 1999. Effect of electric polarization saturation on stress intensity factors in a piezoelectric ceramic. Int. J. Solids Struct. 36, 869-883.
Ru, C.Q., 2000. Eshelby's problem for two-dimensional piezoelectric inclusions of arbitrary shape. Proc. R. Soc. Lond. A 456, 1051-1068.
Ru, C.Q., 2001. A two-dimensional Eshelby problem for two bonded piezoelectric half-planes. Proc. R. Soc. Lond. A 457, 865-883.
Stroh, A.N., 1958. Dislocations and cracks in anisotropic elasticity. Philos. Mag. 3, 625-646.
Suo, Z., Kuo, C.M., Barnett, D.M., Willis, J.R., 1992. Fracture mechanics for piezoelectric ceramics. J. Mech. Phys. Solids 40, 739-765.
Suo, Z., 1993. Models for breakdown-resistant dielectric and ferroelectric ceramics. J. Mech. Phys. Solids 41, 1155-1176.

Ting, T.C.T., 1988. Some identities and the structure of $N_{i}$ in the Stroh formalism of anisotropic elasticity. Q. Appl. Math. 46, 109-120.

Ting, T.C.T., 1996. Anisotropic Elasticity - Theory and Applications. Oxford University Press, New York.
Wang, X., 1994. Trial Discussions on the Mathematical Structure of Inclusion, Dislocation and Crack. Master Thesis, Xi'an Jiaotong Univ.
Wang, X., Pan, E., 2007. Magnetoelectric effects in multiferroic fibrous composite with imperfect interface. Phys. Rev. B76, 214107.
Wang, X., Sudak, L.J., Ru, C.Q., 2007. Elastic fields in two imperfectly bonded halfplanes with a thermal inclusion of arbitrary shape. Zeitschrift fur angewandte Mathematik und Physik (ZAMP) 58, 488-509.
Wang, X., Pan, E., Albrecht, J.D., 2008. Two-dimensional Green's functions in anisotropic multiferroic bimaterials with a viscous interface. J. Mech. Phys. Solids 56, 2863-2875.

Willis, J.R., 1964. Anisotropic elastic inclusion problems. Q. J. Mech. Appl. Math. 17 157-174.
Yang, H.C., Chou, Y.T., 1976. Generalized plane problems of elastic inclusions in anisotropic solids. ASME J. Appl. Mech. 43, 424-430.
Yang, H.C., Chou, Y.T., 1977. Anisotropic strain problems of an elliptic inclusion in an anisotropic medium. ASME J. Appl. Mech. 44, 437-444.
Yoon, J., Ru, C.Q., Mioduchovski, A., 2006. Effect of a thin surface coating layer on thermal stresses within an elastic half-plane. Acta Mech. 185, 227-243.
Yu, H.Y., Sanday, S.C., 1991. Elastic field in jointed semi-infinite solids with an inclusion. Proc. R. Soc. Lond. A 434, 521-530.
Zhang, H.T., Chou, Y.T., 1985. Eigenstrain problem for two-phase anisotropic materials. ASME J. Appl. Mech. 52, 87-90.


[^0]:    * Corresponding author. Tel.: +1 3028310378.

    E-mail address: xuwang_sun@hotmail.com (X. Wang).

