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# Analysis of cracked transversely isotropic and inhomogeneous solids by a special BIE formulation 

C.Y. Dong ${ }^{\text {a }}, \mathrm{X}$. Yang $^{\mathrm{a}}$, E. Pan ${ }^{\mathrm{b}, *}$<br>${ }^{\text {a }}$ Department of Mechanics, School of Aerospace Engineering, Beijing Institute of Technology, China<br>${ }^{\mathrm{b}}$ Department of Civil Engineering, University of Akron, Akron, OH, USA

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#### Abstract

In this paper, a special boundary integral equation (BIE) formation is proposed to analyze the fracture problem in transversely isotropic and inhomogeneous solids. In this formulation, the single-domain boundary element method (BEM) is utilized to discretize the cracked matrix and the displacement BEM to the surface of the embedded inhomogeneity. The two regions are then connected through the continuity conditions along their joint interface. The conventional and three special nine-node quadrilateral elements are utilized to discretize the inhomogeneity-matrix interface and the crack surface. From the crack-opening displacements on the crack surface, the mixed-mode stress intensity factors (SIFs) are calculated, using the well-known asymptotic expression in terms of the Barnett-Lothe tensor. In the numerical analysis, the distance between the inhomogeneity and the crack as well as the orientation of the isotropic plane of the transversely isotropic media is varied to show their influences on the mixed-mode SIFs along the crack fronts.


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## 1. Introduction

Mechanical behaviors of heterogeneous materials such as composites, rock structures, porous and cracked media have been widely investigated, using various boundary integral-related methods. Bush [1] investigated the interaction between a crack and a particle cluster in composites, using the boundary element method (BEM). Also applying the BEM, Knight et al. [2] analyzed the effects of the constituent material properties, fibre spatial distribution and microcrack damage on the localized behavior of fibre-reinforced composites. Dong et al. [3,4] presented a generalpurpose integral formulation in order to study the interaction between the inhomogeneity and crack embedded in two-dimensional (2D) and three-dimensional (3D) isotropic matrices. Based on a symmetric-Galerkin BEM, Kitey et al. [5] investigated the crack growth behavior in materials embedded with a cluster of inhomogeneities. Phan et al. [6] used the symmetric-Galerkin BEM to calculate the stress intensity factors (SIFs) for the 2D crack-inhomogeneity interaction problem. Lee and Tran [7] applied the Eshelby equivalent inclusion method to carry out the stress analysis, when a penny-shaped crack interacts with inhomogeneities and voids. Interface cracks in two or more

[^0]isotropic materials were also studied by Sladek and Sladek [8] and Liu and Xu [9].

So far, however, only a few studies exist when the inhomogeneous material is of anisotropy, e.g., transverse isotropy. Berger and Tewary [10] studied the interface crack problems in 2D anisotropic bimaterials. Huang and Liu [11] used the eigenstrain method to obtain the elastic fields around the inclusion and further studied the interactive energy in the system. Pan and Yuan [12] investigated the fracture mechanics problems in 3D anisotropic solids, using the combined displacement and traction integral representations (i.e., the single-domain BEM). Ariza and Dominguez [13] obtained the boundary traction integral equation for cracked 3D transversely isotropic bodies, in which explicit expressions for the fundamental traction derivatives were presented. Yue et al. [14] calculated the 3D SIFs of an inclined square crack within a bimaterial cuboid, using the single-domain BEM. Chen et al. $[15,16]$ studied the fracture behavior of a cracked transversely isotropic cuboid also using 3D BEM. Benedetti et al. [17] presented a fast dual BEM for cracked 3D problems.

While the interaction between the inhomogeneities and cracks embedded in a transversely isotropic medium is important, there is no existing literature on this topic. Therefore, in this paper, the effect of a spherical inhomogeneity on the SIFs of a square-shaped crack, both being embedded in a transversely isotropic matrix, is studied using a special BIM formulation. The influence of the distance between the inhomogeneity and the square-shaped
crack and the material orientation on the SIFs of the crack fronts is discussed.

## 2. Boundary integral equations

We consider the general case where a transversely isotropic inhomogeneity is embedded in a cracked infinite matrix of transverse isotropy. In order to study the effect of the inhomogeneity on the SIFs of the crack, a special BIE formulation is presented. In our formulation, the displacement and traction boundary integral equations [12]

$$
\begin{align*}
b_{i j} u_{j}\left(y_{S}\right)= & \int_{S} U_{i j}\left(y_{S}, x_{S}\right) t_{j}\left(x_{S}\right) d S\left(x_{S}\right)-\int_{S} T_{i j}\left(y_{S}, x_{S}\right) u_{j}\left(x_{S}\right) d S\left(x_{S}\right) \\
& -\int_{\Gamma^{+}} T_{i j}\left(y_{S}, x_{\Gamma^{+}}\right)\left[u_{j}\left(x_{\Gamma^{+}}\right)-u_{j}\left(x_{\Gamma^{-}}\right)\right] d \Gamma\left(x_{\Gamma^{+}}\right)+u_{i}^{0}\left(y_{S}\right) \tag{1}
\end{align*}
$$

$$
\begin{align*}
& {\left[t_{l}\left(y_{\Gamma^{+}}\right)-t_{l}\left(y_{\Gamma^{-}}\right)\right] / 2+n_{m}\left(y_{\Gamma^{+}}\right) \int_{S} c_{l m i k} T_{i j, k}\left(y_{\Gamma^{+}}, x_{S}\right) u_{j}\left(x_{S}\right) d S\left(x_{S}\right)} \\
& \quad+n_{m}\left(y_{\Gamma^{+}}\right) \int_{\Gamma} c_{l m i k} T_{i j, k}\left(y_{\Gamma^{+}}, x_{\Gamma^{+}}\right)\left[u_{j}\left(x_{\Gamma^{+}}\right)-u_{j}\left(x_{\Gamma^{-}}\right)\right] d \Gamma\left(x_{\Gamma^{+}}\right) \\
& \quad=n_{m}\left(y_{\Gamma^{+}}\right) \int_{S} c_{l m i k} U_{i j, k}^{*}\left(y_{\Gamma^{+}}, x_{S}\right) t_{j}\left(x_{S}\right) d S\left(x_{S}\right)+\left[t_{l}^{0}\left(y_{\Gamma^{+}}\right)-t_{l}^{0}\left(y_{\Gamma^{-}}\right)\right] / 2 \tag{2}
\end{align*}
$$

are applied to the cracked matrix. In Eqs. (1) and (2), $b_{i j}$ are coefficients that depend only on the local geometry of the inhomogeneity-matrix interface $S$ at $y_{s}$. A point on the positive (or negative) side of the crack is denoted by $x_{\Gamma^{+}}$(or $x_{\Gamma^{-}}$), and on the inhomogeneity-matrix interface $S$ by both $x_{S}$ and $y_{S} ; n_{m}$ is the unit outward normal of the positive side of the crack surface at $y_{\Gamma^{+}} ; c_{l m i k}$ is the fourth-order stiffness tensor of the material; $u_{i}^{0}\left(y_{s}\right)$ is the displacement component along the $i$-direction at point $y_{S}$ caused by a given uniform remote loading, and $t_{l}^{0}\left(y_{\Gamma^{+}}\right)$and $t_{l}^{0}\left(y_{\Gamma^{-}}\right)$ are the corresponding traction components along $l$-direction at points $y_{\Gamma^{+}}$and $y_{\Gamma^{-}}$and $t_{i}$ are the displacements and tractions on the inhomogeneity-matrix interface $S$ (or the crack surface $\Gamma$ ); $U_{i j}$ and $T_{i j}$ are the Green's functions of the displacements and tractions; $U_{i j, k}$ and $T_{i j, k}$ are, respectively, the derivatives of the Green's displacements and tractions with respect to the source point. The displacement and traction Green's functions are taken from Pan and Chou [18], whilst their derivatives are taken from Pan and Yuan [12].

The displacement integral equation is applied to the surface of the inhomogeneity as follows:
$b_{i j} u_{j}\left(y_{S}\right)=\int_{S} U_{i j}\left(y_{S}, x_{S}\right) t_{j}\left(x_{S}\right) d S\left(x_{S}\right)-\int_{S} T_{i j}\left(y_{S}, x_{S}\right) u_{j}\left(x_{S}\right) d S\left(x_{S}\right)$

Eqs. (1)-(3) then can be used to investigate the effect of the inhomogeneity on the SIFs of the crack embedded in a transversely isotropic matrix. In discretization of these equations, we apply nine-node quadrilateral curved elements as shown in Fig. 1 to the inhomogeneity-matrix interface and the crack surface with the crack front being discretized by special elements. For any point within each element on the inhomogeneity-matrix interface, the global coordinates, displacements and tractions can be expressed, in terms of the element type I (Fig. 1), as [12,15,16]
$x_{i}=\sum_{k=1}^{9} \phi_{k} x_{i}^{k}, \quad u_{i}=\sum_{k=1}^{9} \phi_{k} u_{i}^{k}, \quad t_{i}=\sum_{k=1}^{9} \phi_{k} t_{i}^{k}, \quad i=1,2,3$
where the subscript $i$ is the Cartesian coordinate component; the superscript $k$ is the nodal number; $\phi_{k}(k=1-9)$ are the shape functions (of the local coordinates $\xi_{1}$ and $\xi_{2}$ ), which are given in





Fig. 1. Four types of elements employed for the discretization of the crack surface [12], where the dash line represents the crack front.

Pan and Yuan [12]; $x_{i}^{k}, t_{i}^{k}, u_{i}^{k}$ are, respectively, the coordinates, tractions and displacements at nodal point $k$.

Similarly, the crack-opening displacements (CODs) $\Delta u_{i}$ ( $=$ $\left.u_{i}\left(x_{\Gamma^{+}}\right)-u_{i}\left(x_{\Gamma^{-}}\right)\right)$on the crack surface can be expressed as
$\Delta u_{i}=\sum_{k=1}^{9} \phi_{k} \Delta u_{i}^{k}, \quad i=1,2,3$
where $\Delta u_{i}^{k}$ are the crack-opening displacements at nodal point $k$. For the crack elements away from the crack front, the shape functions $\phi_{k}(k=1-9)$ are the same as those in Eq. (4). However, for the crack element near the crack front, the corresponding shape functions need to be modified. In other words, the shape functions near the crack front should be multiplied by suitable weight functions to represent the nearfield behavior of the crack. For the element type II shown in Fig. 1, the CODs can be expressed as
$\Delta u_{i}=\sum_{k=1}^{9} \sqrt{1+\xi_{2}} \phi_{k} \Delta u_{i}^{k}, \quad i=1,2,3$ for type II

For the element types III and IV shown in Fig. 1, the CODs have the following expressions
$\Delta u_{i}=\sum_{k=1}^{9} \sqrt{\left(1+\xi_{1}\right)\left(1+\xi_{2}\right)} \phi_{k} \Delta u_{i}^{k}, \quad i=1,2,3$ for type III
$\Delta u_{i}=\sum_{k=1}^{9} \sqrt{\left(1-\xi_{1}\right)\left(1+\xi_{2}\right)} \phi_{k} \Delta u_{i}^{k}, \quad i=1,2,3$ for type IV

We point out that element types II-IV are in general called non-conforming elements, employed to better approximate the field behavior. The concept of this type of elements was introduced and discussed in [19-23]. We further mention that while in this paper, the concerned nodes are fixed at $2 / 3$, other locations, such as the quarter point, could be selected with equal efficiency.

Taking each node in turn as the collocation point and performing the involved integrals, we finally obtain the compact forms of the discretized equations from Eqs. (1)-(3) as

$$
\left[\begin{array}{ll}
\mathbf{H}_{11} & \mathbf{H}_{12}  \tag{8}\\
\mathbf{H}_{21} & \mathbf{H}_{22}
\end{array}\right]\left[\begin{array}{l}
\mathbf{U}_{m} \\
\Delta \mathbf{U}_{c}
\end{array}\right]+\left[\begin{array}{l}
\mathbf{B}_{1} \\
\mathbf{B}_{2}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{G}_{11} & \mathbf{G}_{12} \\
\mathbf{G}_{21} & \mathbf{G}_{22}
\end{array}\right]\left[\begin{array}{l}
\mathbf{T}_{m} \\
\mathbf{T}_{c}
\end{array}\right]
$$

and
$\mathbf{H}_{i} \mathbf{U}_{i}=\mathbf{G}_{i} \mathbf{T}_{i}$
where the subscripts $i$ and $m$ represent, respectively, the inhomogeneity and matrix; $\mathbf{H}$ and $\mathbf{G}$ are, respectively, the
influence coefficient matrices containing integrals of the fundamental Green's function solutions; $\mathbf{B}_{1}$ and $\mathbf{B}_{2}$ are, respectively, the displacement and traction vectors induced by the remote loading; $\mathbf{U}_{m}\left(\mathbf{U}_{i}\right)$ and $\mathbf{T}_{m}\left(\mathbf{T}_{i}\right)$ are, respectively, the node displacement and traction vectors on the matrix side (inhomogeneity side) of the inhomogeneity-matrix interface; $\Delta \mathbf{U}_{c}$ and $\mathbf{T}_{c}$ are, respectively, the discontinuous displacement and traction vectors over the crack surface. In this paper, we assume that the tractions on both sides of the crack are equal and opposite. Therefore $\mathbf{T}_{c}$ is equal to zero.

Using the continuity condition of the displacement and traction vectors along the interface, i.e., $\mathbf{U}_{m}=\mathbf{U}_{i}$ and $\mathbf{T}_{m}=-\mathbf{T}_{i}$, between the inhomogeneity and matrix, we can combine Eqs. (8) and (9) into

$$
\left[\begin{array}{ll}
\mathbf{H}_{11}+\mathbf{G}_{11} \mathbf{G}_{i}^{-1} \mathbf{H}_{i} & \mathbf{H}_{12}  \tag{10}\\
\mathbf{H}_{21}+\mathbf{G}_{21} \mathbf{G}_{i}^{-1} \mathbf{H}_{i} & \mathbf{H}_{22}
\end{array}\right]\left\{\begin{array}{c}
\mathbf{U}_{m} \\
\Delta \mathbf{U}_{c}
\end{array}\right\}=-\left\{\begin{array}{c}
\mathbf{B}_{1} \\
\mathbf{B}_{2}
\end{array}\right\}
$$

Therefore, once the unknowns $\mathbf{U}_{m}$ and $\Delta \mathbf{U}_{c}$ are solved, the SIFs ( $K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{III}}$ ) along the crack front can be evaluated, using the following asymptotic expression [12]
$\left\{\begin{array}{l}\Delta u_{1} \\ \Delta u_{2} \\ \Delta u_{3}\end{array}\right\}=2 \sqrt{\frac{2 r}{\pi}} \mathbf{L}^{-1}\left\{\begin{array}{l}K_{I I} \\ K_{I} \\ K_{I I I}\end{array}\right\}$
where $r$ is the distance behind the crack front; $\mathbf{L}$ is the BarnettLothe tensor [24], which depends only on the anisotropic properties of the solid in the local crack-front coordinates; $\Delta u_{1}$, $\Delta u_{2}$ and $\Delta u_{3}$ are the relative CODs in the local crack-front coordinates.

## 3. Numerical examples

We study the effect of a spherical inhomogeneity on the SIFs along the crack fronts of a square-shaped crack. Both the inhomogeneity and crack are embedded in an infinite matrix, which is under a far-field stress $\sigma^{\infty}=1.0 \mathrm{GPa}$ in the $z$-direction. The side length of the square is $2 a(=2.0 \mathrm{~m})$. The radius of the sphere is $R=1.0 \mathrm{~m}$ and it is made of transversely isotropic marble with the following elastic properties: $E_{X}=90 \mathrm{GPa}, E_{Z}=55$ $\mathrm{GPa}, v_{X Y}=v_{Y Z}=0.3, G_{Y Z}=21 \mathrm{GPa}[15,16]$. The matrix material properties are $E_{X}=12 \mathrm{GPa}, E_{Z}=4 \mathrm{GPa}, v_{X Y}=v_{Y Z}=0.3, G_{Y Z}=1.6 \mathrm{GPa}$. We should point out that all these coefficients are with respect to the local material coordinates with $X, Y$ and $Z$ being, respectively, along the longitudinal, transverse and normal directions of the $X-Y$ plane of isotropy. The space-fixed global coordinates ( $x, y, z$ ) can be related to ( $X, Y, Z$ ), using the orientation and inclined angles $\beta$ and $\Psi$ between them. In other words, the transformation relation between the local ( $X, Y, Z$ ) and global $(x, y, z)$ coordinates is as follows [25]
$\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{lll}-\cos \psi \sin \beta & \cos \beta & \sin \psi \sin \beta \\ -\cos \psi \cos \beta & -\sin \beta & \sin \psi \cos \beta \\ \sin \psi & 0 & \cos \psi\end{array}\right]\left[\begin{array}{l}X \\ Y \\ Z\end{array}\right]$
In the numerical analysis, 24 nine-node quadrilateral elements with 98 nodes (Fig. 2a) and 100 nine-node quadrilateral elements with 441 nodes are employed to discretize the inhomogeneitymatrix interface and the square-shaped crack surface (Fig. 3 below), respectively. A refined mesh with 386 nodes ( 96 elements, Fig. 2b) is also used to discretize the inhomogeneitymatrix interface to check the accuracy of the numerical solution. It is found that SIFs from both refined and coarse meshes are nearly the same (to the third decimal number) and therefore, only the results from the coarse mesh are discussed. We consider two
a

b


Fig. 2. Discretization of a spherical inhomogeneity-matrix interface with 24 ninenode quadrilateral elements (98 nodes) in (a) and with 96 elements ( 386 nodes) in (b).


Fig. 3. A spherical inhomogeneity and a square-shaped crack within an infinite matrix under a far-field stress. The distance between the inhomogeneity and crack is $d$ in the $x$-direction. The $x-z$ plane view in (a) and the $x-y$ plane view in (b). The crack fronts $A B, B C, C D$ and $D A$ are denoted, respectively, by $(-1,1),(1,3),(3,5)$ and $(5,7)$ in the SIF plots.
different relative orientations of the inhomogeneity and crack, and they are discussed below separately.
3.1. The spherical inhomogeneity and square-shaped crack are both in the $x-y$ plane, separated by a distance $d$ in the $x$-direction

The relative locations and orientations of the spherical inhomogeneity and square-shaped crack are shown in Fig. 3. For varying distance $d$ but fixed $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ for both the inhomogeneity and the matrix, the normalized SIF $K I=$ $K_{I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ along the crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square is shown in Fig. 4 (The crack fronts $A B, B C, C D$ and $D A$ are denoted, respectively, by $(-1,1),(1,3),(3,5)$ and $(5,7)$ in all SIF plots). It is obvious that as $d$ decreases, the SIF along the crack


Fig. 4. The normalized SIF $K_{I}$ along the square-shaped crack fronts ( $-1,1$ ), ( 1,3 ), $(3,5)$ and $(5,7)$ for different sphere-square distance $d$ with fixed material orientations $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ for both the spherical inhomogeneity and matrix (Hereafter, $(-1,1),(1,3),(3,5)$ and $(5,7)$ denote, respectively, the crack fronts AB, $B C, C D$ and $D A$ ).


Fig. 5. The normalized SIF $K_{1}$ along the square-shaped crack fronts ( $-1,1$ ), ( 1,3 ), $(3,5)$ and $(5,7)$ for different material orientations $\beta$ and $\Psi$ of the matrix with fixed distance $d=0.5 \mathrm{~m}$, and fixed angles $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ for the inhomogeneity.
front DA (which closes to the inhomogeneity) is significantly decreased, while the SIFs along the other crack fronts (i.e., $\mathrm{AB}, \mathrm{BC}$ and CD) are nearly insensitive to $d$.

For fixed distance $d=0.5 \mathrm{~m}$, fixed $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ for the inhomogeneity, but different angles $\beta$ and $\Psi$ for the matrix, the normalized SIF $K I=K_{I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ along crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square crack is shown in Fig. 5. It is observed that with increasing angle $\Psi$, the SIF $K_{\mathrm{I}}$ along the crack fronts AB and CD decreases, while it increases along the crack fronts $B C$ and $D A$. The maximum SIF $K_{I}$ appears in the middle of the crack front BC , approximately equal to 0.9 , whilst the minimum $K_{\mathrm{I}}$ appears in the middle of the crack fronts AB and CD , approximately equal to 0.6.

Fig. 6 shows the effect of the material orientations $\beta$ and $\psi$ of the inhomogeneity on the SIF $K_{\mathrm{I}}$ along the crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square crack. In this example, the distance is fixed at $d=0.5 \mathrm{~m}$ and the orientations of the matrix are fixed at $\beta=0^{\circ}$ and $\Psi=0^{\circ}$. Contrary to Fig. 5, where the SIF $K_{\mathrm{I}}$ is very sensitive to the matrix anisotropy, here the SIF $K_{\mathrm{I}}$ is nearly independent of the inhomogeneity anisotropy.

For fixed $d=0.5 \mathrm{~m}$, fixed $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ for the inhomogeneity and fixed $\beta=0^{\circ}$ and $\Psi=45^{\circ}$ for the matrix, the normalized SIFs $K I I=K_{I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ and $K I I I=K_{I I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ along the crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square crack is shown in Fig. 7. It is


Fig. 6. The normalized SIF $K_{\mathrm{I}}$ along the square-shaped crack fronts $(-1,1),(1,3)$, $(3,5)$ and $(5,7)$ for different material orientations $\beta$ and $\Psi$ of the inhomogeneity with fixed distance $d=0.5 \mathrm{~m}$, and fixed angles $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of the matrix.


Fig. 7. The normalized SIFs $K_{\text {II }}$ and $K_{\text {III }}$ along the square-shaped crack fronts $(-1,1),(1,3),(3,5)$ and $(5,7)$ for fixed distance $d=0.5 \mathrm{~m}$, fixed angles $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of the inhomogeneity, and fixed angles $\beta=0^{\circ}$ and $\Psi=45^{\circ}$ of the matrix.


Fig. 8. The normalized SIF $K_{\mathrm{I}}$ along the square-shaped crack fronts $(-1,1),(1,3)$, $(3,5)$ and $(5,7)$ for fixed distance $d=0.1 \mathrm{~m}$, but with different material anisotropy pairs for the inhomogeneity and matrix.
observed that the variation of the SIFs $K_{\text {II }}$ and $K_{\text {III }}$ along the crack front is more complicated than the SIF $K_{\mathrm{l}}$.

The effect of material anisotropy on the SIFs is further studied by comparing to the corresponding isotropic case. Shown in Fig. 8
is the normalized SIF $K_{\mathrm{I}}$ along the crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square for fixed $d=0.1 \mathrm{~m}$ with various material pairs. In this figure, $\operatorname{Iso}(m)-\operatorname{Iso}(i)$ denotes the case, where both the inhomogeneity and matrix are of isotropy with $E=4 \mathrm{GPa}$ and $v=0.25$; $\operatorname{Tr}(m)-\operatorname{Iso}(i)$ denotes the case, where the matrix is of transverse isotropy with $E_{X}=12 G P a, E_{Z}=4 G P a, v_{X Y}=v_{Y Z}=0.3, G_{Y Z}=1.6 G P a$, whilst the inhomogeneity is of isotropy with $\mathrm{E}=4 \mathrm{GPa}$ and $v=0.25 ; \operatorname{Iso}(m)-\operatorname{Tr}$ (i) denotes the case, where the matrix is of isotropy with $E=4 \mathrm{GPa}$ and $v=0.25$, whilst the inhomogeneity is of transverse isotropy with $E_{X}=12 \mathrm{GPa}, E_{Z}=4 \mathrm{GPa}, v_{X Y}=v_{Y Z}=0.3$, $G_{Y Z}=1.6 \mathrm{GPa} ; \operatorname{Tr}(m)-\operatorname{Tr}(i)$ denotes the case, where both the inhomogeneity and matrix are of transverse isotropy with $E_{X}=12 G P a, E_{Z}=4 G P a, v_{X Y}=v_{Y Z}=0.3, G_{Y Z}=1.6 \mathrm{GPa}$. The effect of material anisotropy on the SIF $K_{\mathrm{I}}$ can be clearly observed from Fig. 8, where the SIF $K_{\mathrm{I}}$ corresponding to material pair $\operatorname{Tr}(m)-\operatorname{Tr}(i)$ (i.e., both the inhomogeneity and matrix are of transverse isotropy) is smaller than those corresponding to other material pairs. Particularly along the crack front AD, even the behavior of the SIF $K_{\mathrm{I}}$ variation for the material pair $\operatorname{Tr}(m)-\operatorname{Tr}(i)$ is different, as also observed in Fig. 4.
3.2. The spherical inhomogeneity and square-shaped crack are in the $x-y$ plane, separated by a distance $d$ in the $z$-direction.

The relative locations and orientations of the spherical inhomogeneity and square-shaped crack are shown in Fig. 9. All the material parameters, mesh size and remote loading are the same as those in the first case (see Section 3.1) (Fig. 3). For different distance $d$ and fixed $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of both the inhomogeneity and the matrix, the normalized SIF $K I=K_{I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ along crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square is shown in Fig. 10 (again, the crack fronts AB, BC, CD and DA are denoted, respectively, by $(-1,1),(1,3),(3,5)$ and $(5,7)$ in all SIF plots). It is observed from Fig. 10 that the SIF $K_{I}$ distribution of the crack fronts $A B, B C$ and $C D$ is symmetrical with respect to the middle point of each crack front, as expected. Also for this case, different to the first case (see Section 3.1), the normalized SIFs


Fig. 9. A spherical inhomogeneity and a square-shaped crack within an infinite matrix under a far-field stress. The distance between the inhomogeneity and the crack is $d$ in the $z$-direction. The $x-z$ plane view in (a) and the $x-y$ plane view in (b). The crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA are denoted, respectively, by $(-1,1),(1,3),(3,5)$ and $(5,7)$ in the SIF plots.


Fig. 10. The normalized SIF $K_{\mathrm{I}}$ along the square-shaped crack fronts ( $-1,1$ ), ( 1,3 ), $(3,5)$ and $(5,7)$ for different distance $d$ and fixed material orientations $\beta=0^{\circ}$ and $\Psi=0^{\circ}$, for both the inhomogeneity and matrix.


Fig. 11. The normalized SIFs $K_{\text {II }}$ and $K_{\text {III }}$ along the square-shaped crack fronts $(-1,1),(1,3),(3,5)$ and $(5,7)$ for different distance $d$ and fixed material orientations $\beta=0^{\circ}$ and $\Psi=0^{\circ}$, for both the inhomogeneity and matrix.


Fig. 12. The normalized SIF $K_{I}$ along the square-shaped crack fronts ( $-1,1$ ), ( 1,3 ), $(3,5)$ and $(5,7)$ for different material orientations $\beta$ and $\Psi$ of the matrix with fixed distance $d=0.5 \mathrm{~m}$, and fixed angles $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of the inhomogeneity.
$K I I=K_{I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ and $K I I I=K_{I I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ along the crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square are nonzero, as shown in Fig. 11.

For fixed $d=0.5 \mathrm{~m}$, fixed $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of the inhomogeneity and different angles $\beta$ and $\Psi$ of the matrix, the SIF


Fig. 13. The normalized SIF $K_{\text {II }}$ along the square-shaped crack fronts ( $-1,1$ ), ( 1,3 ), $(3,5)$ and $(5,7)$ for different material orientations $\beta$ and $\Psi$ of the matrix with fixed distance $d=0.5 \mathrm{~m}$, and fixed angles $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of the inhomogeneity.


Fig. 14. The normalized SIF $K_{\text {III }}$ along the square-shaped crack fronts ( $-1,1$ ), ( 1,3 ), $(3,5)$ and $(5,7)$ for different material orientations $\beta$ and $\Psi$ of the matrix with fixed distance $d=0.5 \mathrm{~m}$, and fixed angles $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of the inhomogeneity.
$K I=K_{I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ along the crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square is shown in Fig. 12. It is observed that the distribution of the SIF $K_{\mathrm{I}}$ is similar to that in the first case (see Section 3.1) (Fig. 5). In other words, with increasing angle $\Psi$, the SIF $K_{\mathrm{I}}$ along the crack fronts AB and CD decreases, whilst the SIF $K_{\mathrm{I}}$ along the crack fronts BC and DA increases. The maximum value of $K_{\mathrm{I}}$ appears in the middle of the crack fronts $B C$ and $D A$ and is approximately equal to 1.0 , whilst the minimum value of $K_{\mathrm{I}}$ appears in the middle of the crack fronts $A B$ and $C D$, with a value equal to 0.65 . For fixed $d=0.5 \mathrm{~m}$, fixed $\beta=0^{\circ}$ and $\Psi=0^{\circ}$ of the inhomogeneity and different values of $\beta$ and $\Psi$ of the matrix, the normalized SIFs $K I I=K_{I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ and $K I I I=K_{I I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ along the crack fronts $\mathrm{AB}, \mathrm{BC}, \mathrm{CD}$ and DA of the square are shown in Figs. 13 and 14 . It is obvious that relatively larger SIFs $K I I=K_{I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ and $K I I I=K_{I I I} /\left(\sigma^{\infty} \sqrt{\pi a}\right)$ are observed for fixed $\beta=0^{\circ}$ and $\Psi=45^{\circ}$ of the matrix.

## 4. Conclusions

A special BIE formulation is developed for the study of the fracture problem in a transversely isotropic and heterogeneous
medium. In this formulation, the single-domain BEM is applied to the cracked matrix, whilst the displacement BEM to the surface of the inhomogeneity. The continuity conditions along the inhomo-geneity-matrix interface are then used to derive the final system of equations. In the numerical analysis, four sets of nine-node quadrilateral elements are applied to discretize the inhomogene-ity-matrix interface and the square-shaped crack surface. The mixed-mode SIFs are calculated from the solved discontinuous displacements on the crack surface. The effect of the distance between the inhomogeneity and the crack as well as the material anisotropy on the SIFs of crack fronts is investigated. It is observed that accurate SIFs can be obtained with 24 nine-node quadrilateral elements to the spherical surface and 100 elements to the square-shaped crack surface. It is believed that the proposed formulation could be applied to study more complicated interaction problems between inhomogeneities and cracks in 3D anisotropic media.

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[^0]:    * Corresponding author.

    E-mail address: pan2@uakron.edu (E. Pan).

