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A circular inhomogeneity with interface slip and diffusion under in-plane deformation

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ABSTRACT

We consider the in-plane deformation of a circular elastic inhomogeneity embedded in an infinite elastic matrix subjected to remote uniform stresses or uniform heat flow. The inhomogeneity and matrix have different material properties. The rate-dependent slip and mass transport by stress-driven diffusion concurrently occur on the inhomogeneity/matrix interface. For the remote uniform stress case, it is observed that the internal stresses within the inhomogeneity are quadratic functions of the coordinates x and y, and decay with two relaxation times. Interestingly the average mean stress within the circular inhomogeneity is in fact time-independent. As time approaches infinity, the internal stress field within the inhomogeneity becomes uniform and hydrostatic. In addition the change of strain energy due to the introduction of the circular elastic inhomogeneity is derived, containing various existing results as special cases. Furthermore, a simple condition leading to an internal uniform stress state within the inhomogeneity is found. This condition, which is independent of the elastic properties of the inhomogeneity and matrix, gives a simple relationship between the interface drag and diffusion parameters. For the remote heat flow case, the internal thermal stresses are linear functions of the coordinates x and y and decay only with a single relaxation time. Numerical results are presented to demonstrate the obtained solution and the corresponding physics.

1. Introduction

It has been observed that slip and diffusion on the inhomogeneity/matrix interface at high homologous temperatures can reduce the creep strength of the composite [1,2] and will cause stress relaxation in the composite or polycrystal [3–7]. The imperfect interface can now be more precisely described as follows: (i) the slip rate is proportional, in terms of the interface drag parameter, to the interfacial shear stress; and (ii) the normal displacement jump rate is proportional, in terms of the interface diffusion parameter, to a certain differential expression of the interfacial normal stress. Interface slip is due to the viscous nature of the thin interphase; whilst interface diffusion is induced by the gradient in the normal stress along the interface [1–4,8]. Furthermore the heterogeneity in interface diffusion and drag parameters in polycrystals can induce transient high local stress concentrations [8]. To simplify the analysis involved, it was assumed in previous modeling attempts that the inhomogeneity is rigid [1,2,6,7], or the interfacial shear stress is fully relaxed by the interface slip [3], or the inhomogeneity and the matrix have exactly the same elastic moduli [4,5].

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Nomenclature

$A(t)$, $B(t)$	time-dependent complex constants
A_0, B_0	real constants defined by Eq. (81) for heat flow case
D	non-negative interface diffusion parameter
f	volume fraction of the inhomogeneities
F_x, F_y	resultant forces in x- and y-directions
K_{12}^*	effective plane-strain bulk modulus
k_t	thermal conductivity
L	interface between the inhomogeneity and matrix
P, Q	time-independent constants
q_0	remote uniform heat flow
r	radial variable in polar coordinates
R	radius of the cylindrical inhomogeneity
S_1	inhomogeneity domain
S_2	matrix domain
t	time variable
t_0	relaxation time defined by Eq. (80) for heat flow case
t_1, t_2	relaxation times defined by Eq. (25), notice that $t_1 \geqslant t_2 \geqslant 0$
x, y	coordinate variables in the (x,y)-plane
z	complex variable $(z = x + iy)$
α_t	thermal expansion coefficient
α and β	dimensionless parameters defined by Eq. (10)
$\phi(z,t)$ and $\psi(z,t)$	analytic functions of complex variable z and time t
μ, v	shear modulus and Poisson's ratio
$\kappa = 3 - 4v$	parameter for plane strain
$\kappa = (3 - v)/(1 + v)$	parameter for plane stress
θ	angular variable in polar coordinates
η	non-negative interface drag parameter
$\left(\sigma_{xx}^{\infty}, \sigma_{yy}^{\infty}, \sigma_{xy}^{\infty}\right)$	remote uniform stress field
γ	parameter in the dimension of time, defined by Eq. (15)
Γ and Γ'	remote-stress related constants defined by Eq. (6)
χ	parameter in the dimension of time, defined by Eq. (11)
ΔW	change of strain energy per unit length due to the introduction of the inhomogeneity
μ_{12}^*	effective in-plane shear modulus
σ_{xx}, σ_{yy}, σ_{xy}	in-plane stress field in Cartesian coordinates
σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$	in-plane stress field in polar coordinates
ΔW_{SD}	change of strain energy per unit length in Eq. (73)
$\Delta W_S = \Delta W_D$	change of strain energy per unit length in Eq. (74)
ΔW_p	change of strain energy per unit length in Eq. (75)
ΔW_C	change of strain energy per unit length in Eq. (76)

This research is devoted to a rigorous study of the two-dimensional circular elastic inhomogeneity bonded to an infinite matrix through a sharp interface on which both slip and diffusion concurrently occur. By means of the complex variable method ([9,10]), a closed-form solution is developed for the loading case when the matrix is subjected to remote uniform in-plane stresses. As a byproduct, the inhomogeneity/matrix system under a remote uniform heat flow is also solved. Since our solution is very general, various important features associated with this type of imperfect interface are discussed. These features should aid our understanding of the material or composite behaviors which containing such an interface.

2. Formulation

As shown in Fig. 1, we consider a domain in $\Re^2$, infinite in extent, containing a single circular elastic inhomogeneity with elastic properties different than those of the surrounding matrix. The inhomogeneity S_1: $x^2 + y^2 < R^2$ is bonded to the matrix S_2: $x^2 + y^2 > R^2$ through a sharp interface L: $x^2 + y^2 = R^2$. The matrix is subjected to an in-plane remote uniform stress field $(\sigma_{xx}^{\infty}, \sigma_{yy}^{\infty}, \sigma_{xy}^{\infty})$ (Fig. 1a). The subscripts 1 and 2 (or the superscripts (1) and (2)) will refer to the regions S_1 and S_2, respectively.

For an in-plane deformation of an isotropic elastic material, the stresses, displacements and resultant forces can be expressed in terms of two analytic functions $\phi(z,t)$ and $\psi(z,t)$ as [9]

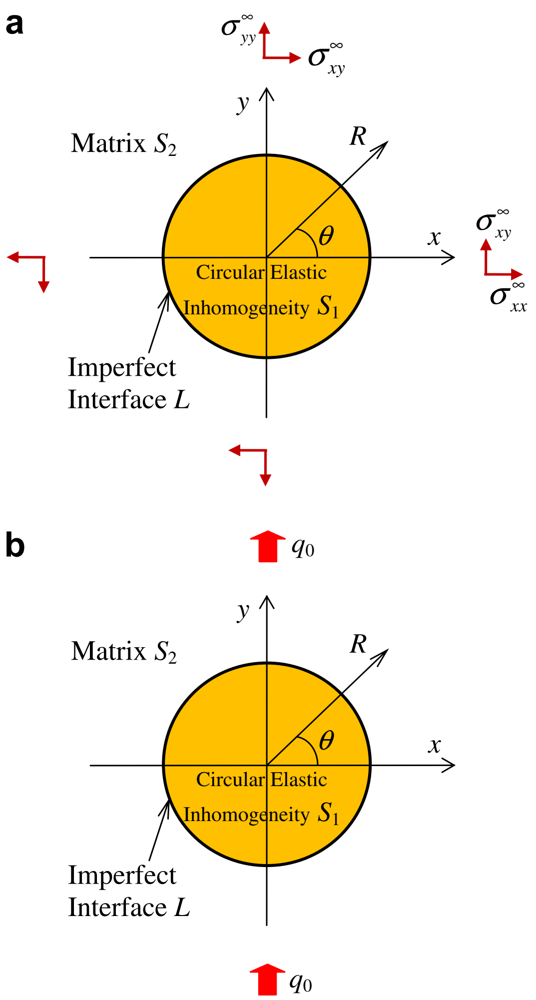

Fig. 1. A circular elastic inhomogeneity bonded to an infinite elastic matrix through a sharp interface on which slip and diffusion concurrently occur. The elastic matrix is subjected to an in-plane remote uniform stress field $\left(\sigma_{xx}^{\infty}, \sigma_{yy}^{\infty}, \sigma_{xy}^{\infty}\right)$ in (a) or to a remote uniform heat flow q_0 in (b).

$$\begin{aligned}
&\sigma_{rr} + \sigma_{\theta\theta} = 2[\phi'(z,t) + \overline{\phi'(z,t)}],\\
&\sigma_{rr} - i\sigma_{r\theta} = \phi'(z,t) + \overline{\phi'(z,t)} - e^{2i\theta}[\bar{z}\phi''(z,t) + \psi'(z,t)],\\
&2\mu(u_r + iu_\theta) = e^{-i\theta}\left[\kappa\phi(z,t) - z\overline{\phi'(z,t)} - \overline{\psi(z,t)}\right],\\
&F_x + iF_y = -i\left[\phi(z,t) + z\overline{\phi'(z,t)} + \overline{\psi(z,t)}\right],
\end{aligned} \tag{1}$$

where $\kappa = 3 - 4\nu$ for plane-strain deformation, which is assumed in this investigation, and $\kappa = (3 - \nu)/(1 + \nu)$ for plane-stress deformation; μ and ν, where $\mu > 0$ and $0 \leqslant \nu \leqslant 0.5$, are the shear modulus and Poisson's ratio, respectively; t is the real time variable, whilst $z = x + iy = re^{i\theta}$ is the complex variable. The appearance of the real time in the above expression is solely due to the rate-dependent interface slip and diffusion on L. The interface slip and diffusion boundary conditions can be expressed as [2–4,8]

$$\sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} = \eta\left(\dot{u}_\theta^{(2)} - \dot{u}_\theta^{(1)}\right), \quad \frac{D}{R^2}\frac{\mathrm{d}^2\sigma_{rr}^{(1)}}{\mathrm{d}\theta^2} = \frac{D}{R^2}\frac{\mathrm{d}^2\sigma_{rr}^{(2)}}{\mathrm{d}\theta^2} = \dot{u}_r^{(1)} - \dot{u}_r^{(2)}, \quad \text{on} \quad L, \tag{2}$$

where an overdot denotes the derivative with respect to time t, η is the non-negative interface drag parameter, and D is the non-negative interface diffusion parameter. Here we assume that both η and D are constant along the circular interface. For the shearing component, it is noted that $\eta = 0$ corresponds to a slipping interface where the shear stress is zero whilst $\eta \to \infty$ corresponds to the condition where the shear traction and displacement are continuous along the interface. On the other hand, for the normal component, it is noted that $D = 0$ corresponds to the condition where the normal traction and displacement are continuous along the interface whilst $D \to \infty$ corresponds to the "opening interface" where the normal stress is zero.

3. Analytical solutions

To facilitate the analysis involved, we first introduce the following analytic continuations

$$\phi_i(z,t) = -z\overline{\phi'}_i(R^2/z,t) - \bar{\psi}_i(R^2/z,t), \quad i = 1,2. \tag{3}$$

As a result, the continuity condition of tractions across the interface can be simply expressed as:

$$\phi_1^+(z,t) - \phi_1^-(z,t) = \phi_2^-(z,t) - \phi_2^+(z,t), \quad \text{on} \quad |z| = R, \tag{4}$$

where the superscripts "+" and "−" denote the limit values from the inhomogeneity and matrix sides of the interface $|z| = R$, respectively.

By employing the Liouville's theorem, we arrive at the following expression defined in the whole complex z-plane

$$\phi_1(z,t) + \phi_2(z,t) = \left[\Gamma - \overline{\phi_1'(0,t)}\right]z - R^2\overline{\Gamma'}z^{-1}, \tag{5}$$

where Γ and Γ' are two time-independent constants related to the remote uniform stresses through

$$\Gamma = \frac{\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty}}{4}, \quad \Gamma' = \frac{\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty} + 2i\sigma_{xy}^{\infty}}{2}. \tag{6}$$

Meanwhile the interface slip condition can be expressed as:

$$\mathrm{Im}\{\phi_1'^{+}(z,t) - \phi_1'^{-}(z,t)\} = \frac{\eta R}{2\mu_2}\mathrm{Im}\left\{z^{-1}\left[\kappa_2\dot{\phi}_2^-(z,t) + \dot{\phi}_2^+(z,t)\right]\right\} - \frac{\eta R}{2\mu_1}\mathrm{Im}\left\{z^{-1}\left[\kappa_1\dot{\phi}_1^+(z,t) + \dot{\phi}_1^-(z,t)\right]\right\}, \quad \text{on} \quad |z| = R. \tag{7}$$

Substituting Eq. (5) into the above expression and also applying the Liouville's theorem, we arrive at

$$\dot{\phi}_1(z,t) - \alpha\frac{z^2}{R^2}\bar{\dot{\phi}}_1(R^2/z,t) + \frac{1}{\chi}z\left[\phi_1'(z,t) + \overline{\phi_1'}(R^2/z,t)\right] = \left[(1+\alpha)\overline{\dot{\phi}_1'(0,t)} + 2i\beta\mathrm{Im}\left\{\dot{\phi}_1'(0,t)\right\}\right]z, \quad (|z| < R), \tag{8}$$

$$\alpha\dot{\phi}_1(z,t) - \frac{z^2}{R^2}\bar{\dot{\phi}}_1(R^2/z,t) - \frac{1}{\chi}z\left[\phi_1'(z,t) + \bar{\phi}_1'(R^2/z,t)\right] = -(1+\alpha)\overline{\dot{\phi}_1'(0,t)}z, \quad (|z| > R), \tag{9}$$

where α and β are two dimensionless parameters given by

$$\alpha = \frac{\kappa_2\mu_1 + \mu_2}{\kappa_1\mu_2 + \mu_1}, \quad \beta = \frac{\kappa_2\mu_1 + \mu_1}{\kappa_1\mu_2 + \mu_1}, \quad \left(\frac{1}{3} \leqslant \frac{1}{\kappa_1} \leqslant \alpha \leqslant \kappa_2 \leqslant 3\right) \tag{10}$$

and χ is defined by

$$\chi = \frac{\eta R(\kappa_1\mu_2 + \mu_1)}{2\mu_1\mu_2}, \tag{11}$$

which has the time dimension [1].

The interface diffusion condition can be expressed as:

$$\begin{aligned}\frac{D}{R^2}\frac{\mathrm{d}^2\left[\mathrm{Re}\{\phi_1'^{+}(z,t) - \phi_1'^{-}(z,t)\}\right]}{\mathrm{d}\theta^2} &= \frac{R}{2\mu_1}\mathrm{Re}\left\{z^{-1}\left[\kappa_1\dot{\phi}_1^+(z,t) + \dot{\phi}_1^-(z,t)\right]\right\} \\ &\quad - \frac{R}{2\mu_2}\mathrm{Re}\left\{z^{-1}\left[\kappa_2\dot{\phi}_2^-(z,t) + \dot{\phi}_2^+(z,t)\right]\right\}, \quad \text{on} \quad |z| = R.\end{aligned} \tag{12}$$

Substituting Eq. (5) into the above expression and also applying the Liouville's theorem, we finally arrive at

$$\begin{aligned}&\dot{\phi}_1(z,t) + \alpha\frac{z^2}{R^2}\bar{\dot{\phi}}_1(R^2/z,t) + \frac{1}{4\gamma}\left[z^2\phi_1''(z,t) + z^3\phi_1'''(z,t) - R^2\overline{\phi_1''}(R^2/z,t) - \frac{R^4}{z}\overline{\phi_1'''}(R^2/z,t)\right] \\ &\quad = \left[(\alpha-1)\overline{\dot{\phi}_1'(0,t)} - 2\beta\mathrm{Re}\left\{\dot{\phi}_1'(0,t)\right\}\right]z, \quad (|z| < R),\end{aligned} \tag{13}$$

$$\alpha\dot{\phi}_1(z,t) + \frac{z^2}{R^2}\bar{\dot{\phi}}_1(R^2/z,t) - \frac{1}{4\gamma}\left[z^2\phi_1''(z,t) + z^3\phi_1'''(z,t) - R^2\overline{\phi_1''}(R^2/z,t) - \frac{R^4}{z}\overline{\phi_1'''}(R^2/z,t)\right] = (1-\alpha)\overline{\dot{\phi}_1'(0,t)}z, \quad (|z| > R), \tag{14}$$

where γ is defined by

$$\gamma = \frac{R^3(\kappa_1\mu_2 + \mu_1)}{8\mu_1\mu_2 D}, \tag{15}$$

which also has the time dimension [1].

The compatibility condition between Eqs. (8) and (9) yields

$$\mathrm{Im}\left\{\dot{\phi}_1'(0,t)\right\} = 0, \tag{16}$$

and the compatibility condition between Eqs. (13) and (14) yields

$$\mathrm{Re}\left\{\dot{\phi}_1'(0,t)\right\} = 0. \tag{17}$$

Eqs. (16) and (17) imply that $\phi_1'(0,t)$ is in fact time-independent. Thus we can write $\phi_1'(0,t) \equiv \phi_1'(0) = \frac{\beta\Gamma}{1-\alpha+\beta}$ due to the fact that the interface is *perfect* at the initial time $t = 0$. Utilizing this result, Eqs. (8), (9), (13) and (14) can be simplified into

$$\dot{\phi}_1(z,t) - \alpha\frac{z^2}{R^2}\dot{\bar{\phi}}_1(R^2/z,t) + \frac{1}{\chi}z\left[\phi_1'(z,t) + \bar{\phi}_1'(R^2/z,t)\right] = 0, \quad (|z| < R), \tag{18}$$

$$\alpha\dot{\phi}_1(z,t) - \frac{z^2}{R^2}\dot{\bar{\phi}}_1(R^2/z,t) - \frac{1}{\chi}z\left[\phi_1'(z,t) + \bar{\phi}_1'(R^2/z,t)\right] = 0, \quad (|z| > R), \tag{19}$$

$$\dot{\phi}_1(z,t) + \alpha\frac{z^2}{R^2}\dot{\bar{\phi}}_1(R^2/z,t) + \frac{1}{4\gamma}\left[z^2\phi_1''(z,t) + z^3\phi_1'''(z,t) - R^2\overline{\phi_1''}(R^2/z,t) - \frac{R^4}{z}\overline{\phi_1'''}(R^2/z,t)\right] = 0, \quad (|z| < R), \tag{20}$$

$$\alpha\dot{\phi}_1(z,t) + \frac{z^2}{R^2}\dot{\bar{\phi}}_1(R^2/z,t) - \frac{1}{4\gamma}\left[z^2\phi_1''(z,t) + z^3\phi_1'''(z,t) - R^2\overline{\phi_1''}(R^2/z,t) - \frac{R^4}{z}\overline{\phi_1'''}(R^2/z,t)\right] = 0, \quad (|z| > R). \tag{21}$$

The analytic function $\phi_1(z,t)$ in its original region $|z| < R$ and in its continuation region $|z| > R$ is assumed to take the following forms

$$\begin{aligned} &\phi_1(z,t) = \tfrac{\beta\Gamma}{1-\alpha+\beta}z + A(t)z^3, \quad (|z| < R),\\ &\phi_1(z,t) = -\tfrac{\beta\Gamma}{1-\alpha+\beta}z + R^4\overline{B(t)}z^{-1}, \quad (|z| > R), \end{aligned} \tag{22}$$

where $A(t)$ and $B(t)$ are two time-dependent complex constants to be determined. Consequently we arrive at the following state-space equation

$$\begin{bmatrix} \chi & -\chi\alpha \\ \gamma & \gamma\alpha \end{bmatrix}\begin{bmatrix} \dot{A}(t) \\ \dot{B}(t) \end{bmatrix} = -\begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} A(t) \\ B(t) \end{bmatrix}, \tag{23}$$

whose solution can be easily given by

$$\begin{aligned} &A(t) = P(\chi\alpha - t_1)\exp\left(-\tfrac{t}{t_1}\right) + Q(\chi\alpha - t_2)\exp\left(-\tfrac{t}{t_2}\right),\\ &B(t) = P(\chi - 3t_1)\exp\left(-\tfrac{t}{t_1}\right) + Q(\chi - 3t_2)\exp\left(-\tfrac{t}{t_2}\right), \end{aligned} \tag{24}$$

where P and Q are two time-independent constants determined by the initial conditions, and t_1 and t_2 are the two relaxation times determined by

$$t_{1,2} = \frac{(3\alpha+1)(\chi+\gamma) \pm \sqrt{(3\alpha+1)^2(\chi+\gamma)^2 - 48\alpha\chi\gamma}}{12}, \tag{25}$$

where $t_1 \geqslant t_2 \geqslant 0$.

Because at the initial time the interface is perfect, we then arrive at the following explicit expressions of $\phi_1(z,t)$ in its original region $|z| < R$ and in its continuation region $|z| > R$ as

$$\begin{aligned} &\phi_1(z,t) = \tfrac{\beta\Gamma}{1-\alpha+\beta}z + \frac{\beta\Gamma'(\chi\alpha-t_1)(\chi\alpha-t_2)\left[\exp\left(-\frac{t}{t_1}\right)-\exp\left(-\frac{t}{t_2}\right)\right]}{\chi\alpha R^2(3\alpha-1)(t_1-t_2)}z^3, \quad (|z| < R),\\ &\phi_1(z,t) = -\tfrac{\beta\Gamma}{1-\alpha+\beta}z + \frac{\beta R^2\overline{\Gamma'}\left[(\chi-3t_1)(\chi\alpha-t_2)\exp\left(-\frac{t}{t_1}\right)-(\chi-3t_2)(\chi\alpha-t_1)\exp\left(-\frac{t}{t_2}\right)\right]}{\chi\alpha(3\alpha-1)(t_1-t_2)}z^{-1}, \quad (|z| > R). \end{aligned} \tag{26}$$

Then the explicit expressions of $\phi_2(z,t)$ in its original region $|z| > R$ and in its continuation region $|z| < R$ can be easily obtained as

$$\begin{aligned}
&\phi_2(z,t) = -R^2\bar{\Gamma}'z^{-1} + \tfrac{1-\alpha-\beta}{1-\alpha+\beta}\Gamma z - \frac{\beta\Gamma'(\chi\alpha-t_1)(\chi\alpha-t_2)\left[\exp\left(-\frac{t}{t_1}\right)-\exp\left(-\frac{t}{t_2}\right)\right]}{\chi\alpha R^2(3\alpha-1)(t_1-t_2)}z^3, \quad (|z|<R),\\
&\phi_2(z,t) = \Gamma z - R^2\overline{\Gamma'}\left[1+\frac{\beta\left[(\chi-3t_1)(\chi\alpha-t_2)\exp\left(-\frac{t}{t_1}\right)-(\chi-3t_2)(\chi\alpha-t_1)\exp\left(-\frac{t}{t_2}\right)\right]}{\chi\alpha(3\alpha-1)(t_1-t_2)}\right]z^{-1}, \quad (|z|>R).
\end{aligned} \tag{27}$$

Thus, the two original analytic functions $\psi_1(z,t)$ and $\psi_2(z,t)$ can be obtained as

$$\psi_1(z,t) = \frac{\beta\Gamma'\left[[(3\alpha+1)\chi-6t_1](\chi\alpha-t_2)\exp\left(-\frac{t}{t_1}\right)-[(3\alpha+1)\chi-6t_2](\chi\alpha-t_1)\exp\left(-\frac{t}{t_2}\right)\right]}{\alpha\chi(1-3\alpha)(t_1-t_2)}z, \quad (|z|<R), \tag{28}$$

$$\begin{aligned}
\psi_2(z,t) = {} & \Gamma' z - \frac{2(1-\alpha)R^2\Gamma}{1-\alpha+\beta}z^{-1}\\
& - R^4\bar{\Gamma}'\left[1+\frac{\beta\left[(\chi\alpha-t_2)[\chi(1-\alpha)-2t_1]\exp\left(-\frac{t}{t_1}\right)-(\chi\alpha-t_1)[\chi(1-\alpha)-2t_2]\exp\left(-\frac{t}{t_2}\right)\right]}{\chi\alpha(3\alpha-1)(t_1-t_2)}\right]z^{-3}, \quad (|z|>R).
\end{aligned} \tag{29}$$

It is observed that the internal stress components within the inhomogeneity are quadratic functions of the coordinates x and y when $t > 0$ and decay with the two relaxation times t_1 and t_2. In addition the average mean stress within the inhomogeneity is given by

$$\left(\sigma_{xx}^{(1)}+\sigma_{yy}^{(1)}\right)_{\text{Ave}} = \frac{\beta(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})}{1-\alpha+\beta}, \tag{30}$$

which is in fact time-independent. Furthermore, it can be easily shown that, as time approaches infinity, the internal stress field within the inhomogeneity is uniform and hydrostatic such that

$$\sigma_{xx}^{(1)} = \sigma_{yy}^{(1)} = \frac{\beta(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})}{2(1-\alpha+\beta)}, \quad \sigma_{xy}^{(1)} = 0, \quad \text{for} \quad r<R \quad \text{as} \quad t\to\infty. \tag{31}$$

With the derived analytical displacement and stress fields, the strain energy change can be also obtained. This can be done by using the Eshelby's formula [11–14]. Actually, the change of strain energy per unit length ΔW due to the introduction of the circular inhomogeneity with the remote uniform stresses being held constant can be finally evaluated as

$$\begin{aligned}
\frac{\Delta W}{\pi R^2} = {} & \frac{(1-\alpha)(1-\nu_2)(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})^2}{4\mu_2(1-\alpha+\beta)} + \frac{(1-\nu_2)\left[(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})^2+4(\sigma_{xy}^{\infty})^2\right]}{2\mu_2}\\
& \times\left[1+\frac{\beta\left[(\chi-3t_1)(\chi\alpha-t_2)\exp\left(-\frac{t}{t_1}\right)-(\chi-3t_2)(\chi\alpha-t_1)\exp\left(-\frac{t}{t_2}\right)\right]}{\chi\alpha(3\alpha-1)(t_1-t_2)}\right],
\end{aligned} \tag{32}$$

This important result can be employed to predict the effective property of the corresponding composite. We assume that the composite contains a dilute and random dispersion of the same circular inhomogeneities with the same interface conditions. Then the effective moduli of the composite, i.e., the effective plane-strain bulk modulus K_{12}^* and effective in-plane shear modulus μ_{12}^* can be easily obtained as

$$\frac{1}{K_{12}^*} = \frac{1-2\nu_2}{\mu_2} + \frac{2f(1-\alpha)(1-\nu_2)}{\mu_2(1-\alpha+\beta)}, \tag{33}$$

$$\frac{1}{\mu_{12}^*} = \frac{1}{\mu_2} + \frac{4f(1-\nu_2)}{\mu_2}\left[1+\frac{\beta\left[(\chi-3t_1)(\chi\alpha-t_2)\exp\left(-\frac{t}{t_1}\right)-(\chi-3t_2)(\chi\alpha-t_1)\exp\left(-\frac{t}{t_2}\right)\right]}{\chi\alpha(3\alpha-1)(t_1-t_2)}\right], \tag{34}$$

where f is the volume fraction of the inhomogeneities. Eqs. (33) and (34) clearly indicate that the effective plane-strain bulk modulus is in fact time-independent, whereas the time-dependent effective in-plane shear modulus decays with the two relaxation times t_1 and t_2. The time-independence of the effective bulk modulus lies in the fact that the hydrostatic part of the remote stresses produces uniform normal stress and vanishing tangential stress along the entire circular interface which cannot be relaxed or influenced by the interface slip and diffusion [3]. In addition the effective in-plane shear modulus at the initial time $t = 0$ can be determined from Eq. (34) as

$$\frac{1}{\mu_{12}^*} = \frac{1}{\mu_2} + \frac{4f(1-\nu_2)}{\mu_2}\left(1-\frac{\beta}{\alpha}\right). \tag{35}$$

It can be strictly proved that Eqs. (33) and (35) recover the classical dilute result [12,15] and are consistent with the Mori–Tanaka predictions [16] for small values of f ($f \ll 1$).

4. Discussions

Besides the brief discussion on the effective property, our solution contains various important results associated with the interface behavior. Many of these results are new and should be very useful to our understanding of the material/composite features involving imperfect interface.

4.1. The interface diffusion is absent ($\gamma \to \infty$)

In this case the two relaxation times can be determined as

$$t_1 \to \infty, \quad t_2 = \frac{2\alpha\chi}{3\alpha+1}. \tag{36}$$

Consequently the two analytic functions $\phi_1(z,t)$ and $\psi_1(z,t)$ defined in the inhomogeneity are

$$\begin{aligned} \phi_1(z,t) &= \frac{\beta\Gamma}{1-\alpha+\beta} z - \frac{\beta\Gamma'\left[1-\exp\left(-\frac{t}{t_2}\right)\right]}{R^2(3\alpha+1)} z^3, \\ \psi_1(z,t) &= \frac{\beta\Gamma'\left[6\alpha+(1-3\alpha)\exp\left(-\frac{t}{t_2}\right)\right]}{\alpha(3\alpha+1)} z, \end{aligned} \quad (|z|<R) \tag{37}$$

and the two analytic functions $\phi_2(z,t)$ and $\psi_2(z,t)$ defined in the matrix are

$$\begin{aligned} \phi_2(z,t) &= \Gamma z + \frac{R^2\bar{\Gamma}'}{3\alpha+1}\left[3(\beta-\alpha) - 1 + \frac{\beta}{\alpha}\exp\left(-\frac{t}{t_2}\right)\right] z^{-1}, \\ \psi_2(z,t) &= \Gamma' z - \frac{2(1-\alpha)R^2\Gamma}{1-\alpha+\beta} z^{-1} + \frac{R^4\bar{\Gamma}'}{3\alpha+1}\left[2\beta - 3\alpha - 1 + \frac{\beta(1+\alpha)}{\alpha}\exp\left(-\frac{t}{t_2}\right)\right] z^{-3}, \end{aligned} \quad (|z|>R). \tag{38}$$

In this case, the internal stresses within the inhomogeneity are given by

$$\begin{aligned} \sigma_{xx}^{(1)} &= \frac{\beta(\sigma_{xx}^\infty+\sigma_{yy}^\infty)}{2(1-\alpha+\beta)} + \frac{\beta\left(\sigma_{xx}^\infty-\sigma_{yy}^\infty\right)}{2\alpha(3\alpha+1)}\left[6\alpha+(1-3\alpha)\exp\left(-\frac{t}{t_2}\right)\right] + \frac{6\beta\left[y^2(\sigma_{yy}^\infty-\sigma_{xx}^\infty)+2xy\sigma_{xy}^\infty\right]}{R^2(3\alpha+1)}\left[1-\exp\left(-\frac{t}{t_2}\right)\right], \\ \sigma_{yy}^{(1)} &= \frac{\beta\left(\sigma_{xx}^\infty+\sigma_{yy}^\infty\right)}{2(1-\alpha+\beta)} + \frac{\beta\left(\sigma_{yy}^\infty-\sigma_{xx}^\infty\right)}{2\alpha(3\alpha+1)}\left[6\alpha+(1-3\alpha)\exp\left(-\frac{t}{t_2}\right)\right] + \frac{6\beta\left[x^2\left(\sigma_{xx}^\infty-\sigma_{yy}^\infty\right)+2xy\sigma_{xy}^\infty\right]}{R^2(3\alpha+1)}\left[1-\exp\left(-\frac{t}{t_2}\right)\right], \\ \frac{\sigma_{xy}^{(1)}}{\sigma_{xy}^\infty} &= \frac{\beta\left[6\alpha+(1-3\alpha)\exp\left(-\frac{t}{t_2}\right)\right]}{\alpha(3\alpha+1)} - \frac{6\beta(x^2+y^2)}{R^2(3\alpha+1)}\left[1-\exp\left(-\frac{t}{t_2}\right)\right], \quad (x^2+y^2<R^2) \end{aligned} \tag{39}$$

and the tractions are distributed along the interface as

$$\begin{aligned} \sigma_{rr}^{(1)} = \sigma_{rr}^{(2)} &= \frac{\beta\left(\sigma_{xx}^\infty+\sigma_{yy}^\infty\right)}{2(1-\alpha+\beta)} + \frac{\beta\left[6\alpha+(1-3\alpha)\exp\left(-\frac{t}{t_2}\right)\right]\left[\left(\sigma_{xx}^\infty-\sigma_{yy}^\infty\right)\cos 2\theta+2\sigma_{xy}^\infty\sin 2\theta\right]}{2\alpha(3\alpha+1)}, \\ \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} &= \frac{\beta\left[\left(\sigma_{yy}^\infty-\sigma_{xx}^\infty\right)\sin 2\theta+2\sigma_{xy}^\infty\cos 2\theta\right]}{2\alpha}\exp\left(-\frac{t}{t_2}\right), \end{aligned} \quad \text{on} \quad L, \tag{40}$$

which clearly indicates that the interfacial shear stress $\sigma_{r\theta}$ decays to zero as the time approaches infinity.

In addition the change of strain energy due to the introduction of the inhomogeneity can be obtained as

$$\frac{\Delta W}{\pi R^2} = \frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^\infty+\sigma_{yy}^\infty\right)^2}{4\mu_2(1-\alpha+\beta)} + \frac{(1-\nu_2)\left[\left(\sigma_{yy}^\infty-\sigma_{xx}^\infty\right)^2+4\left(\sigma_{xy}^\infty\right)^2\right]}{2\mu_2(3\alpha+1)}\left[3(\alpha-\beta)+1-\frac{\beta}{\alpha}\exp\left(-\frac{t}{t_2}\right)\right]. \tag{41}$$

4.2. The interface slip is absent ($\chi \to \infty$)

In this case the two relaxation times can be determined as

$$t_1 \to \infty, \quad t_2 = \frac{2\alpha\gamma}{3\alpha+1}. \tag{42}$$

Remark 1. When the inhomogeneity is rigid ($\mu_1 \to \infty$), the above expression reduces to

$$t_2 = \frac{R^3(3-4\nu_2)}{8D\mu_2(5-6\nu_2)}, \tag{43}$$

which is just the result obtained by He and Zhao [6]. We point out that there is a typo in [6]. M appearing in Eq. (32) should be in the denominator, but not in the numerator.

When the interface slip is absent, the two analytic functions $\phi_1(z,t)$ and $\psi_1(z,t)$ defined in the inhomogeneity are

$$\begin{aligned}\phi_1(z,t) &= \tfrac{\beta\Gamma}{1-\alpha+\beta}z + \frac{\beta\Gamma'\left[1-\exp\left(-\frac{t}{t_2}\right)\right]}{R^2(3\alpha+1)}z^3,\\ \psi_1(z,t) &= \tfrac{\beta\Gamma'}{\alpha}\exp\left(-\tfrac{t}{t_2}\right)z,\end{aligned} \quad (|z|<R) \tag{44}$$

and the two analytic functions $\phi_2(z,t)$ and $\psi_2(z,t)$ defined in the matrix are

$$\begin{aligned}\phi_2(z,t) &= \Gamma z + \tfrac{R^2\bar{\Gamma}'}{3\alpha+1}\left[3(\beta-\alpha)-1+\tfrac{\beta}{\alpha}\exp\left(-\tfrac{t}{t_2}\right)\right]z^{-1},\\ \psi_2(z,t) &= \Gamma' z - \tfrac{2(1-\alpha)R^2\Gamma}{1-\alpha+\beta}z^{-1} + \tfrac{R^4\bar{\Gamma}'}{3\alpha+1}\left[4\beta-3\alpha-1+\tfrac{\beta(1-\alpha)}{\alpha}\exp\left(-\tfrac{t}{t_2}\right)\right]z^{-3},\end{aligned} \quad (|z|>R). \tag{45}$$

In this case, the internal stresses within the inhomogeneity are given by

$$\begin{aligned}\sigma_{xx}^{(1)} &= \tfrac{\beta\left(\sigma_{xx}^\infty+\sigma_{yy}^\infty\right)}{2(1-\alpha+\beta)} + \tfrac{\beta\left(\sigma_{xx}^\infty-\sigma_{yy}^\infty\right)}{2\alpha}\exp\left(-\tfrac{t}{t_2}\right) + \tfrac{6\beta\left[y^2\left(\sigma_{xx}^\infty-\sigma_{yy}^\infty\right)-2xy\sigma_{xy}^\infty\right]}{R^2(3\alpha+1)}\left[1-\exp\left(-\tfrac{t}{t_2}\right)\right],\\ \sigma_{yy}^{(1)} &= \tfrac{\beta\left(\sigma_{xx}^\infty+\sigma_{yy}^\infty\right)}{2(1-\alpha+\beta)} + \tfrac{\beta\left(\sigma_{yy}^\infty-\sigma_{xx}^\infty\right)}{2\alpha}\exp\left(-\tfrac{t}{t_2}\right) + \tfrac{6\beta\left[x^2\left(\sigma_{yy}^\infty-\sigma_{xx}^\infty\right)-2xy\sigma_{xy}^\infty\right]}{R^2(3\alpha+1)}\left[1-\exp\left(-\tfrac{t}{t_2}\right)\right], \quad (x^2+y^2<R^2)\\ \tfrac{\sigma_{xy}^{(1)}}{\sigma_{xy}^\infty} &= \tfrac{\beta}{\alpha}\exp\left(-\tfrac{t}{t_2}\right) + \tfrac{6\beta(x^2+y^2)}{R^2(3\alpha+1)}\left[1-\exp\left(-\tfrac{t}{t_2}\right)\right],\end{aligned} \tag{46}$$

and the tractions are distributed along the circular interface as

$$\begin{aligned}\sigma_{rr}^{(1)} = \sigma_{rr}^{(2)} &= \tfrac{\beta\left(\sigma_{xx}^\infty+\sigma_{yy}^\infty\right)}{2(1-\alpha+\beta)} + \tfrac{\beta\left[\left(\sigma_{xx}^\infty-\sigma_{yy}^\infty\right)\cos 2\theta+2\sigma_{xy}^\infty\sin 2\theta\right]}{2\alpha}\exp\left(-\tfrac{t}{t_2}\right),\\ \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} &= \frac{\beta\left[6\alpha+(1-3\alpha)\exp\left(-\frac{t}{t_2}\right)\right]}{2\alpha(3\alpha+1)}\left[\left(\sigma_{yy}^\infty-\sigma_{xx}^\infty\right)\sin 2\theta+2\sigma_{xy}^\infty\cos 2\theta\right],\end{aligned} \quad \text{on} \quad L, \tag{47}$$

which indicates that the interfacial normal stress will finally become uniform along the whole interface as time approaches infinity.

In addition, the change of strain energy due to the introduction of the inhomogeneity can be obtained as

$$\frac{\Delta W}{\pi R^2} = \frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^\infty+\sigma_{yy}^\infty\right)^2}{4\mu_2(1-\alpha+\beta)} + \frac{(1-\nu_2)\left[\left(\sigma_{yy}^\infty-\sigma_{xx}^\infty\right)^2+4\left(\sigma_{xy}^\infty\right)^2\right]}{2\mu_2(3\alpha+1)}\left[3(\alpha-\beta)+1-\frac{\beta}{\alpha}\exp\left(-\frac{t}{t_2}\right)\right]. \tag{48}$$

Interestingly Eq. (41) for a purely slip interface and Eq. (48) for a purely diffusion interface are similar except that the definitions of t_2 for the two cases are different.

4.3. The interface slip occurs much faster than the interface diffusion ($\chi \to 0$)

In this case the two relaxation times can be determined as

$$t_1 = \frac{\gamma(3\alpha+1)}{6} = \frac{R^3[\mu_1(5-6\nu_2)+\mu_2(3-2\nu_1)]}{24\mu_1\mu_2 D}, \quad t_2 = 0, \tag{49}$$

which is identical to that obtained by Koeller and Raj [3].

Consequently the two analytic functions $\phi_1(z,t)$ and $\psi_1(z,t)$ defined in the inhomogeneity are

$$\phi_1(z,t) = \frac{\beta\Gamma}{1-\alpha+\beta}z - \frac{\beta\Gamma'\exp\left(-\frac{t}{t_1}\right)}{R^2(3\alpha+1)}z^3, \quad \psi_1(z,t) = \frac{6\beta\Gamma'}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right)z, \quad (|z|<R) \tag{50}$$

and the two analytic functions $\phi_2(z,t)$ and $\psi_2(z,t)$ defined in the matrix are

$$\begin{aligned}\phi_2(z,t) &= \Gamma z - R^2\bar{\Gamma}'\left[1-\tfrac{3\beta}{3\alpha+1}\exp\left(-\tfrac{t}{t_1}\right)\right]z^{-1},\\ \psi_2(z,t) &= \Gamma' z - \tfrac{2(1-\alpha)R^2\Gamma}{1-\alpha+\beta}z^{-1} - R^4\bar{\Gamma}'\left[1-\tfrac{2\beta}{3\alpha+1}\exp\left(-\tfrac{t}{t_1}\right)\right]z^{-3},\end{aligned} \quad (|z|>R). \tag{51}$$

Thus the internal stresses within the inhomogeneity can be determined as

$$\begin{aligned}
&\sigma_{xx}^{(1)} = \frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)} + \frac{3\beta\left(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty}\right)}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right) + \frac{6\beta\left[y^2\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)+2xy\sigma_{xy}^{\infty}\right]}{R^2(3\alpha+1)}\exp\left(-\frac{t}{t_1}\right),\\
&\sigma_{yy}^{(1)} = \frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)} + \frac{3\beta\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right) + \frac{6\beta\left[x^2\left(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty}\right)+2xy\sigma_{xy}^{\infty}\right]}{R^2(3\alpha+1)}\exp\left(-\frac{t}{t_1}\right), \quad (x^2+y^2<R^2)\\
&\frac{\sigma_{xy}^{(1)}}{\sigma_{xy}^{\infty}} = \frac{6\beta\left[R^2-(x^2+y^2)\right]}{R^2(3\alpha+1)}\exp\left(-\frac{t}{t_1}\right),
\end{aligned} \tag{52}$$

and the tractions are distributed along the circular interface as

$$\begin{aligned}
&\sigma_{rr}^{(1)} = \sigma_{rr}^{(2)} = \frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)} + \frac{3\beta\left[\left(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty}\right)\cos 2\theta+2\sigma_{xy}^{\infty}\sin 2\theta\right]}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right), \quad \text{on} \quad L,\\
&\sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} = 0,
\end{aligned} \tag{53}$$

which indicates that the interfacial shear stress is fully relaxed by the interface slip. Interestingly it follows from Eq. $(52)_3$ that $\sigma_{xy}^{(1)} \equiv 0$ along the whole interface L.

In addition the change of strain energy due to the introduction of the inhomogeneity can be obtained as

$$\frac{\Delta W}{\pi R^2} = \frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)^2}{4\mu_2(1-\alpha+\beta)} + \frac{(1-\nu_2)\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)^2+4\left(\sigma_{xy}^{\infty}\right)^2\right]}{2\mu_2}\left[1-\frac{3\beta}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right)\right]. \tag{54}$$

4.4. The interface diffusion occurs much faster than the interface slip ($\gamma \to 0$)

In this case the two relaxation times can be determined as

$$t_1 = \frac{\chi(3\alpha+1)}{6}, \quad t_2 = 0. \tag{55}$$

Consequently the two analytic functions $\phi_1(z,t)$ and $\psi_1(z,t)$ defined in the inhomogeneity are

$$\phi_1(z,t) = \frac{\beta\Gamma}{1-\alpha+\beta}z + \frac{\beta\Gamma'}{R^2(3\alpha+1)}\exp\left(-\frac{t}{t_1}\right)z^3, \quad \psi_1(z,t) = 0, \quad (|z|<R) \tag{56}$$

and the two analytic functions $\phi_2(z,t)$ and $\psi_2(z,t)$ defined in the matrix are

$$\begin{aligned}
&\phi_2(z,t) = \Gamma z - R^2\bar{\Gamma}'\left[1-\frac{3\beta}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right)\right]z^{-1},\\
&\psi_2(z,t) = \Gamma' z - \frac{2(1-\alpha)R^2\Gamma}{1-\alpha+\beta}z^{-1} - R^4\bar{\Gamma}'\left[1-\frac{4\beta}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right)\right]z^{-3},
\end{aligned} \quad (|z|>R). \tag{57}$$

Thus the internal stresses within the inhomogeneity can be determined as

$$\begin{aligned}
&\sigma_{xx}^{(1)} = \frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)} + \frac{6\beta\left[y^2\left(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty}\right)-2xy\sigma_{xy}^{\infty}\right]}{R^2(3\alpha+1)}\exp\left(-\frac{t}{t_1}\right),\\
&\sigma_{yy}^{(1)} = \frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)} + \frac{6\beta\left[x^2(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})-2xy\sigma_{xy}^{\infty}\right]}{R^2(3\alpha+1)}\exp\left(-\frac{t}{t_1}\right), \quad (x^2+y^2<R^2)\\
&\frac{\sigma_{xy}^{(1)}}{\sigma_{xy}^{\infty}} = \frac{6\beta(x^2+y^2)}{R^2(3\alpha+1)}\exp\left(-\frac{t}{t_1}\right),
\end{aligned} \tag{58}$$

and the tractions are distributed along the circular interface as

$$\begin{aligned}
&\sigma_{rr}^{(1)} = \sigma_{rr}^{(2)} = \frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)},\\
&\sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} = \frac{3\beta\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)\sin 2\theta+2\sigma_{xy}^{\infty}\cos 2\theta\right]}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right),
\end{aligned} \quad \text{on} \quad L, \tag{59}$$

which indicates that there is no gradient in interfacial normal stress (i.e., the normal stress is constant along the interface).

In addition the change of strain energy due to the introduction of the inhomogeneity can be obtained as

$$\frac{\Delta W}{\pi R^2} = \frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)^2}{4\mu_2(1-\alpha+\beta)} + \frac{(1-\nu_2)\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)^2+4\left(\sigma_{xy}^{\infty}\right)^2\right]}{2\mu_2}\left[1-\frac{3\beta}{3\alpha+1}\exp\left(-\frac{t}{t_1}\right)\right]. \tag{60}$$

Interestingly Eqs. (54) and (60) are similar except that the definitions of t_1 for the two cases are different.

4.5. The inhomogeneity and the matrix have the same elastic properties

When the inhomogeneity and the matrix have exactly the same elastic properties ($\mu_1 = \mu_2 = \mu$ and $\nu_1 = \nu_2 = \nu$), the two relaxation times are given by

$$t_{1,2} = \frac{\chi + \gamma \pm \sqrt{(\chi+\gamma)^2 - 3\chi\gamma}}{3}, \tag{61}$$

where

$$\chi = \frac{2\eta R(1-\nu)}{\mu}, \quad \gamma = \frac{R^3(1-\nu)}{2\mu D}. \tag{62}$$

It can be easily checked that Eq. (61) coincides with the result obtained by Onaka et al. [4, Eqs. (34) and (35)] for the case of $f = 0$.

In this case the change of strain energy due to the introduction of the inhomogeneity can be obtained as

$$\frac{\Delta W}{\pi R^2} = \frac{(1-\nu_2)\left[\left(\sigma_{yy}^\infty - \sigma_{xx}^\infty\right)^2 + 4\left(\sigma_{xy}^\infty\right)^2\right]}{2\mu_2}\left[1 + \frac{(\chi - 3t_1)(\chi - t_2)\exp\left(-\frac{t}{t_1}\right) - (\chi - 3t_2)(\chi - t_1)\exp\left(-\frac{t}{t_2}\right)}{2\chi(t_1 - t_2)}\right], \tag{63}$$

which indicates that the hydrostatic part $\left(\sigma_{xx}^\infty + \sigma_{yy}^\infty\right)$ does not contribute to the energy change.

4.6. Both the inhomogeneity and the matrix are incompressible ($\nu_1 = \nu_2 = 0.5$)

In this case the two relaxation times are determined as

$$t_{1,2} = \frac{\chi + \gamma \pm \sqrt{(\chi+\gamma)^2 - 3\chi\gamma}}{3}, \tag{64}$$

where

$$\chi = \frac{\eta R(\mu_1 + \mu_2)}{2\mu_1\mu_2}, \quad \gamma = \frac{R^3(\mu_1 + \mu_2)}{8\mu_1\mu_2 D}. \tag{65}$$

The change of strain energy due to the introduction of the inhomogeneity can be obtained as

$$\frac{\Delta W}{\pi R^2} = \frac{\left(\sigma_{yy}^\infty - \sigma_{xx}^\infty\right)^2 + 4\left(\sigma_{xy}^\infty\right)^2}{4\mu_2}\left[1 + \frac{\beta\left[(\chi - 3t_1)(\chi - t_2)\exp\left(-\frac{t}{t_1}\right) - (\chi - 3t_2)(\chi - t_1)\exp\left(-\frac{t}{t_2}\right)\right]}{2\chi(t_1 - t_2)}\right], \tag{66}$$

which indicates that the hydrostatic part $\left(\sigma_{xx}^\infty + \sigma_{yy}^\infty\right)$ also does not contribute to the energy change.

4.7. $\chi = \gamma$ ($4\eta D = R^2$)

In this case the two relaxation times can be determined as

$$t_1 = \alpha\chi, \quad t_2 = \frac{\chi}{3}. \tag{67}$$

Consequently the two analytic functions $\phi_1(z,t)$ and $\psi_1(z,t)$ defined in the inhomogeneity are

$$\phi_1(z,t) = \frac{\beta\Gamma}{1-\alpha+\beta}z, \quad \psi_1(z,t) = \frac{\beta\Gamma'}{\alpha}\exp\left(-\frac{t}{t_1}\right)z, \quad (|z| < R) \tag{68}$$

and the two analytic functions $\phi_2(z,t)$ and $\psi_2(z,t)$ defined in the matrix are

$$\begin{aligned}\phi_2(z,t) &= \Gamma z - R^2\bar{\Gamma}'\left[1 - \tfrac{\beta}{\alpha}\exp\left(-\tfrac{t}{t_1}\right)\right]z^{-1},\\ \psi_2(z,t) &= \Gamma' z - \tfrac{2(1-\alpha)R^2\Gamma}{1-\alpha+\beta}z^{-1} - R^4\bar{\Gamma}'\left[1 - \tfrac{\beta}{\alpha}\exp\left(-\tfrac{t}{t_1}\right)\right]z^{-3},\end{aligned} \quad (|z| > R). \tag{69}$$

Interestingly in this case the stresses are *uniformly* distributed within the inhomogeneity as

$$\begin{aligned}
&\sigma_{xx}^{(1)}=\frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)}+\frac{\beta}{2\alpha}\left(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty}\right)\exp\left(-\frac{t}{t_1}\right),\\
&\sigma_{yy}^{(1)}=\frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)}+\frac{\beta}{2\alpha}\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)\exp\left(-\frac{t}{t_1}\right),\quad (x^2+y^2<R^2),\\
&\frac{\sigma_{xy}^{(1)}}{\sigma_{xy}^{\infty}}=\frac{\beta}{\alpha}\exp\left(-\frac{t}{t_1}\right),
\end{aligned}\tag{70}$$

which decay only with the larger relaxation time $t_1=\alpha\chi$. Meanwhile the tractions are distributed along the circular interface as

$$\left.\begin{aligned}
&\sigma_{rr}^{(1)}=\sigma_{rr}^{(2)}=\frac{\beta\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{2(1-\alpha+\beta)}+\frac{\beta\left[\left(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty}\right)\cos 2\theta+2\sigma_{xy}^{\infty}\sin 2\theta\right]}{2\alpha}\exp\left(-\frac{t}{t_1}\right),\\
&\sigma_{r\theta}^{(1)}=\sigma_{r\theta}^{(2)}=\frac{\beta\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)\sin 2\theta+2\sigma_{xy}^{\infty}\cos 2\theta\right]}{2\alpha}\exp\left(-\frac{t}{t_1}\right),
\end{aligned}\right.\quad \text{on}\quad L.\tag{71}$$

It can be easily proved that $\chi=\gamma$ is also the sufficient condition leading to an internal uniform stress state within the inhomogeneity. $\chi=\gamma$ can be equivalently written in terms of the interface drag and diffusion parameters as $4\eta D=R^2$. It is observed that this condition is independent of the elastic properties of both the inhomogeneity and matrix.

In this case the change of strain energy due to the introduction of the inhomogeneity can be obtained as

$$\frac{\Delta W}{\pi R^2}=\frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)^2}{4\mu_2(1-\alpha+\beta)}+\frac{(1-\nu_2)\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)^2+4\left(\sigma_{xy}^{\infty}\right)^2\right]}{2\mu_2}\left[1-\frac{\beta}{\alpha}\exp\left(-\frac{t}{t_1}\right)\right].\tag{72}$$

4.8. Energy comparison

It is of interest to compare the values of the change of strain energy for the following several cases:

(a) The steady-state ($t\to\infty$) of Eq. (32) for a slip and diffusion interface

$$\frac{\Delta W_{SD}}{\pi R^2}=\frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)^2}{4\mu_2(1-\alpha+\beta)}+\frac{(1-\nu_2)\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)^2+4\left(\sigma_{xy}^{\infty}\right)^2\right]}{2\mu_2}.\tag{73}$$

(b) The steady-state of Eqs. (41) or (48) for a purely slip or purely diffusion interface

$$\frac{\Delta W_S}{\pi R^2}=\frac{\Delta W_D}{\pi R^2}=\frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)^2}{4\mu_2(1-\alpha+\beta)}+\frac{(1-\nu_2)\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)^2+4\left(\sigma_{xy}^{\infty}\right)^2\right]}{2\mu_2(3\alpha+1)}[3(\alpha-\beta)+1].\tag{74}$$

(c) The change of strain energy due to the introduction of a *perfectly bonded* inhomogeneity

$$\frac{\Delta W_P}{\pi R^2}=\frac{(1-\alpha)(1-\nu_2)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)^2}{4\mu_2(1-\alpha+\beta)}+\frac{(1-\nu_2)\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)^2+4\left(\sigma_{xy}^{\infty}\right)^2\right]}{2\mu_2}\left(1-\frac{\beta}{\alpha}\right).\tag{75}$$

(d) The change of strain energy due to the introduction of a circular cavity ($\mu_1=0$)

$$\frac{\Delta W_C}{\pi R^2}=\frac{(1-\nu_2)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)^2}{4\mu_2}+\frac{(1-\nu_2)\left[\left(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}\right)^2+4\left(\sigma_{xy}^{\infty}\right)^2\right]}{2\mu_2}.\tag{76}$$

It can be easily found that the following inequality holds

$$\Delta W_P\leqslant\Delta W_S=\Delta W_D\leqslant\Delta W_{SD}\leqslant\Delta W_C.\tag{77}$$

Remark 2. It can be easily checked that the factor $2[3(\alpha-\beta)+1]/(3\alpha+1)$ appearing in Eq. (74) is equal to β_2 defined in Eq. (24) by Koeller and Raj [3]. Consequently Koeller and Raj's result of the effective shear modulus can be modified to become exactly the same as ours if we replace $2/\pi$ in Eqs. (22) and (23) in [3] by 1. Keep in mind that our prediction is based on the energy approach [11–14], whilst Koeller and Raj's result was obtained by "integrating the displacements along the mid-distance line between the particles, where the particles are assumed to be arranged in a square array." [3] In addition it is found that Eq. (74) is in agreement with the result in [15].

4.9. The matrix is subjected to a remote uniform heat flow

When the matrix is only subjected to a remote steady uniform heat flow q_0 (Fig. 1b) along the positive y-direction [17,18], the two analytic functions $\phi_1(z,t)$ and $\psi_1(z,t)$ defined in the inhomogeneity can be determined as

$$\phi_1(z,t) = iA_0 \exp\left(-\frac{t}{t_0}\right)z^2, \quad \psi_1(z,t) = 0, \quad (|z| < R) \tag{78}$$

and the two analytic functions $\phi_2(z,t)$ and $\psi_2(z,t)$ defined in the matrix are given by

$$\begin{aligned} &\phi_2(z,t) = \phi_2(z) = iR^2 B_0 \ln z, \\ &\psi_2(z,t) = -iR^2 B_0 \ln z - iR^4\left[B_0 + A_0 \exp\left(-\tfrac{t}{t_0}\right)\right]z^{-2}, \end{aligned} \quad (|z| > R), \tag{79}$$

where the relaxation time t_0 and the two time-independent real constants A_0 and B_0 are

$$t_0 = \frac{4\chi\gamma}{\chi + 4\gamma} \geqslant 0, \tag{80}$$

$$A_0 = \frac{q_0\mu_1\mu_2}{\kappa_1\mu_2 + \mu_1}\left[\frac{\alpha_{t2}(1+\nu_2)}{k_{t2}} - \frac{2\alpha_{t1}(1+\nu_1)}{k_{t1}+k_{t2}}\right], \quad B_0 = \frac{q_0\mu_2\alpha_{t2}(1+\nu_2)}{2k_{t2}(1-\nu_2)}\frac{k_{t2}-k_{t1}}{k_{t2}+k_{t1}}, \tag{81}$$

with k_t being the thermal conductivity and α_t the thermal expansion coefficient.

During the above theoretical developments, a perfect thermal interface contact [17,18] and the plane-strain condition have been assumed. It is observed from the above expressions that:

(i) the internal thermal stresses are now *linear* functions of the coordinates x and y and decay only with the single relaxation time t_0 such that

$$\sigma_{xx}^{(1)} = -6A_0 y \exp\left(-\frac{t}{t_0}\right), \quad \sigma_{yy}^{(1)} = -2A_0 y \exp\left(-\frac{t}{t_0}\right), \quad \sigma_{xy}^{(1)} = 2A_0 x \exp\left(-\frac{t}{t_0}\right), \quad (x^2 + y^2 < R^2), \tag{82}$$

(ii) the relaxation time t_0 is zero when the interface slip occurs much faster than the interface diffusion ($\chi = 0$) or when the interface diffusion occurs much faster than the interface slip ($\gamma = 0$);

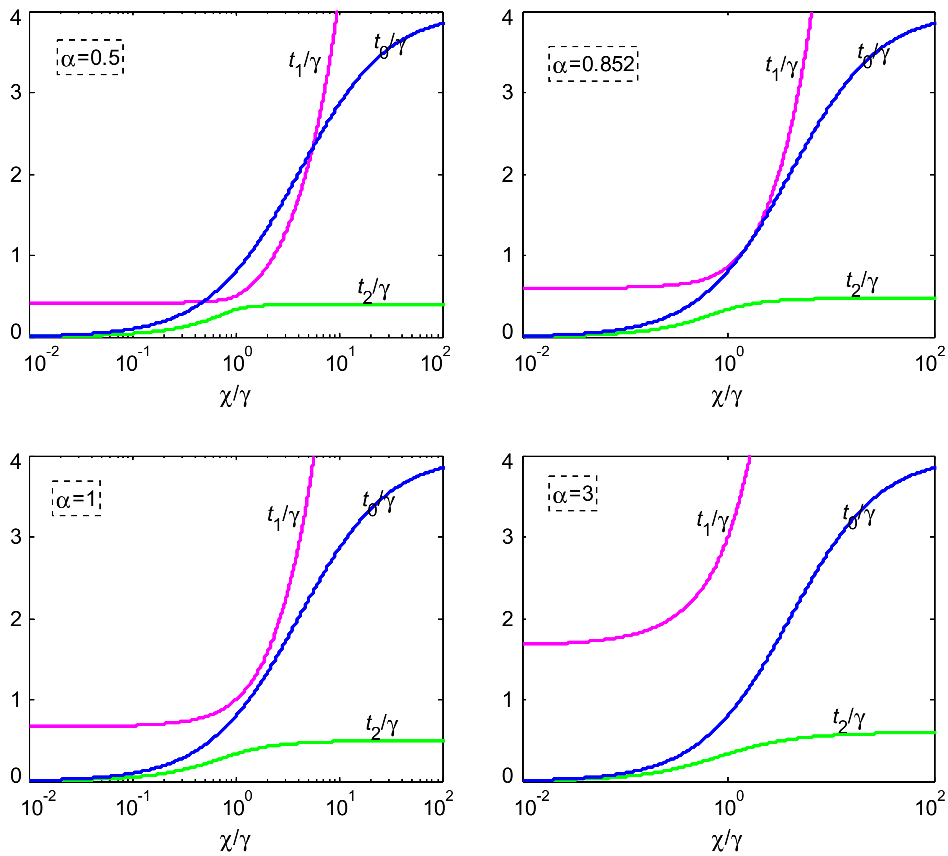

Fig. 2. Variation of the three relaxation times t_1, t_2 and t_0 as functions of χ/γ for four different values of $\alpha = 0.5, 0.852, 1, 2$.

(iii) the mean stress within the matrix is in fact time-independent and is given by $\sigma_{xx}^{(2)} + \sigma_{yy}^{(2)} = 4R^2 B_0 \sin\theta/r$, $(r > R)$; and
(iv) as time approaches infinity, all the internal thermal stresses within the inhomogeneity become zero whilst the steady distributions of the external thermal stresses in the matrix are similar in form to those caused by an insulating circular cavity perturbing a remote uniform heat flow [19].

5. Numerical results

In this section, we present numerical results to demonstrate the obtained solution and the corresponding physics.

5.1. The relaxation times

Here we first look into the relaxation times determined by Eqs. (25) and (80). Fig. 2 illustrates the variation of the three relaxation times t_1, t_2 and t_0 as functions of χ/γ for four typical values of α. Apparently t_0 determined by Eq. (80) is independent of α. It is observed from Fig. 2 that:

(i) For a fixed value of α, all the three relaxation times are increasing functions of χ/γ. More specifically the value of t_1 monotonically increases from $\gamma\,(3\alpha + 1)/6$ to infinity as χ/γ increases from zero to infinity; the value of t_2 monotonically increases from zero to $2\alpha\gamma/(3\alpha + 1)$ as χ/γ increases from zero to infinity; and the value of t_0 monotonically increases from zero to 4γ as χ/γ increases from zero to infinity.
(ii) When $\alpha < 0.852$, the curve of t_1 intersects with that of t_0 at two points (see the upper-left subplot for $\alpha = 0.5$); when $\alpha = 0.852$, the curve of t_1 is just tangential to that of t_0 (see the upper-right subplot); when $\alpha > 0.852$, there is no intersection between the curves t_1 and t_0 (see the lower two subplots for $\alpha = 1$ and $\alpha = 3$).
(iii) For a fixed value of α, the value of t_0 is always greater than that of t_2.

5.2. The time-dependent interfacial tractions

Figs. 3 and 4 illustrate the time-dependent distributions of the tractions σ_{rr} and $\sigma_{r\theta}$ along the circular interface L for four different values of $\chi/\gamma = 0, 1, 5, \infty$ when the composite is subjected to remote shear stress $\sigma_{xy}^{\infty}\left(\sigma_{xx}^{\infty} = \sigma_{yy}^{\infty} = 0\right)$ with $\mu_1 = 10\mu_2$ and $\nu_1 = \nu_2 = 1/3$. In view of the fact that γ is finite and non-zero, then $\chi/\gamma = \infty$ corresponds to the case in which the interface

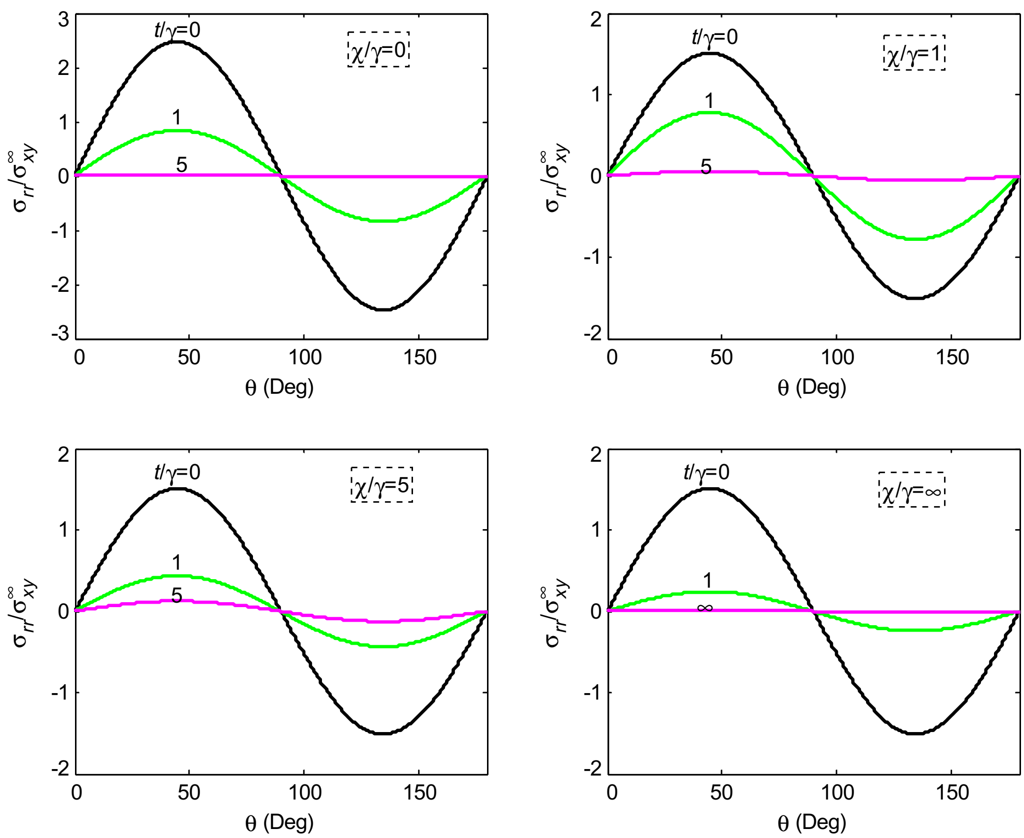

Fig. 3. Time-dependent distribution of the interfacial normal stress σ_{rr} along the circular interface L for four different values of $\chi/\gamma = 0, 1, 5, \infty$ when the composite is only subjected to the remote shear stress σ_{xy}^{∞} with fixed $\mu_1 = 10\mu_2$ and $\nu_1 = \nu_2 = 1/3$.

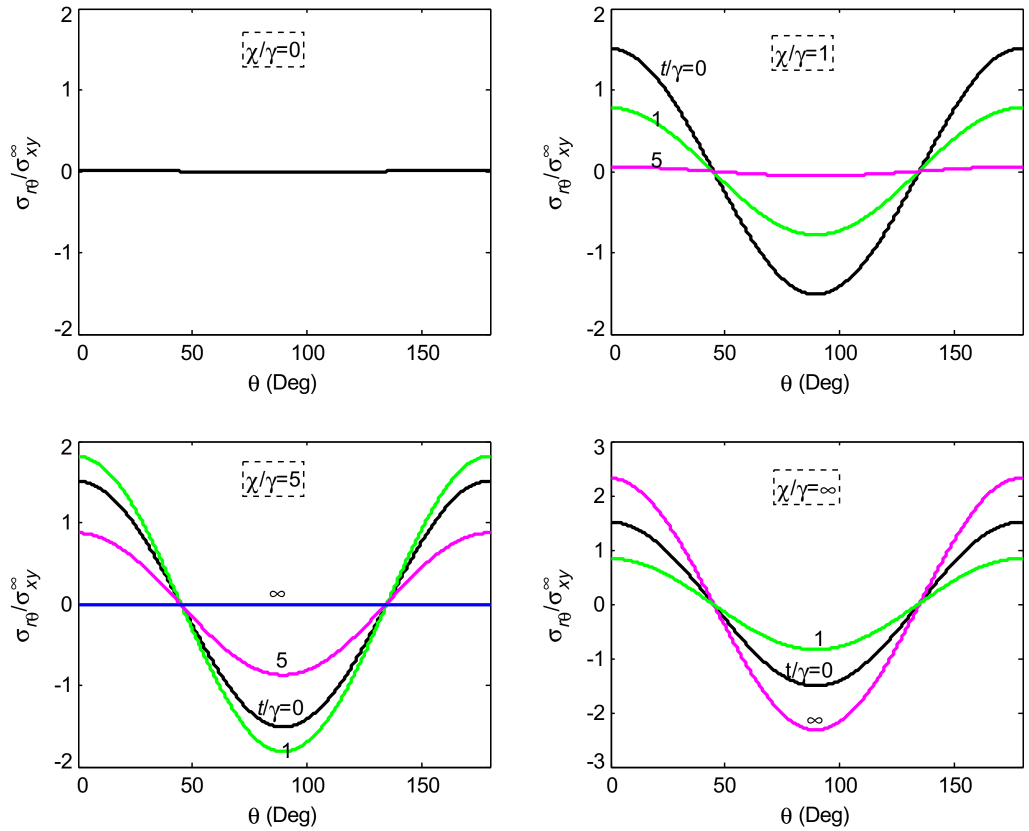

Fig. 4. Time-dependent distribution of the interfacial shear stress $\sigma_{r\theta}$ along the circular interface L for four different values of $\chi/\gamma = 0, 1, 5, \infty$ when the composite is only subjected to the remote shear stress σ_{xy}^{∞} with fixed $\mu_1 = 10\mu_2$ and $\nu_1 = \nu_2 = 1/3$.

slip is absent (see Section 4.2), and $\chi/\gamma = 0$ corresponds to the case in which the interface slip occurs much faster than the interface diffusion (see Section 4.3). It is observed from Fig. 3 that at the initial moment, the maximum value of the normal

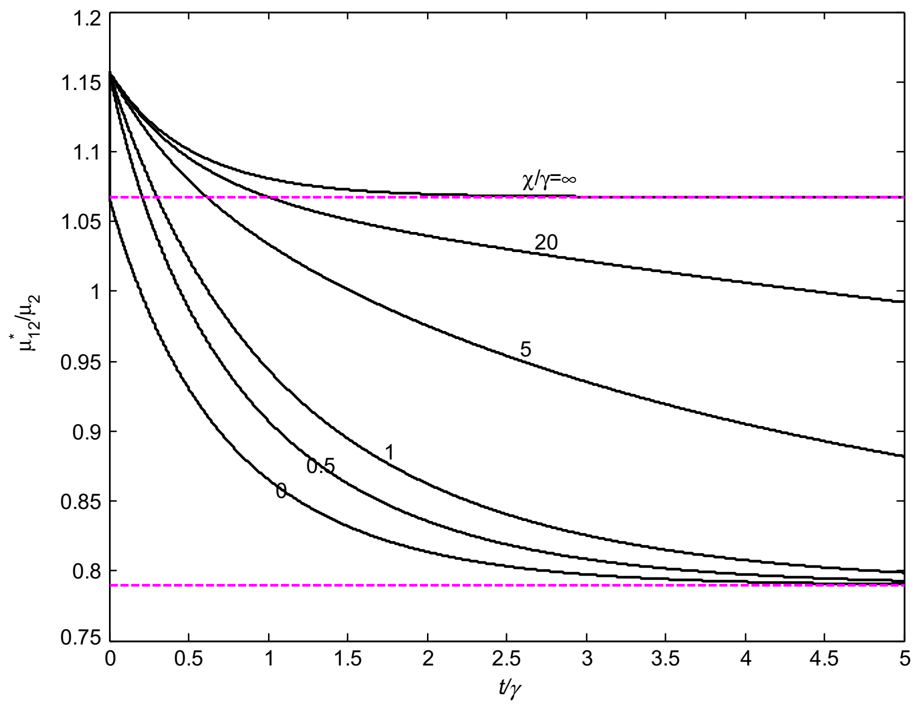

Fig. 5. Variation of the effective in-plane shear modulus with the normalized time t/γ for six different values of χ/γ. Other fixed parameters are $\mu_1 = 10\mu_2$, $\nu_1 = \nu_2 = 1/3$ and $f = 0.1$.

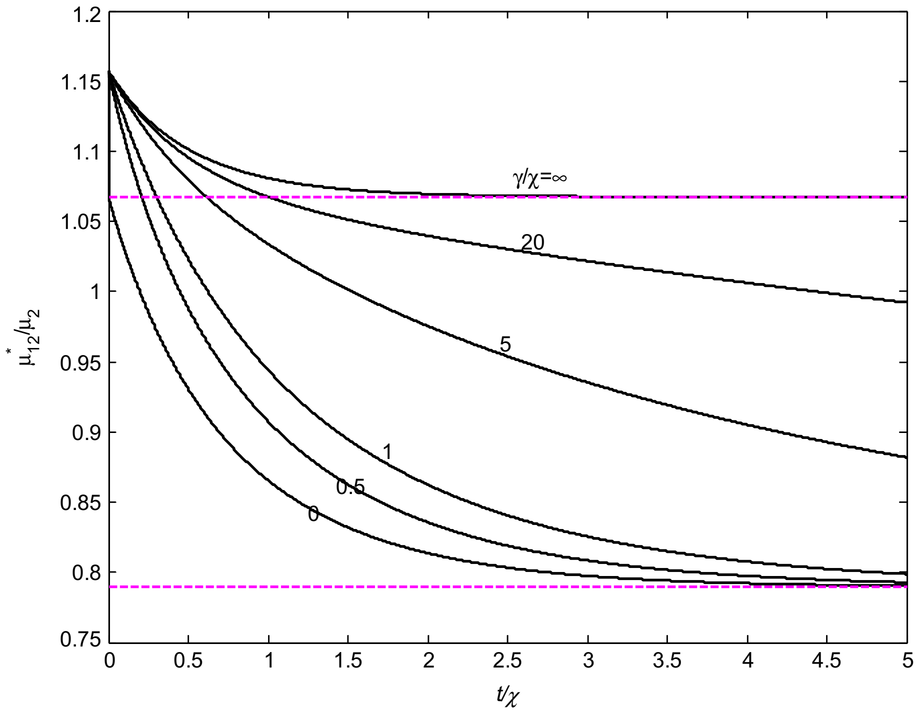

Fig. 6. Variation of the effective in-plane shear modulus with the normalized time t/χ for six different values of γ/χ. Other fixed parameters are $\mu_1 = 10\mu_2$, $\nu_1 = \nu_2 = 1/3$ and $f = 0.1$.

stress $\sigma_{rr}/\sigma_{xy}^{\infty}$ along the interface is $6\beta/(3\alpha+1) = 2.4742$ when $\chi/\gamma = 0$ and is $\beta/\alpha = 1.5094$ when $\chi/\gamma \neq 0$. It is observed from Fig. 4 that: (i) the interfacial shear stress is always zero when $\chi/\gamma = 0$ due to the fact that the interfacial shear stress is fully relaxed by the interface slip; (ii) as $t \to \infty$, the steady state of the interfacial shear stress $\sigma_{r\theta}/\sigma_{xy}^{\infty}$ is zero when $\chi/\gamma < \infty$, and is the sinusoidal distribution $6\beta\cos 2\theta/(3\alpha+1) = 2.4742\cos 2\theta$ when $\chi/\gamma = \infty$.

5.3. The time-dependent effective shear modulus

Next we show in Fig. 5 the variation of the effective in-plane shear modulus with the normalized time t/γ for six different values of the ratio χ/γ. Here we set $\mu_1 = 10\mu_2$, $\nu_1 = \nu_2 = 1/3$, and the volume fraction is chosen as $f = 0.1$. It is observed that: (i) in the case of $\chi/\gamma = 0$, the initial value of the effective shear modulus is $\mu^*_{12}/\mu_2 = 1/\left[1 + \frac{4f(1-\nu_2)[3(\alpha-\beta)+1]}{3\alpha+1}\right] = 1.0675$. When $\chi/\gamma \neq 0$, the initial value of the effective shear modulus is $\mu^*_{12}/\mu_2 = 1/[1 + 4f(1-\nu_2)(1-\beta/\alpha)] = 1.1572$; and (ii) in the case of $\chi/\gamma = \infty$, the effective shear modulus monotonically decreases to $\mu^*_{12}/\mu_2 = 1.0675$ as time approaches infinity. When $\chi/\gamma < \infty$, the effective shear modulus monotonically decreases to $\mu^*_{12}/\mu_2 = 1/[1 + 4f(1-\nu_2)] = 0.7895$ as time approach infinity.

Finally we illustrate in Fig. 6 the variation of the effective in-plane shear modulus with the normalized time t/χ for six different values of the ratio γ/χ. Here we also set $\mu_1 = 10\mu_2$, $\nu_1 = \nu_2 = 1/3$, and f=0.1. In view of the fact that χ is finite and non-zero, then $\gamma/\chi = \infty$ corresponds to the case in which the interface diffusion is absent (see Section 4.1), and $\gamma/\chi = 0$ corresponds to the case in which the interface diffusion occurs much faster than the interface slip (see Section 4.4). A comparison of Fig. 5 with Fig. 6 reveals a very interesting feature: the effective in-plane shear modulus curves in the two figures are exactly the same, which means that this effective modulus has the same dependence on the normalized times t/γ and t/χ.

6. Conclusions

We have derived, in this paper, a rigorous closed-form solution for the benchmark problem of an isolated circular elastic inhomogeneity bonded to an infinite matrix through a sharp interface on which both diffusion and slip concurrently occur. The special cases in which only interface slip or interface diffusion occurs, or interface slip occurs much faster than interface diffusion or vice versa were also discussed through a limiting procedure. When the interface slip occurs much faster than the interface diffusion, or when the inhomogeneity and the matrix have the same elastic properties, or when the inhomogeneity is rigid, our results can be reduced to previous ones [3,4,6]. Interestingly, we have found that the simple condition $4\eta D = R^2$ would lead to an internal uniform but time-decaying stress field within the inhomogeneity, a new phenomenon which has not been observed before. Furthermore, as a byproduct, we have also considered the case in which the matrix is subjected to a remote uniform heat flow. While our work also suggests that the complex variable method can be conveniently applied to investigate two-dimensional inhomogeneity problems with rate-dependent imperfect interface, numerical examples are included to demonstrate the obtained solution with the corresponding physics.

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