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Solitary waves in a magneto-electro-elastic circular rod

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Abstract
A simple nonlinear model is proposed in this paper to study the solitary wave in a circular magneto-electro-elastic rod. Based on the constitutive relation for transversely isotropic piezoelectric and piezomagnetic materials, combined with the differential equations of motion, we derive the longitudinal wave motion equation in a long circular rod. The nonlinearity considered is geometrically associated with the nonlinear normal strain in the longitudinal rod direction and the transverse Poisson's effect is included by introducing the effective Poisson's ratio. The nonlinear solitary wave equation is solved by the Jacobi elliptic function expansion method and numerical examples demonstrate not only the existence of such a wave but also some interesting characteristics of the solitary wave in the rod made of different multiphase coupled materials.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A soliton is a solitary, traveling wave pulse which is the solution to certain nonlinear partial differential equations. This special wave could have various important applications due to its remarkable stability properties [1]. In particular, while most dispersive waves would scatter inelastically and lose 'energy' due to radiation, solitary waves remerge, retaining their identities with nearly the same speed and shape after full nonlinear interaction. For example, the propagation of a powerful laser beam through an optical crystal/fiber is often accompanied by the wave breaking phenomenon and formation of an optical shock, which plays an important role in the propagation of light pulses through digital communication fiber optic systems.

The first observation of a solitary wave was documented in 1834 by the Scottish engineer John Russell, who followed a water wave traveling through a canal for as much as two miles without experiencing any noticeable distortion by dispersion [2]. Solitary wave solutions to nonlinear wave equations have also been used to describe deep water waves [3], the nonlinear lattice of DNA double-strand dynamics in biophysics [4], coupled waveguide arrays in nonlinear optics [5], blood flow in arteries [6], and ultrashort pulses in metamaterials [7].

In the last two decades, nonlinear elastic effects on solitary waves have received considerable attention in solid mechanics [8–11]. On the basis of classical linear theory, Zhang [12] derived the nonlinear equations, for a thin elastic rod, of the longitudinal, torsional and flexural waves using the Hamilton variation principle. Liu [13] solved the nonlinear wave equation in an elastic rod by the Jacobi elliptic function expansion method. Christov *et al* [14] considered the propagation of a stationary solitary wave in an elastic rod over an elastic foundation.

With increasing usage of magneto-electro-elastic (MEE) structures in various engineering fields (such as sensors, actuators, etc), wave propagation in MEE media has also attracted many researchers. Using the propagator matrix and state-vector approaches, Chen *et al* [15] presented an analytical treatment for the propagation of harmonic waves in MEE multilayered plates. Chen and Shen [16] obtained the effective wave velocity and attenuation factor when axial shear MEE

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waves propagate in piezoelectric–piezomagnetic composites. Wu *et al* [17] derived a dynamic solution for the propagation of harmonic waves in inhomogeneous (functionally graded) MEE plates.

So far, however, there has been no report on nonlinear solitary waves in MEE solids, which motivates this study. Therefore, in this paper we present the solution and the corresponding numerical results for a solitary wave in a long MEE circular rod. This paper is organized as follows. In section 2, we review the basic equations for MEE materials. In section 3, the longitudinal wave equations in a MEE circular rod are derived. Based on these equations, the corresponding solitary wave equation is derived in section 4. Numerical examples are given in section 5 and conclusions are drawn in section 6.

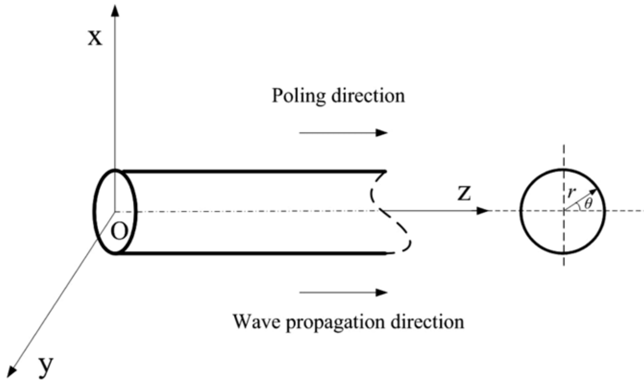

Figure 1. Schematic of a long circular MEE rod.

2. Basic equations

We assume that the rod is made of a transversely isotropic MEE material with its axis of symmetry being along the z-direction (i.e., the rod direction). Therefore, the constitutive relations in the cylindrical coordinate system (r, θ, z) can be written as

$$\begin{aligned}\sigma_r &= c_{11}\varepsilon_r + c_{12}\varepsilon_\theta + c_{13}\varepsilon_z - e_{31}E_z - q_{31}H_z,\\ \sigma_\theta &= c_{12}\varepsilon_r + c_{11}\varepsilon_\theta + c_{13}\varepsilon_z - e_{31}E_z - q_{31}H_z,\\ \sigma_z &= c_{13}\varepsilon_r + c_{13}\varepsilon_\theta + c_{33}\varepsilon_z - e_{33}E_z - q_{33}H_z,\\ \tau_{rz} &= c_{44}\gamma_{rz} - e_{15}E_r - q_{15}H_r,\\ \tau_{\theta z} &= c_{44}\gamma_{\theta z} - e_{15}E_\theta - q_{15}H_\theta,\\ \tau_{r\theta} &= c_{66}\gamma_{r\theta},\end{aligned} \tag{1}$$

$$\begin{aligned}D_r &= e_{15}\gamma_{rz} + \varepsilon_{11}E_r + d_{11}H_r,\\ D_\theta &= e_{15}\gamma_{\theta z} + \varepsilon_{11}E_\theta + d_{11}H_\theta,\\ D_z &= e_{31}\varepsilon_r + e_{31}\varepsilon_\theta + e_{33}\varepsilon_z + \varepsilon_{33}E_z + d_{33}H_z,\end{aligned} \tag{2}$$

$$\begin{aligned}B_r &= q_{15}\gamma_{rz} + d_{11}E_r + \mu_{11}H_r,\\ B_\theta &= q_{15}\gamma_{\theta z} + d_{11}E_\theta + \mu_{11}H_\theta,\\ B_z &= q_{31}\varepsilon_r + q_{31}\varepsilon_\theta + q_{33}\varepsilon_z + d_{33}E_z + \mu_{33}H_z,\end{aligned} \tag{3}$$

where σ_i and τ_{ij} are the normal and shear stresses; ε_i and γ_{ij} are the normal and shear strains; E_i, H_i, D_i, and B_i are, respectively, the electric field, magnetic field, electric displacement, and magnetic induction; c_{ij}, ε_{ij}, e_{ij}, q_{ij}, d_{ij}, and μ_{ij} are, respectively, the elastic, dielectric, piezoelectric, piezomagnetic, magneto-electric, and magnetic coefficients. It is noted that for the transversely isotropic material, the relation $c_{11} = c_{12} + 2c_{66}$ holds.

Furthermore, in the absence of body forces and without electric and magnetic charges, the equations of motion in the rod are

$$\begin{aligned}&\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho\frac{\partial^2 U_r}{\partial t^2},\\ &\frac{\partial\tau_{r\theta}}{\partial r} + \frac{\partial\sigma_\theta}{r\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = \rho\frac{\partial^2 U_\theta}{\partial t^2},\\ &\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\tau_{\theta z}}{r\partial\theta} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho\frac{\partial^2 U_z}{\partial t^2},\end{aligned} \tag{4}$$

$$\frac{\partial D_r}{\partial r} + \frac{\partial D_\theta}{r\partial\theta} + \frac{\partial D_z}{\partial z} = 0, \tag{5}$$

$$\frac{\partial B_r}{\partial r} + \frac{\partial B_\theta}{r\partial\theta} + \frac{\partial B_z}{\partial z} = 0, \tag{6}$$

where U_r, U_θ, and U_z are, respectively, the mechanical displacements in the r-, θ-, and z-directions.

The finite (nonlinear) elastic strain–displacement, electric field-potential and magnetic field-potential relations can be expressed as

$$\begin{aligned}&\varepsilon_r = \frac{\partial U_r}{\partial r}, \qquad \varepsilon_\theta = \frac{\partial U_\theta}{r\partial\theta} + \frac{U_r}{r},\\ &\varepsilon_z = \frac{\partial U_z}{\partial z} + \frac{1}{2}\left(\frac{\partial U_z}{\partial z}\right)^2, \qquad \gamma_{r\theta} = \frac{\partial U_r}{r\partial\theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r},\\ &\gamma_{\theta z} = \frac{\partial U_z}{r\partial\theta} + \frac{\partial U_\theta}{\partial z}, \qquad \gamma_{rz} = \frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z},\end{aligned} \tag{7}$$

$$E_r = -\frac{\partial\phi}{\partial r}, \qquad E_\theta = -\frac{\partial\phi}{r\partial\theta}, \qquad E_z = -\frac{\partial\phi}{\partial z}, \tag{8}$$

$$H_r = -\frac{\partial\psi}{\partial r}, \qquad H_\theta = -\frac{\partial\psi}{r\partial\theta}, \qquad H_z = -\frac{\partial\psi}{\partial z}, \tag{9}$$

where ϕ and ψ are the electric and magnetic potentials, respectively. It should be noted that in writing equations (7) we have assumed that the normal strain component in the longitudinal rod direction (z-direction) is finite.

3. Longitudinal wave equations in a MEE circular rod

We consider the wave propagation in a long MEE circular rod as shown in figure 1. In the cylindrical coordinate system (r, θ, z), z is along the rod direction, i.e., the wave propagation direction, and $\theta \in [0, 2\pi]$, $0 \leqslant r \leqslant R$. To facilitate our study, the following assumptions are made: (1) the cross section of the rod remains plane before and after the deformation; (2) the lateral surface of the rod has axial symmetry, which implies that $U_\theta = 0$ and $\partial/\partial\theta = 0$; (3) to consider the Poisson's effect, the gradient of the longitudinal displacement U_z and the radial displacement U_r are connected by $U_r = -\nu_{\text{eff}} r\,\partial U_z/\partial z$, where ν_{eff} is the effective Poisson's ratio to be determined later.

Furthermore, since the problem is one-dimensional, the extended tractions on the lateral boundary of the rod should be zero. In other words, one has $\sigma_r = 0$, $\tau_{rz} = 0$, $\tau_{r\theta} = 0$, $D_r = 0$, $B_r = 0$, from which the following relations are obtained:

$$\begin{aligned} &\gamma_{rz} = \gamma_{\theta z} = 0, \qquad E_r = E_\theta = 0, \\ &H_r = H_\theta = 0, \qquad D_\theta = B_\theta = 0, \end{aligned} \tag{10}$$

$$\varepsilon_r = \frac{e_{31}E_z + q_{31}H_z - c_{12}\varepsilon_\theta - c_{13}\varepsilon_z}{c_{11}}. \tag{11}$$

We now let $U = U_z$ be the longitudinal displacement, then the equations of motion are reduced to

$$-\frac{\sigma_\theta}{r} = \rho\frac{\partial^2 U_r}{\partial t^2}, \qquad \frac{\partial \sigma_z}{\partial z} = \rho\frac{\partial^2 U}{\partial t^2}, \tag{12a}$$

$$\frac{\partial D_z}{\partial z} = 0, \tag{12b}$$

$$\frac{\partial B_z}{\partial z} = 0. \tag{12c}$$

In terms of U, ϕ and ψ, these equations become

$$A_1\frac{\partial\phi}{\partial z} + A_2\frac{\partial\psi}{\partial z} - A_3\nu_{\text{eff}}\frac{\partial U}{\partial z} + A_4\varepsilon_z = \nu_{\text{eff}}\rho r^2\frac{\partial^3 U}{\partial t^2\partial z}, \tag{13}$$

$$\frac{\partial}{\partial z}\left[B_1\frac{\partial\phi}{\partial z} + B_2\frac{\partial\psi}{\partial z} - A_4\nu_{\text{eff}}\frac{\partial U}{\partial z} + B_4\varepsilon_z\right] = \rho\frac{\partial^2 U}{\partial t^2}, \tag{14}$$

$$\frac{\partial}{\partial z}\left[C_1\frac{\partial\phi}{\partial z} + C_2\frac{\partial\psi}{\partial z} + A_1\nu_{\text{eff}}\frac{\partial U}{\partial z} - B_1\varepsilon_z\right] = 0, \tag{15}$$

$$\frac{\partial}{\partial z}\left[C_2\frac{\partial\phi}{\partial z} + D_2\frac{\partial\psi}{\partial z} + A_2\nu_{\text{eff}}\frac{\partial U}{\partial z} - B_2\varepsilon_z\right] = 0, \tag{16}$$

where

$$\begin{aligned} &A_1 = \left(1 - \frac{c_{12}}{c_{11}}\right)e_{31}, \qquad A_2 = \left(1 - \frac{c_{12}}{c_{11}}\right)q_{31}, \\ &A_3 = c_{11} - \frac{c_{12}^2}{c_{11}}, \qquad A_4 = c_{13}\left(1 - \frac{c_{12}}{c_{11}}\right), \end{aligned} \tag{17a}$$

$$\begin{aligned} &B_1 = e_{33} - \frac{c_{13}}{c_{11}}e_{31}, \qquad B_2 = q_{33} - \frac{c_{13}}{c_{11}}q_{31}, \\ &B_4 = c_{33} - \frac{c_{13}^2}{c_{11}}, \end{aligned} \tag{17b}$$

$$\begin{aligned} &C_1 = \frac{e_{31}}{c_{11}}e_{31} + \varepsilon_{33}, \qquad C_2 = \frac{e_{31}}{c_{11}}q_{31} + d_{33}, \\ &D_2 = \frac{q_{31}}{c_{11}}q_{31} + \mu_{33}. \end{aligned} \tag{17c}$$

Taking the derivative of equation (13) with respect to z, we obtain

$$A_1\frac{\partial^2\phi}{\partial z^2} + A_2\frac{\partial^2\psi}{\partial z^2} - A_3\nu_{\text{eff}}\frac{\partial^2 U}{\partial z^2} + A_4\frac{\partial\varepsilon_z}{\partial z} = \rho\nu_{\text{eff}}r^2\frac{\partial^4 U}{\partial t^2\partial z^2}. \tag{18}$$

Integrating both sides of equation (18) over the cross section of the rod, we arrive at

$$A_1\frac{\partial^2\phi}{\partial z^2} + A_2\frac{\partial^2\psi}{\partial z^2} - A_3\nu_{\text{eff}}\frac{\partial^2 U}{\partial z^2} = -A_4\frac{\partial\varepsilon_z}{\partial z} + \frac{1}{2}\rho\nu_{\text{eff}}R^2\frac{\partial^4 U}{\partial t^2\partial z^2}. \tag{19}$$

Then, equations (15), (16) and (19) can be solved for ϕ, ψ and U as

$$\frac{\partial^2\phi}{\partial z^2} = \frac{\Delta_1}{\Delta}, \qquad \frac{\partial^2\psi}{\partial z^2} = \frac{\Delta_2}{\Delta}, \qquad \frac{\partial^2 U}{\partial z^2} = \frac{\Delta_3}{\Delta}, \tag{20}$$

where

$$\Delta = \begin{vmatrix} A_1 & A_2 & -\nu_{\text{eff}}A_3 \\ C_1 & C_2 & \nu_{\text{eff}}A_1 \\ C_2 & D_2 & \nu_{\text{eff}}A_2 \end{vmatrix}, \tag{21a}$$

$$\Delta_1 = \begin{vmatrix} \frac{1}{2}\rho\nu_{\text{eff}}R^2\frac{\partial^4 U}{\partial t^2\partial z^2} - A_4\frac{\partial\varepsilon_z}{\partial z} & A_2 & -\nu_{\text{eff}}A_3 \\ B_1\frac{\partial\varepsilon_z}{\partial z} & C_2 & \nu_{\text{eff}}A_1 \\ B_2\frac{\partial\varepsilon_z}{\partial z} & D_2 & \nu_{\text{eff}}A_2 \end{vmatrix}, \tag{21b}$$

$$\Delta_2 = \begin{vmatrix} A_1 & \frac{1}{2}\rho\nu_{\text{eff}}R^2\frac{\partial^4 U}{\partial t^2\partial z^2} - A_4\frac{\partial\varepsilon_z}{\partial z} & -\nu_{\text{eff}}A_3 \\ C_1 & B_1\frac{\partial\varepsilon_z}{\partial z} & \nu_{\text{eff}}A_1 \\ C_2 & B_2\frac{\partial\varepsilon_z}{\partial z} & \nu_{\text{eff}}A_2 \end{vmatrix}, \tag{21c}$$

$$\Delta_3 = \begin{vmatrix} A_1 & A_2 & \frac{1}{2}\rho\nu_{\text{eff}}R^2\frac{\partial^4 U}{\partial t^2\partial z^2} - A_4\frac{\partial\varepsilon_z}{\partial z} \\ C_1 & C_2 & B_1\frac{\partial\varepsilon_z}{\partial z} \\ C_2 & D_2 & B_2\frac{\partial\varepsilon_z}{\partial z} \end{vmatrix}. \tag{21d}$$

Substituting equations (20) into equation (14) gives

$$\frac{1}{\Delta}(B_1\Delta_1 + B_2\Delta_2 - \nu_{\text{eff}}A_4\Delta_3) = \rho\frac{\partial^2 U}{\partial t^2} - B_4\frac{\partial\varepsilon_z}{\partial z}. \tag{22}$$

Finally, taking the derivation of equation (22) with respect to z and making use of the finite elastic strain–displacement relation in equations (7), we obtain the following longitudinal wave equation for a MEE circular rod:

$$\frac{1}{\Delta}(B_1\Delta_1^* + B_2\Delta_2^* - \nu_{\text{eff}}A_4\Delta_3^*) = \rho\frac{\partial^2 u}{\partial t^2} - B_4\frac{\partial^2}{\partial z^2}\left(u + \frac{1}{2}u^2\right), \tag{23}$$

where $u = \partial U/\partial z$, and

$$\begin{aligned} \Delta_1^* = {} & \frac{1}{2}\rho\nu_{\text{eff}}^2R^2(C_2A_2 - A_1D_2)\frac{\partial^4 u}{\partial t^2\partial z^2} \\ & + \nu_{\text{eff}}[-A_4(C_2A_2 - A_1D_2) - B_1(A_2^2 + A_3D_2) \\ & + B_2(A_1A_2 + C_2A_3)]\frac{\partial^2}{\partial z^2}\left(u + \frac{1}{2}u^2\right), \end{aligned} \tag{24a}$$

$$\begin{aligned} \Delta_2^* = {} & -\frac{1}{2}\rho\nu_{\text{eff}}^2R^2(C_1A_2 - A_1C_2)\frac{\partial^4 u}{\partial t^2\partial z^2} \\ & + \nu_{\text{eff}}[A_4(C_1A_2 - A_1C_2) + B_1(A_1A_2 + C_2A_3) \\ & - B_2(A_1^2 + C_1A_3)]\frac{\partial^2}{\partial z^2}\left(u + \frac{1}{2}u^2\right), \end{aligned} \tag{24b}$$

$$\begin{aligned} \Delta_3^* = {} & \frac{1}{2}\rho\nu_{\text{eff}}R^2(C_1D_2 - C_2^2)\frac{\partial^4 u}{\partial t^2\partial z^2} \\ & + [-A_4(C_1D_2 - C_2^2) - B_1(A_1D_2 - C_2A_2) \\ & + B_2(A_1C_2 - C_1A_2)]\frac{\partial^2}{\partial z^2}\left(u + \frac{1}{2}u^2\right). \end{aligned} \tag{24c}$$

Equation (23) can be also changed to the following standard nonlinear wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left(\frac{c_0^2}{2}u^2 + N\frac{\partial^2 u}{\partial t^2}\right), \tag{25}$$

with

$$\begin{aligned} c_0^2 = \frac{\nu_{\rm eff}}{\Delta\rho}\{&B_1[-A_4(C_2A_2 - A_1D_2) - B_1(A_2^2 + A_3D_2) \\ &+ B_2(A_1A_2 + C_2A_3)] + B_2[A_4(C_1A_2 - A_1C_2) \\ &+ B_1(A_1A_2 + C_2A_3) - B_2(A_1^2 + C_1A_3)] \\ &- A_4[-A_4(C_1D_2 - C_2^2) - B_1(A_1D_2 - C_2A_2) \\ &+ B_2(A_1C_2 - C_1A_2)]\} + B_4/\rho, \end{aligned} \tag{26}$$

$$\begin{aligned} N = \tfrac{1}{2}\nu_{\rm eff}R^2&[B_1(C_2A_2 - A_1D_2) - B_2(C_1A_2 - A_1C_2) \\ &- A_4(C_1D_2 - C_2^2)][A_1(C_2A_2 - A_1D_2) \\ &- A_2(C_1A_2 - A_1C_2) - A_3(C_1D_2 - C_2^2)]^{-1}, \end{aligned} \tag{27}$$

where c_0 is the linear longitudinal wave velocity for a MEE circular rod and N is the dispersion parameter, both depending on the material properties as well as the geometry of the rod.

Equation (25) is a nonlinear wave equation with dispersion caused by the transverse Poisson's effect. The material properties c_0 and N are for a circular rod made of general MEE materials, and, therefore, contain the following special cases.

(1) The piezoelectric (PE) rod if we neglect the magnetic coupling. For this case equations (26) and (27) are reduced to

$$\begin{aligned} c_0^2 &= \frac{1}{\rho}\left(B_4 - \frac{A_4^2C_1 - A_3B_1^2 + 2B_1A_1A_4}{A_1^2 + A_3C_1}\right), \\ N &= \frac{1}{2}\nu_{\rm eff}R^2\frac{B_1A_1 + A_4C_1}{A_1^2 + A_3C_1}. \end{aligned} \tag{28a}$$

(2) The piezomagnetic (PM) rod if we neglect the electric coupling. For this case equations (26) and (27) are reduced to

$$\begin{aligned} c_0^2 &= \frac{1}{\rho}\left(B_4 - \frac{A_4^2D_2 - A_3B_2^2 + 2B_2A_2A_4}{A_2^2 + A_3D_2}\right), \\ N &= \frac{1}{2}\nu_{\rm eff}R^2\frac{B_2A_2 + A_4D_2}{A_2^2 + A_3D_2}. \end{aligned} \tag{28b}$$

(3) The purely elastic transversely isotropic (TI) rod if we neglect both the electric and magnetic couplings. For this case equations (26) and (27) are reduced to

$$\begin{aligned} c_0^2 &= \frac{1}{\rho}\left(B_4 - \frac{A_4^2}{A_3}\right) = \frac{1}{\rho}\left(c_{33} - \frac{2c_{13}^2}{c_{11} + c_{12}}\right), \\ N &= \frac{1}{2}\nu_{\rm eff}R^2\frac{A_4}{A_3} = \frac{1}{2}\nu_{\rm eff}R^2\frac{c_{13}}{c_{11} + c_{12}}. \end{aligned} \tag{28c}$$

(4) The purely elastic and isotropic (EI) rod if we neglect both the electric and magnetic couplings, and let $c_{11} = c_{33} = E(1-\nu_{\rm eff})/((1+\nu_{\rm eff})(1-2\nu_{\rm eff}))$ and $c_{12} = c_{13} = E\nu_{\rm eff}/((1+\nu_{\rm eff})(1-2\nu_{\rm eff}))$. For this case equations (26) and (27) are reduced to

$$c_0^2 = \frac{E}{\rho}, \qquad N = \frac{1}{2}\nu_{\rm eff}^2R^2. \tag{28d}$$

We point out that only the purely elastic isotropic rod case was studied before and, furthermore, our results (equations (28*d*)) are exactly the same as in [13].

In deriving the nonlinear wave equation (25), we have made use of the effective Poisson's ratio $\nu_{\rm eff}$. Its expression can be simply obtained from the conditions we derived above: $\sigma_\theta = \sigma_r = 0$, $\tau_{rz} = \tau_{r\theta} = \tau_{\theta z} = 0$, $D_r = D_\theta = D_z = 0$, $B_r = B_\theta = B_z = 0$, which gives $\nu_{\rm eff}$ as

$$\begin{aligned} \nu_{\rm eff} = -\frac{\varepsilon_r}{\varepsilon_z} = [&q_{33}q_{31} + \mu_{33}c_{13} \\ &+ (d_{33}q_{31} - e_{31}\mu_{33})(e_{33}\mu_{33} - q_{33}d_{33})/(d_{33}^2 - \varepsilon_{33}\mu_{33})] \\ &\times [2q_{31}^2 + \mu_{33}(c_{11} + c_{12}) + 2(d_{33}q_{31} - e_{31}\mu_{33}) \\ &\times (e_{31}\mu_{33} - q_{31}d_{33})/(d_{33}^2 - \varepsilon_{33}\mu_{33})]^{-1}. \end{aligned} \tag{29}$$

4. Solitary waves in a MEE circular rod

We assume that the traveling wave solution of equation (25) is in the following form (recall that $u = \partial U/\partial z$):

$$u = u(\xi), \qquad \xi = k(z - ct), \tag{30}$$

where k and c are the wavenumber and wave velocity, respectively. Then equation (25) can be transformed into an ordinary differential equation:

$$k^2u_{\xi\xi\xi\xi} - \frac{c^2 - c_0^2}{Nc^2}u_{\xi\xi} + \frac{c_0^2}{2Nc^2}(u^2)_{\xi\xi} = 0. \tag{31}$$

Integrating equation (31) twice with respect to ξ, and letting the integral constants be zero for convenience, we then have

$$k^2u_{\xi\xi} - \frac{c^2 - c_0^2}{Nc^2}u + \frac{c_0^2}{2Nc^2}u^2 = 0. \tag{32}$$

The solution to equation (32) can be found in terms of the Jacobi elliptic cosine function expansion ($\mathrm{cn}(\xi, m) \equiv \mathrm{cn}(\xi)$ for brevity). In other words, the solution of equation (32) can be written as

$$u(\xi) = \sum_{j=0}^{n} a_j\mathrm{cn}^j\xi. \tag{33}$$

To facilitate the following discussion, we list below a couple of important relations for the three kinds of Jacobi elliptic functions and their asymptotic behaviors:

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \qquad \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1, \tag{34}$$

$$\begin{gathered} \mathrm{d}(\mathrm{sn}\xi)/\mathrm{d}\xi = \mathrm{cn}\xi\mathrm{dn}\xi, \qquad \mathrm{d}(\mathrm{cn}\xi)/\mathrm{d}\xi = -\mathrm{sn}\xi\mathrm{dn}\xi, \\ \mathrm{d}(\mathrm{dn}\xi)/\mathrm{d}\xi = -m^2\mathrm{sn}\xi\mathrm{cn}\xi, \end{gathered} \tag{35}$$

$$\mathrm{sn}(\xi, 0) = \sin\xi, \qquad \mathrm{cn}(\xi, 0) = \cos\xi, \qquad \mathrm{dn}(\xi, 0) = 1, \tag{36}$$

$$\mathrm{sn}(\xi, 1) = \tanh\xi, \qquad \mathrm{cn}(\xi, 1) = \mathrm{dn}(\xi, 1) = \mathrm{sech}\,\xi, \tag{37}$$

where $\mathrm{sn}\xi$ and $\mathrm{dn}\xi$ are, respectively, the Jacobi elliptic sine function and the third kind of Jacobi elliptic function, and m is the modulus ($0 \leqslant m \leqslant 1$).

From equation (33), we observe that the highest order of $u(\xi)$ is n, i.e.,

$$O(u(\xi)) = n. \tag{38}$$

Based on the differential relations in equation (35), it is easy to show that

$$O(\mathrm{d}u/\mathrm{d}\xi) = n+1, \qquad O(\mathrm{d}^2u/\mathrm{d}\xi^2) = n+2. \tag{39}$$

Therefore, the highest order of u^2 is

$$O(u^2) = 2n. \tag{40}$$

The highest order, n, in equation (38) is determined by the homogeneous balance principle, i.e., the order balance between the highest order derivative terms and the highest degree nonlinear term in the nonlinear equation (32). As such we have $2n = n + 2$ so that $n = 2$. Thus equation (33) is rewritten as

$$u(\xi) = a_0 + a_1\mathrm{cn}\xi + a_2\mathrm{cn}^2\xi. \tag{41}$$

Taking the derivative of equation (41) twice with respect to ξ, we have

$$\mathrm{d}^2u/\mathrm{d}\xi^2 = 2a_2(1-m^2) - a_1(1-2m^2)\mathrm{cn}\xi + 4a_2(2m^2-1)\mathrm{cn}^2\xi - 2a_1m^2\mathrm{cn}^3\xi - 6a_2m^2\mathrm{cn}^4\xi. \tag{42}$$

Substituting equations (41) and (42) into equation (32) and comparing the terms with the same powers of cnξ, we find the expressions for the expansion coefficients as

$$a_0 = \frac{c^2 - c_0^2 - 4Nc^2k^2(2m^2-1)}{c_0^2}, \qquad a_1 = 0, \qquad a_2 = \frac{12Nc^2k^2m^2}{c_0^2}, \tag{43}$$

Therefore, the solution to the nonlinear wave equation (31) is

$$u(\xi) = \frac{c^2 - c_0^2 - 4Nc^2k^2(2m^2-1)}{c_0^2} + \frac{12Nc^2k^2m^2}{c_0^2}\mathrm{cn}^2(\xi, m). \tag{44}$$

It can be noted that this is an exact periodical solution! To obtain the solitary wave solution, we take the limit of the modulus $m \to 1$, so that cn$\xi \to$ sech ξ. Thus, equation (44) is finally reduced to the solitary wave solution as

$$u(\xi) = \frac{c^2 - c_0^2 - 4Nc^2k^2}{c_0^2} + \frac{12Nc^2k^2}{c_0^2}\mathrm{sech}^2\xi. \tag{45}$$

To discuss only the wave feature, we let the constant term be zero. In other words,

$$k^2 = \frac{c^2 - c_0^2}{4Nc^2}. \tag{46}$$

Substituting equation (46) into equation (45), a standard solitary wave solution of equation (32) is obtained:

$$u(\xi) = A\mathrm{sech}^2\frac{z-ct}{\Lambda}, \tag{47}$$

where A is the wave amplitude and Λ is the wavelength, expressed as

$$A = \frac{3(c^2 - c_0^2)}{c_0^2}, \qquad \Lambda = \frac{2\pi}{k} = 4\pi\sqrt{\frac{Nc^2}{c^2 - c_0^2}} = 4\pi\sqrt{\frac{3N}{A}}, \tag{48}$$

where $c > c_0$ for the existence of a solitary wave. It should be also noticed that the wavelength is inversely proportional to the square root of the wave amplitude.

Table 1. Material coefficients for the $BaTiO_3$–$CoFe_2O_4$ MEE composite rod based on the simple rule of mixture $M_C = M_E v_f + M_M(1 - v_f)$. Units: elastic constants, c_{ij}, in 10^9 N m^{-2}, piezoelectric constants, e_{ij}, in C m^{-2}, piezomagnetic constants, q_{ij}, in N Am^{-1}, dielectric constants, ε_{ij}, in 10^{-9} C^2 Nm^{-2}, magnetic constants, μ_{ij}, in 10^{-4} Ns^2 C^{-2}, magneto-electric coefficients, d_{ij}, in 10^{-12} N s V^{-1} C^{-1} and ρ in 10^3 kg m^{-3}.

v_f	0% (PM)	50% (MEE)	100% (PE)
c_{11}	286	225	166
c_{12}	173	125	77
c_{13}	170	124	78
c_{33}	269.5	216	162
c_{44}	45.3	44	43
e_{31}	0	−2.2	−4.4
e_{33}	0	9.3	18.6
e_{15}	0	5.8	11.6
ε_{11}	0.08	5.64	11.2
ε_{33}	0.093	6.35	12.6
μ_{11}	5.9	2.97	0.05
μ_{33}	1.57	0.835	0.1
q_{31}	580	290.2	0
q_{33}	700	350	0
q_{15}	550	275	0
d_{11}	0	0	0
d_{33}	0	0	0
ρ	5.3	5.55	5.8

5. Numerical results and discussion

In our numerical examples, we assume that the infinite homogeneous MEE circular rod is made of the composite $BaTiO_3$–$CoFe_2O_4$ with different volume fractions (v_f) of $BaTiO_3$. The rod has a radius $R = 0.05$ m. The material properties of the composite are estimated using the simple rule of mixture according to the volume fraction [18, 19]. Denoting for the composite the volume fraction of $BaTiO_3$ as v_f, and that of $CoFe_2O_4$ as $1 - v_f$, we then have

$$M_C = M_E v_f + M_M(1 - v_f), \tag{49}$$

where M represents an arbitrary material constant, and the subscripts C, E, and M indicate the composite, piezoelectric phase and piezomagnetic phase, respectively.

In the following, we consider three different cases of material combinations, by taking the volume fraction of $BaTiO_3$ as 0% (PM), 50% (MEE) and 100% (PE), respectively. Obviously, when $v_f = 0$, the composite is piezomagnetic (PM), whilst $v_f = 100\%$ corresponds to a piezoelectric (PE) material [20]. The MEE material properties are listed in table 1. Another two purely elastic materials are also considered. One is the transversely isotropic elastic material (TI) taking from 50% (MEE) only the elastic coefficients. The other one is the effective elastic isotropy (EI) obtained from the TI by making it isotropic (i.e., letting $c_{11} = c_{33}$ and $c_{12} = c_{13}$).

For the rod made of the five different materials, we have also calculated the linear wave velocity c_0, effective Poisson's ratio ν_{eff}, dispersion parameter N, and the wavelength Λ, as listed in table 2. It is observed that the wave velocity c_0 in the PM rod is the highest, followed by that in the 50% MEE rod and then that in the PE rod. The wave velocity in the

Table 2. Linear wave velocity c_0, effective Poisson's ratio ν_{eff}, dispersion parameter N, and wavelength Λ for piezoelectric $BaTiO_3$ (PE), piezoelectric–piezomagnetic $BaTiO_3$–$CoFe_2O_4$ (MEE), piezomagnetic $CoFe_2O_4$ (PM), purely transversely isotropic (TI) material with elastic properties from 50% MEE, and purely elastic isotropy (EI) reduced from TI as discussed in the text. (Note: TI stands for the purely elastic transverse isotropy with its elastic material properties being directly taken from 50% MEE; EI stands for the purely elastic isotropy where the isotropic material properties are obtained from TI by forcing the isotropy condition.)

v_f	0% (PM)	50% (MEE)	100% (PE)	Transverse isotropy (TI)	Elastic isotropy (EI)
c_0 ($\times 10^3$ m s^{-1})	5.2131	5.1446	5.0498	4.8003	4.8398
ν_{eff}	0.3725	0.3451	0.2906	0.3543	0.36
N ($\times 10^{-4}$ m^2)	1.735	1.489	1.056	1.570	1.620
Λ (m) ($c = 1.1c_0$)	0.3973	0.3680	0.3099	0.3729	0.3839

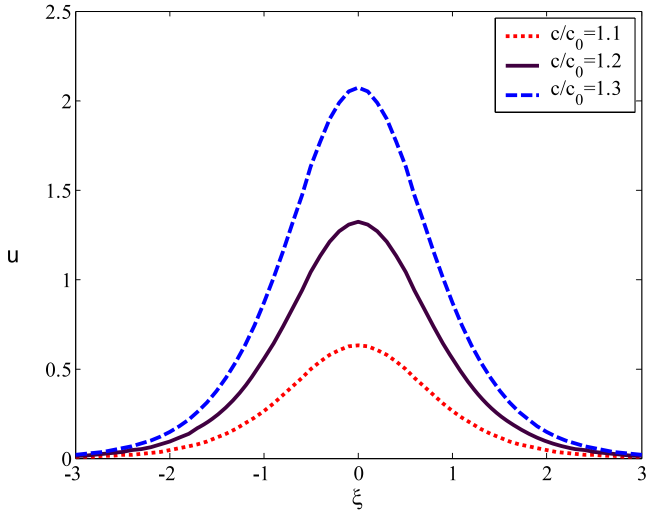

Figure 2. Solitary waves in a 50% MEE rod as a function of $\xi = k(z - ct)$ for three different wave velocity ratios c/c_0.

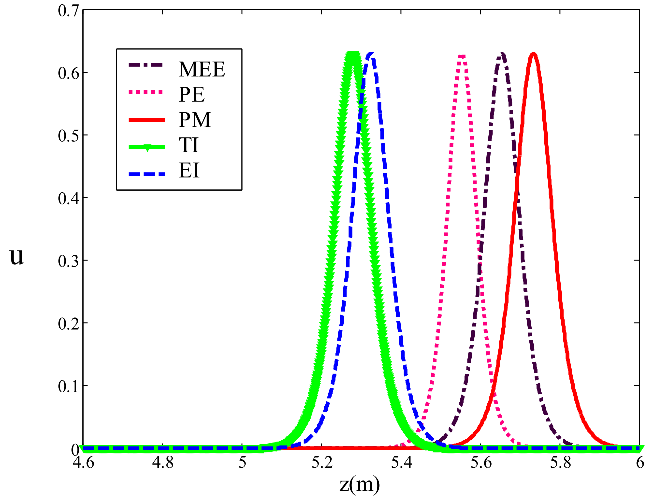

Figure 3. Solitary waves in a rod made of different materials with fixed $t = 0.001$ s and $c = 1.1c_0$.

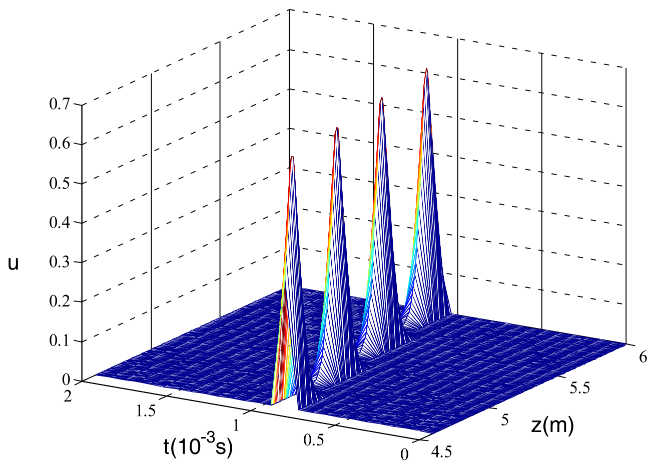

Figure 4. Solitary waves versus time t and coordinate z in a 50% MEE rod with fixed $c = 1.1c_0$.

corresponding purely elastic rod (TI and EI) is clearly lower than those in the coupled rod. It is interesting to note further that the largest values for ν_{eff}, N and Λ are also associated with the PM rod.

Figure 2 shows the solitary wave u in the 50% MEE rod versus the variable $\xi = k(z - ct)$ for different velocity ratios c/c_0. It is clear that the maximum of u is reached at the center $\xi = 0$, and that its amplitude is symmetrical about the center. We also notice that, with increasing velocity ratio c/c_0, its amplitude increases, but its wavelength decreases. In other words, a solitary wave with larger amplitude will have a narrower wavelength, a typical dispersion characteristic in a nonlinear wave.

Figure 3 shows the solitary wave u as a function of the composite rod coordinate z. The time is fixed at $t = 0.001$ s and the nonlinear velocity at $c = 1.1c_0$. The five different materials MEE, PE, PM, TI, and EI are those listed in table 2. It is interesting that while their amplitudes are very close to each other, these waves are clearly grouped into two classes: the piezoelectric and piezomagnetic coupled class (PM, MEE, and PE) with a relatively higher wave velocity and the purely elastic class (EI and TI) with a relatively lower wave velocity.

The comprehensive solitary wave features are presented in figure 4 for a rod made of 50% MEE with fixed $c = 1.1c_0$. It is shown that the solitons appear only at a certain given combination of the time t and coordinate z, and that the same wave pattern will repeat itself at these special combinations of t and z. This again demonstrates that a solitary wave in a MEE rod is very stable and therefore could have potential applications, say in non-destructive evaluation of structures made of the advanced MEE material.

Figure 5 plots the relations between the wave velocity c and wavenumber k for the five different materials in table 2.

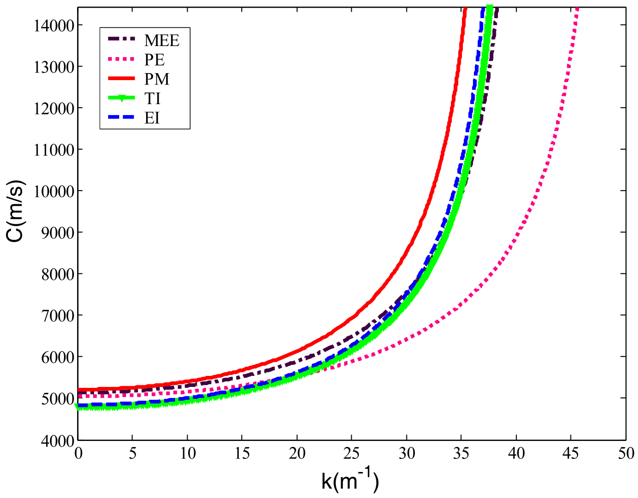

Figure 5. Wave velocity c versus wavenumber k in a rod made of different materials.

It is observed that when the wavenumber is small, the wave velocity in the coupled class (PM, MEE, and PE) is higher than that in the purely elastic class (EI and TI). However, with increasing wavenumber, these five materials form three new classes: PM is the first with the highest velocity; in the middle, we have MEE, EI and TI; and finally PE has the lowest velocity.

6. Conclusions

In this paper, by assuming geometric nonlinearity in the longitudinal direction and by introducing the effective Poisson's ratio for the transverse Poisson's effect, we have derived the nonlinear solitary wave equation in a long MEE circular rod. The wave equation is then solved by the Jacobi elliptic function expansion method and numerical examples are further presented for the wave in a rod made of five different materials: the three-phase fully coupled MEE, coupled piezoelectric PE, coupled piezomagnetic PM, purely elastic but transverse isotropy TI and purely elastic isotropy EI. It is demonstrated that the solitary wave not only exists in such rods but also shows different features in different materials, which could have potential applications in non-destructive evaluation of structures made of the advanced MEE material.

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References

[1] Benjamin T B 1972 The stability of solitary waves *Proc. R. Soc.* A **328** 153–83
[2] Russell J S 1844 Report on waves *14th Mtg of the British Association for the Advancement of Science*
[3] Lake B M, Yuen H C, Rungaldier H and Ferguson W E 1977 Nonlinear deep-water waves: theory and experiment. Part 2. Evolution of a continuous wave train *J. Fluid Mech.* **83** 49–74
[4] Peyrard M 2004 Nonlinear dynamics and statistical physics of DNA *Nonlinearity* **17** 1–40
[5] Kivshar Y S and Agrawal G P 2003 *Optical Solitons: From Fibers to Photonic Crystals* (San Diego, CA: Academic)
[6] Bakırtas I and Antar N 2005 Effect of stenosis on solitary waves in arteries *Int. J. Eng. Sci.* **43** 730–43
[7] Yang R and Zhang Y 2011 Exact combined solitary wave solutions in nonlinear metamaterials *J. Opt. Soc. Am.* B **28** 123–7
[8] Zhang S Y and Zhuang W 1988 The strain solitary waves in a nonlinear elastic rod *Acta Mech. Sin.* **20** 58–67 (in Chinese)
[9] Guo J G, Zhou L J and Zhang S Y 2005 The geometrical nonlinear waves in finite deformation elastic rods *Appl. Math. Mech.* **26** 667–74
[10] Samsonov A M 2001 *Strain Solitons in Solids and How to Construct Them* (New York: Chapman and Hall/CRC)
[11] Porubov A V and Velarde M G 2000 Dispersive–dissipative solitons in nonlinear solids *Wave Motion* **31** 197–207
[12] Zhang S Y and Liu Z F 2008 Three kinds of nonlinear dispersive waves in elastic rods with finite deformation *Appl. Math. Mech.* **29** 909–17
[13] Liu Z F and Zhang S Y 2006 Nonlinear waves and periodic solution in finite deformation elastic rod *Acta Mech. Solida Sin.* **9** 1–8
[14] Christov C I, Marinov T T and Marinova R S 2009 Identification of solitary wave solutions as an inverse problem: application to shapes with oscillatory tails *Math. Comput. Simul.* **80** 56–65
[15] Chen J Y, Pan E and Chen H L 2007 Wave propagation in magneto-electro-elastic multilayered plates *Int. J. Solids Struct.* **44** 1073–85
[16] Chen P and Shen Y 2007 Propagation of axial shear magneto-electro-elastic waves in piezoelectric–piezomagnetic composites with randomly distributed cylindrical inhomogeneities *Int. J. Solids Struct.* **44** 1511–32
[17] Wu B, Yu J G and He C F 2008 Wave propagation in non-homogeneous magneto-electro-elastic plates *J. Sound Vib.* **317** 250–64
[18] Aboudi J 2001 Micromechanical analysis of fully coupled electro-magneto-thermal-elastic multiphase composite *Smart Mater. Struct.* **10** 867–77
[19] Sih G C and Chen E P 2003 Dilatational and distortional behavior of cracks in magnetoelectroelastic materials *Theor. Appl. Fract. Mech.* **40** 1–21
[20] Chen W Q, Pan E, Wang H M and Zhang C Z 2010 Theory of indentation on multiferroic composite materials *J. Mech. Phys. Solids* **58** 1524–51