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# Eshelby's problem in an anisotropic multiferroic bimaterial plane 

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#### Abstract

We solve analytically the Eshelby's problem in an anisotropic multiferroic bimaterial plane. The solution is based on the extended Stroh formalism of complex variables, and is valid for the inclusion of arbitrary shapes, described by a Laurent polynomial, a polygon, or the one bounded by a Jordan curve. Furthermore, the results in the corresponding half plane and full plane can be reduced directly from the bima-terial-plane solution. As such, the solution unifies the complex variable method and the Green's function method, extending further to the multiferroic bimaterial plane of general anisotropy. The essential eigenfunctions are also identified by which the induced fields can be simply determined. Numerical results are presented to investigate the features of these eigenfunctions as well as the strain, electric and magnetic fields (components of the extended Eshelby tensor). Particularly, we present the values of these fields at the center of the $N$-side regular polygonal inclusion and also the average values of these fields over the inclusion area. The effect of the half-plane traction-free surface condition as well as the effect of various couplings on the induced fields is discussed in detail. For the N -side regular polygonal inclusion, it is found that, when the inclusion is in the full plane, both the center and average values of the Eshelby tensor are independent of the side number $N$, except for $N=4$. We further show that the piezoelectric and piezomagnetic coupling coefficients could significantly affect the Eshelby tensor. These features should be useful in controlling the Eshelby tensor for the design of better multiferroic composites. Typical contours of the field quantities in and around the inclusion bounded by both straight and curved line segments in a multiferroic bimaterial plane are also presented.


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## 1. Introduction

The Eshelby's problem is concerned with determining the elastic field of a linearly elastic, homogeneous, and infinite solid containing a subdomain called inclusion which is subjected to a prescribed uniform strain or eigenstrain. Through his celebrated inclusion solution of an ellipsoidal inclusion, called the (classical) Eshelby tensor in the literature, Eshelby $(1957,1959)$ introduced the equivalent inclusion concept to transform the problem of analyzing the stress field in matrix-inclusion solids into an algebraic operation problem. This method now becomes an indispensable part of the theoretical foundation of contemporary composite mechanics and materials (e.g., Mura, 1982; Nemat-Nasser and Hori, 1993), and has many applications in today's nano-science and nano-technologies (e.g., Li and Wang, 2008).

Material anisotropy and piezoelectric coupling are two common features in composites, and thus, research of the corresponding

[^0]Eshelby's inclusion problem in these composites becomes necessary. Under two-dimensional (2D) deformation, the Eshleby's inclusion problem in anisotropic elastic and piezoelectric planes were solved by $\mathrm{Ru}(2000,2003)$ using the conformal mapping method, by Pan and coworkers using the Green's function method (e.g., Pan, 2004), and by Zou and coworkers using a unified approach (Zou et al., 2011), among others. Also employing the Green's function approach, Jiang and Pan (2004) solved the Eshelby problem in the corresponding magnetoelectroelastic bimaterial plane where the inclusion was assumed to be bounded by a finite number of straight line segments.

Recently, due to their special multiphase coupling features, multiferroic materials/composites are attracting intensive research from scientists and engineers. Various interesting results have demonstrated that the magnetoelectric coupling effect can be enhanced by adjusting the relative volume fractions of the singlephase piezoelectric and magnetostrictive materials and by increasing properly the grading factor in the multiferroic composition (i.e., Petrov and Srinivasan, 2008; Wang et al., 2009). Since one of the common composites is the particulate one (made of fibre-reinforced or particles-reinforced composites), the Eshelby's inclusion problem in the corresponding 2D multiferroic plane becomes very
important. With this inclusion problem being solved, the corresponding inhomogeneity problem can be solved based on the micromechanics theory, and the effective property of the composites can be predicted, providing the important parameters for the best design of multiferroic composites.

Thus, in this paper, we present the analytical solution of the Eshelby's inclusion problem in an anisotropic multiferroic bimaterial plane. The inclusion can be of an arbitrary shape and can be bounded by straight and curved line segments. Since the solution is in an explicit and closed form, various physical features associated with the inclusion can be directly extracted from the solution. In particular, the center and average values of the induced fields in the inclusion are investigated in details and results are shown via both tables and contours.

The paper is organized as follows: In Section 2, we briefly present the governing equations and the general solutions in terms of the extended Stroh formalism. In Section 3, solutions are derived for the Eshelby's inclusion problem in an anisotropic multiferroic full plane where the inclusion is of any geometric shape. In Section 4, we solve the corresponding Eshelby's problem in a multiferroic bimaterial plane. In Section 5, explicit expressions of the eigenstrain-induced fields are obtained for various shapes of the inclusion. Numerical examples are presented in Section 6, and conclusions are drawn in Section 7.

## 2. Governing equations and general solutions in terms of the extended Stroh formalism

### 2.1. Governing equations of anisotropic multiferroic media

The governing equations for a linear multiferroic solid are given by (e.g., Chen et al. 2010)

$$
\left.\begin{array}{l}
\sigma_{i j}=C_{i j k l} u_{k, l}+e_{k i j} \phi_{, k}+q_{k i j} \varphi_{, k},  \tag{1}\\
D_{k}=e_{k i j} u_{i, j}-\kappa_{k l} \phi_{, l}-d_{k l} \varphi_{, l} \\
B_{k}=q_{k i j} u_{i, j}-d_{l k} \phi_{, l}-\mu_{k l} \varphi_{, l} \\
\sigma_{i j j}=0, D_{k, k}=0, B_{k, k}=0
\end{array}\right\}
$$

where we have assumed that there is no body force, no electric charge density, and no electric current density; repeated indices mean summation, a comma followed by $i(=1,2,3)$ denotes the partial derivative with respect to the $i$ th spatial coordinate; $u_{i}, \varphi$ and $\phi$ are the elastic displacements, electric potential, and magnetic potential; $\sigma_{i j}, D_{i}$ and $B_{i}$ are the stress, electric displacement, and magnetic induction (i.e., magnetic flux); $C_{i j k l}, \kappa_{l j}$ and $\mu_{i j}$ are the elastic, dielectric and magnetic permeability constants; $e_{i j k}, q_{i j k}$ and $d_{i j}$ are the piezoelectric, piezomagnetic, and magnetoelectric constants.

We now define the extended displacement and stress components by
$u_{I}=\left\{\begin{array}{ll}u_{i}, & I=i=1,2,3, \\ \phi, & I=4, \\ \varphi, & I=5,\end{array} \quad \sigma_{I j}=\left\{\begin{array}{lc}\sigma_{i j}, & I=i=1,2,3, \\ D_{j}, & I=4, \\ B_{j}, & I=5,\end{array}\right.\right.$
and adopt the extended stiffness notation as below

$$
C_{l j K l}= \begin{cases}C_{i j k l}, & I, K=i, k=1,2,3 \\ e_{i j l}, & K=4, I=i=1,2,3 \\ e_{j k l}, & I=4, K=k=1,2,3 \\ q_{i j l}, & K=5, I=i=1,2,3 \\ q_{j k l}, & I=5, K=k=1,2,3 \\ -\kappa_{j l}, & I=K=4 \\ -\mu_{j l}, & I=K=5 \\ -d_{i j} & I=4, K=5 \text { or } I=5, K=4\end{cases}
$$

Then Eq. (1) can be recast into
$\sigma_{l j}=C_{l j k l} u_{K, l}, \quad \sigma_{I j, j}=0$,

### 2.2. General solutions in terms of the extended Stroh formalism

We assume that the extended 2D problem depends only on coordinates $x_{1}$ and $x_{2}$; then the general solution of Eq. (4) can be obtained by virtue of the extended Stroh formalism (Kuo and Barnett, 1991; Suo et al., 1992; Liang and Hwu, 1996; Ting, 1996; Jiang and Pan, 2004). More precisely, we seek the solution in the form
$\mathbf{u}=\left(u_{1}, u_{2}, u_{3}, \phi, \varphi\right)^{T}=\mathbf{a} f\left(x_{1}+p x_{2}\right)$,
where $\mathbf{a}$ is a five-dimensional vector; $p$ is a complex number; $f\left({ }^{*}\right)$ is an analytic function of its variable '*' and the superscript ' $T$ ' denotes the transpose of a matrix or vector. Thus, Eq. (4) is satisfied by the arbitrary analytic function $f$ given in Eq. (5) if (see, e.g., Chung and Ting 1996)

$$
\begin{equation*}
\left[\mathbf{Q}+p\left(\mathbf{R}+\mathbf{R}^{T}\right)+p^{2} \mathbf{T}\right] \mathbf{a}=0 \tag{6}
\end{equation*}
$$

where the $5 \times 5$ matrix $\mathbf{R}$ and the $5 \times 5$ symmetric matrices $\mathbf{Q}$ and $\mathbf{T}$ are defined by
$R_{I K}=C_{I 1 K 2}, \quad Q_{I K}=C_{I 1 K 1}, \quad T_{I K}=C_{I 2 K 2}$.
For the existence of a non-zero vector a, the characteristic equation of the eigenvalue problem (6), namely
$\operatorname{det}\left[\mathbf{Q}+p\left(\mathbf{R}+\mathbf{R}^{T}\right)+p^{2} \mathbf{T}\right]=0$
must be satisfied. Furthermore, for a stable material, the roots of Eq. (8) form five conjugate pairs with non-zero imaginary parts (e.g., Eshelby et al., 1953). Assuming that $p_{I}(I=1,2,3,4,5)$ are the five distinct roots with positive imaginary parts and $\mathbf{a}_{I}(I=1,2,3,4,5)$ the corresponding eigenvectors, then the general solution (the extended displacement $\mathbf{u}$ and the extended stress function $\psi$ ) of Eq. (4) can be written as

$$
\left.\begin{array}{l}
\mathbf{u}=\left(u_{1}, u_{2}, u_{3}, \phi, \varphi\right)^{T}=2 \operatorname{Re}[\mathbf{A f}(z)]  \tag{9}\\
\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}, \psi_{4}, \psi_{5}\right)^{T}=2 \operatorname{Re}[\mathbf{B f}(z)]
\end{array}\right\}
$$

where 'Re' stands for the real part and the constant matrices $\mathbf{A}$ and $\mathbf{B}$ are defined through $\mathbf{a}_{I}$ as follows:
$\left.\begin{array}{l}\mathbf{b}_{I}=\left(\mathbf{R}^{T}+p_{I} \mathbf{T}\right) \mathbf{a}_{I}=-p_{I}^{-1}\left(\mathbf{Q}+p_{I} \mathbf{R}\right) \mathbf{a}_{I}, I=1,2,3,4,5, \\ \mathbf{A}=\left(\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \mathbf{a}_{4}, \mathbf{a}_{5}\right), \mathbf{B}=\left(\mathbf{b}_{1}, \mathbf{b}_{2}, \mathbf{b}_{3}, \mathbf{b}_{4}, \mathbf{b}_{5}\right) .\end{array}\right\}$
In Eq. (9), the five-dimensional vector $\mathbf{f}(z)$ is formed by five arbitrary analytic functions $f_{I}\left(z_{I}\right)(I=1,2,3,4,5)$ as
$\left.\begin{array}{l}\mathbf{f}(z)=\left[f_{1}\left(z_{1}\right), f_{2}\left(z_{2}\right), f_{3}\left(z_{3}\right), f_{4}\left(z_{4}\right), f_{5}\left(z_{5}\right)\right]^{T}, \\ z_{I}=x_{1}+p_{I} x_{2}, p_{I}=\alpha_{I}+i \beta_{I}, \beta_{I}>0, I=1,2,3,4,5,\end{array}\right\}$
Furthermore, the extended stress function vector $\psi$ is related to the extended stress components by
$\sigma_{I 1}=-\psi_{I, 2}, \quad \sigma_{I 2}=\psi_{I, 1}, \quad I=1,2,3,4,5$.
The eigenvalues $\left\{p_{I}\right\}$ and eigenvectors $\left\{\mathbf{a}_{l}, \mathbf{b}_{I}\right\}$ depend on the extended material stiffness matrix $C_{l j K l}$ and can be equivalently determined by the following eigenrelation (Chung and Ting, 1996):

$$
\begin{equation*}
\mathbf{N} \boldsymbol{\xi}=p \boldsymbol{\xi} \tag{13}
\end{equation*}
$$

where $\mathbf{N}$ is a $10 \times 10$ fundamental matrix and $\xi$ is a $10 \times 1$ column vector, both defined by
$\mathbf{N}=\left[\begin{array}{ll}\mathbf{N}_{1} & \mathbf{N}_{2} \\ \mathbf{N}_{3} & \mathbf{N}_{1}^{T}\end{array}\right], \quad \boldsymbol{\xi}=\binom{\mathbf{a}}{\mathbf{b}}$,
where
$\mathbf{N}_{1}=-\mathbf{T}^{-1} \mathbf{R}^{T}, \quad \mathbf{N}_{2}=\mathbf{T}^{-1}, \quad \mathbf{N}_{3}=\mathbf{R T}^{-1} \mathbf{R}^{T}-\mathbf{Q}$,
with $\mathbf{Q}, \mathbf{R}, \mathbf{T}$ being the real matrices defined by Eq. (7). Another approach to compute the eigenvalues $\left\{p_{l}\right\}$ and eigenvectors $\left\{\mathbf{a}_{l}, \mathbf{b}_{I}\right\}$ is via the Lekhnitskii formalism (Lekhnitskii, 1963), where the eigenvectors can be given explicitly after the eigenvalues $\left\{p_{I}\right\}$ are solved.

### 2.3. Alternative expressions

It is often useful to write the extended stress and strain fields explicitly in a different form (a symmetric form in elasticity as in Mantic and Paris, 1997; Ting 1998, 2000). Actually, from
$\sigma_{12}=2 \operatorname{Re}\left[\sum_{M} B_{I M} f_{M}^{\prime}\left(z_{M}\right)\right], \quad \sigma_{I 1}=-2 \operatorname{Re}\left[\sum_{M} B_{I M} p_{M} f_{M}^{\prime}\left(z_{M}\right)\right]$,
and
$B_{1 M}=-p_{M} B_{2 M} \quad$ (no summation)
which are given by Eq. (12) and $\sigma_{21}=\sigma_{12}$, we can reach
$\boldsymbol{\sigma}_{I j}=2 \operatorname{Re} \sum_{M} B_{I M} B_{2 M}^{-1} f_{M}^{\prime}\left(z_{M}\right) B_{j M} \quad($ for $j=1,2)$.
Similarly, from
$u_{I, 1}=2 \operatorname{Re}\left[\sum_{M} A_{I M} f_{M}^{\prime}\left(z_{M}\right)\right], \quad u_{I, 2}=2 \operatorname{Re}\left[\sum_{M} A_{I M} p_{M} f_{M}^{\prime}\left(z_{M}\right)\right]$,
using the notation

$$
\left[\begin{array}{lllll}
B_{21} & -B_{11} & 0 & 0 & 0  \tag{20}\\
B_{22} & -B_{12} & 0 & 0 & 0 \\
B_{23} & -B_{13} & 0 & 0 & 0 \\
B_{24} & -B_{14} & 0 & 0 & 0 \\
B_{25} & -B_{15} & 0 & 0 & 0
\end{array}\right]=\mathbf{B}^{T}\left[\begin{array}{ccccc}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right] \equiv \mathbf{B}^{T} \mathbf{K},
$$

and the geometric relations $\varepsilon_{I J}=\left(u_{I J}+u_{J, I}\right) / 2$ with $u_{\mathrm{I}, 4}=0$ and $u_{\mathrm{I}, 5}=0$, one can further arrive at
$\varepsilon_{l j}=\operatorname{Re}\left[\sum_{M} A_{I M} B_{2 M}^{-1} f_{M}^{\prime}\left(z_{M}\right) B_{N M} K_{N j}+A_{j M} B_{2 M}^{-1} f_{M}^{\prime}\left(z_{M}\right) B_{N M} K_{N I}\right]$,
for $j=1,2$.

## 3. General solution to Eshelby's inclusion problem in an anisotropic multiferroic full plane

### 3.1. An inclusion in a full plane

We will solve the problem via the extended Stroh formalism. In so doing, we first define the following matrix notations
$\boldsymbol{\sigma}_{p}=\left[\sigma_{1 p}, \sigma_{2 p}, \sigma_{3 p}, D_{p}, B_{p}\right]^{T}$,
$\boldsymbol{\varepsilon}_{p}=\left[\varepsilon_{1 p}, \varepsilon_{2 p}, \varepsilon_{3 p},-0.5 E_{p},-0.5 H_{p}\right]^{T}, \quad p=1,2$,
where $E_{p}$ and $H_{p}$ are the electric and magnetic fields defined by $E_{p}=\varphi_{p}, H_{p}=\varphi_{p}$, respectively; and
$\mathbf{f}^{\prime}=\left[f_{1}^{\prime}\left(z_{1}\right), f_{2}^{\prime}\left(z_{2}\right), f_{3}^{\prime}\left(z_{3}\right), f_{4}^{\prime}\left(z_{4}\right), f_{5}^{\prime}\left(z_{5}\right)\right]^{T}$,
where $f_{I}^{\prime}\left(z_{1}\right)(I=1,2,3,4,5)$ are the derivatives of the eigenfunctions $f_{l}\left(z_{l}\right)$ with respect to $z_{l}$. Furthermore, a diagonal matrix composed of five elements, say $\left\{p_{I}\right\}$, is denoted by $\left\langle p_{*}\right\rangle$.

Let $\Omega$ be the $x_{1}-\chi_{2}$ plane made of a homogeneous but anisotropic multiferroic medium. It contains a subdomain, say $\omega$, which undergoes an extended uniform eigenstrain $\boldsymbol{\varepsilon}^{*}$ (i.e., elastic eigenstrain $\left\{\varepsilon_{i p}^{*}, i=1,2,3 ; p=1,2\right\}$, plus eigenelectric and eigenmagnetic fields $\left.\left\{E_{p}^{*}, H_{p}^{*}, p=1,2\right\}\right)$. Let $\varpi$ denote the supplement of $\omega$


Fig. 1. An arbitrary inclusion $\omega$ in a full plane, with its inner point $\mathbf{z}$, boundary point $\mathbf{y}$, and the increasing direction of $d \mathbf{y}$.
in the $x_{1}-x_{2}$ plane, $\Gamma=\partial \omega$ the curve separating $\omega$ and $\varpi$ (Fig. 1), and with $\omega$ and $\sigma$ being defined as open sets. Throughout this paper, we indicate the quantities in $\omega$ and $\varpi$ with the superscripts (or subscripts) 'in' and 'out', respectively. The extended eigendisplacement field $\mathbf{u}^{*}$ in $\omega$ corresponding to the extended eigenstrain fields in $\omega$ can be expressed as
$\mathbf{u}^{*}=\left(\begin{array}{c}\varepsilon_{11}^{*} x_{1}+\varepsilon_{12}^{*} x_{2} \\ \varepsilon_{22}^{*} x_{2}+\varepsilon_{12}^{*} x_{1} \\ 2 \varepsilon_{13}^{*} x_{1}+2 \varepsilon_{23}^{*} x_{2} \\ -\left(E_{1}^{*} x_{1}+E_{2}^{*} x_{2}\right) \\ -\left(H_{1}^{*} x_{1}+H_{2}^{*} x_{2}\right)\end{array}\right)$,
where $\left\{\varepsilon_{11}^{*}, \varepsilon_{12}^{*}, \varepsilon_{22}^{*}\right\}$ are the in-plane eigenstrains, $\left\{\varepsilon_{13}^{*}, \varepsilon_{23}^{*}\right\}$ the antiplane eigenstrains, $\left\{E_{1}^{*}, E_{2}^{*}\right\}$ the eigenelectric field, and $\left\{H_{1}^{*}, H_{2}^{*}\right\}$ the eigenmagnetic field. It is convenient to introduce a diagonal matrix

$$
\begin{equation*}
\mathbf{L}=\langle 1,1,2,2,2\rangle \tag{25}
\end{equation*}
$$

and notation
$\tilde{\boldsymbol{\varepsilon}}_{p}^{*}=\mathbf{L} \boldsymbol{\varepsilon}_{p}^{*}, \quad p=1,2$.
Let $\left\{u_{i}, \varphi, \phi\right\}$ be the elastic displacement, electric potential and magnetic potential caused by the eigenstrains, eigenelectric and eigenmagnetic fields, $\mathbf{n}$ the unit normal of the boundary $\Gamma$ pointing from $\omega$ to $\varpi$ (Fig. 1). The continuity conditions for the displacement and traction vectors across the boundary are
$u_{i}^{\text {out }}=u_{i}^{\text {in }}+u_{i}^{*}, \quad n_{j} \sigma_{i j}^{\text {out }}=n_{j} \sigma_{i j}^{\text {in }}$.
The continuity conditions for the tangential electric and magnetic fields and normal electric displacement and magnetic induction read
$\mathbf{n} \times \mathbf{E}^{\text {out }}=n \times\left(\mathbf{E}^{\text {in }}+\mathbf{E}^{*}\right), \quad \mathbf{n} \cdot \mathbf{D}^{\text {out }}=\mathbf{n} \cdot \mathbf{D}^{\text {in }} ;$
$\mathbf{n} \times \mathbf{H}^{\text {out }}=\mathbf{n} \times\left(\mathbf{H}^{\text {in }}+\mathbf{H}^{*}\right), \quad \mathbf{n} \cdot \mathbf{B}^{\text {out }}=\mathbf{n} \cdot \mathbf{B}^{\text {in }}$.
As illustrated in Fig. 1, the increasing direction of $d \boldsymbol{y}$ is to keep $\omega$ on the left-hand side as the Cartesian coordinate system is coun-ter-clockwise orientated. This implies
$n_{1} d s=d x_{2}, \quad n_{2} d s=-d x_{1}$,
where $d s$ is an infinitesimal arc length element at the boundary point ( $x_{1}, x_{2}$ ). Substituting Eqs. (12), (30), and $E_{i}=-\varphi_{, i}, H_{i}=-\phi, i$ into Eqs. (27) $)_{2}$, (28), and (29) gives
$\left.\begin{array}{l}d\left(\psi_{I}^{\text {out }}-\psi_{I}^{\text {in }}\right) / d s=0 \text { with } I=1,2,3,4,5, \\ d\left(\phi^{\text {out }}-\phi^{\text {in }}-\phi^{*}\right) / d s=d\left(\varphi^{\text {out }}-\varphi^{\text {in }}-\varphi^{*}\right) / d s=0 .\end{array}\right\}$
Making use of the continuity conditions of the relevant qualities, we must have
$\psi_{I}^{\text {out }}=\psi_{I}^{\text {in }}$ with $I=1,2,3,4,5 ;$
$\phi^{\text {out }}=\phi^{\text {in }}+\phi^{*}, \varphi^{\text {out }}=\varphi^{\text {in }}+\varphi^{*}$.
Combining Eqs. (32) and (27) provides the equivalent continuity conditions of the extended displacement and stress function vectors across the interface:
$\mathbf{u}_{\text {out }}(y)=\mathbf{u}_{\text {in }}(y)+\mathbf{u}^{*}(y), \quad \psi_{\text {out }}(y)=\psi_{\text {in }}(y)$,
where $y=x_{1}+i x_{2} \in \boldsymbol{\Gamma}$.
Making use of the general solution equation (9), the continuity condition equation (33) can be expressed by

$$
\left.\begin{array}{l}
\mathbf{A} \mathbf{f}_{\text {out }}(y)+\overline{\mathbf{A}} \overline{\mathbf{f}_{\text {out }}(y)}=\mathbf{A} \mathbf{f}_{\text {in }}(y)+\overline{\mathbf{A}} \overline{\mathbf{f}_{\text {in }}(y)}+\mathbf{u}^{*}  \tag{34}\\
\mathbf{B} f_{\text {out }}(y)+\overline{\mathbf{B}} \overline{\mathbf{f}_{\text {out }}(y)}=\mathbf{B} \mathbf{B}_{\text {in }}(y)+\overline{\mathbf{B}} \overline{\mathbf{f}_{\text {in }}(y)},
\end{array}\right\}
$$

where $y \in \Gamma$ and the overbar denotes the complex conjugate. Multiplying the two vector relations of Eq. (34) by $\mathbf{B}^{T}$ and $\mathbf{A}^{T}$, respectively, and adding the results, we obtain
$\mathbf{f}_{\text {out }}(y)=\mathbf{f}_{\text {in }}(y)+\mathbf{B}^{T} \mathbf{u}^{*}(y), \quad y \in \Gamma$,
where use is made of the extended Stroh orthogonality relation (e.g., Chung and Ting 1996)

$$
\left[\begin{array}{cc}
\mathbf{B}^{T} & \mathbf{A}^{T}  \tag{36}\\
\overline{\mathbf{B}}^{T} & \overline{\mathbf{A}}^{T}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{A} & \overline{\mathbf{A}} \\
\mathbf{B} & \overline{\mathbf{B}}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{A} & \overline{\mathbf{A}} \\
\mathbf{B} & \overline{\mathbf{B}}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{B}^{T} & \mathbf{A}^{T} \\
\overline{\mathbf{B}}^{T} & \overline{\mathbf{A}}^{T}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{1}
\end{array}\right],
$$

with 1 being the $5 \times 5$ identity matrix.
Since $f_{I}\left(z_{I}\right)(I=1,2,3,4,5)$ are five functions which are sectionally analytic with respect to $z_{I}$ in the entire complex plane except for the boundary $\Gamma$, it is helpful to write $\mathbf{B}^{T} \mathbf{u}^{*}(y)$ as functions of $y_{i}$. Using
$x_{1}=\frac{p_{I} \bar{y}_{I}-\bar{p}_{I} y_{I}}{p_{I}-\bar{p}_{I}}, \quad x_{2}=\frac{y_{I}-\bar{y}_{I}}{p_{I}-\bar{p}_{I}}$
on the interface, we have
$\mathbf{B}^{T} \mathbf{u}^{*}(y)=\mathbf{B}^{T} \tilde{\boldsymbol{\varepsilon}}_{p}^{*} x_{p}=-\left(\begin{array}{l}c_{1} y_{1}+d_{1} \bar{y}_{1} \\ c_{2} y_{2}+d_{2} \bar{y}_{2} \\ c_{3} y_{3}+d_{3} \bar{y}_{3} \\ c_{4} y_{4}+d_{4} \bar{y}_{4} \\ c_{5} y_{5}+d_{5} \bar{y}_{5}\end{array}\right)$,
which gives us a decoupled form of the condition as in Eq. (35) across the interface:
$f_{I}^{\text {out }}\left(y_{I}\right)=f_{I}^{\text {in }}\left(y_{I}\right)+\left(c_{I} y_{I}+d_{I} \bar{y}_{I}\right), \quad I=1,2,3,4,5$.
The detailed expressions of $c_{I}$ and $d_{I}$ are given in Eq. (123) below.

We now recall the following Lemma (Henrici, 1986; Ablowitz and Fokas, 2003): Let $\Gamma$ be a simple, closed, regular, positively oriented curve enclosing the origin, and let $b(t), t \in \Gamma$ be a Hölder continuous function (namely for $t, \tau \in \Gamma$, we have $|b(t)-b(t)| \leqslant$ $\left.C|t-\tau|^{\alpha}, C>0, \alpha \in(0,1)\right)$ on $\Gamma$, the degenerated Privalov (or Rie-mann-Hilbert) problem $f^{\text {out }}(t)=f^{\text {in }}(t)+b(t)$ has the general solution (Muskhelishvili, 1963)
$f(z)=\frac{1}{2 \pi i} \oint_{\Gamma} \frac{b(t)}{t-z} d t$.
The jumping relation (39) over the boundary
$\Gamma_{I}=\left\{y_{I}=x_{1}+p_{I} x_{2} \mid y=x_{1}+i x_{2} \in \Gamma\right\}, \quad I=1,2,3,4,5$,
directly yields

$$
\begin{align*}
f_{I}\left(z_{I}\right) & =\frac{1}{2 \pi i} \oint_{\Gamma_{I}} \frac{c_{I} y_{I}+d_{I} \bar{y}_{I}}{y_{I}-z_{I}} d y_{I} \\
& =c_{I} z_{I} \chi^{\omega}+\frac{d_{I}}{2 \pi i} \oint_{\Gamma_{I}} \frac{\bar{y}_{I}}{y_{I}-z_{I}} d y_{I}, \quad I=1,2,3,4,5 \tag{42}
\end{align*}
$$

where $\chi^{\omega}$ is the characteristic function of $\omega$ that equals to 1 or 0 according to whether $z$ is inside or outside $\omega$.

The extended strain and stress components are then given by
$\mathbf{u}_{, 1}=2 \operatorname{Re}\left[\mathbf{A f}^{\prime}\right], \quad \mathbf{u}_{, 2}=2 \operatorname{Re}\left[\mathbf{A}\left\langle p_{*}\right\rangle \mathbf{f}^{\prime}\right]$,
$\boldsymbol{\sigma}_{2}=2 \operatorname{Re}\left[\mathbf{B f}^{\prime}\right], \quad \boldsymbol{\sigma}_{1}=-2 \operatorname{Re}\left[\mathbf{B}\left\langle p_{*}\right\rangle \mathbf{f}^{\prime}\right]$,
where for $I=1,2,3,4,5$,
$f_{I}^{\prime}\left(z_{I}\right)=c_{I} \chi^{\omega}+\frac{d_{I}}{2 \pi i} \oint_{\Gamma_{I}} \frac{\bar{y}_{I}}{\left(y_{I}-z_{I}\right)^{2}} d y_{I}=c_{I} \chi^{\omega}+d_{I} g\left(p_{I} ; z_{I}\right)$
with
$g\left(p_{I} ; z_{I}\right)=\frac{1}{2 \pi i} \oint_{\Gamma_{I}} \frac{d \bar{y}_{I}}{y_{I}-z_{I}}$.
Sometimes, it is convenient to write $g\left(p_{I} ; z_{I}\right)$ as $g(p ; z, \bar{z})$ which takes the form
$g(p ; z, \bar{z})=\frac{1}{2 \pi i} \oint_{\Gamma} \frac{(1+i \bar{p}) d \bar{y}+(1-i \bar{p}) d y}{(1-i p)(y-z)+(1+i p)(\bar{y}-\bar{z})}$.

### 3.2. General expressions of Eshelby tensors in the multiferroic inclusion problem

In this subsection, summation convention for repeated indices does not apply. From Eqs. (38) and (45), we have
$f_{I}^{\prime}\left(z_{I}\right)=-\sum_{K, p} B_{K I} \tilde{\varepsilon}_{K p}^{*} F_{I p}\left(z_{I}\right)$
in which
$F_{I 1}\left(z_{I}\right)=\frac{p_{I} g\left(p_{I} ; z_{I}\right)-\bar{p}_{I} \chi^{\omega}}{p_{I}-\bar{p}_{I}}, \quad F_{I 2}\left(z_{I}\right)=\frac{\chi^{\omega}-g\left(p_{I} ; z_{I}\right)}{p_{I}-\bar{p}_{I}}$.
Substituting Eqs. (48) and (26) into Eq. (21) results in

$$
\begin{align*}
\varepsilon_{l j}= & -\sum_{L, M, N, K, p} \operatorname{Re}\left\{\left(A_{I M} B_{L M} K_{L j}\right.\right. \\
& \left.\left.+K_{L I} B_{L M} A_{j M}\right) B_{2 M}^{-1} B_{K M} L_{K N} F_{M p}\left(z_{M}\right)\right\} \varepsilon_{N p}^{*} \tag{50}
\end{align*}
$$

from which we deduce the Eshelby tensor $\boldsymbol{\Sigma}^{\omega}$, defined by $\varepsilon_{l j}=\sum_{l j N p}^{\omega} \varepsilon_{N p}^{*}$, as
$\Sigma_{l j N p}^{\omega}=-\operatorname{Re}\left\{\sum_{L, M, K}\left(A_{I M} B_{L M} K_{L j}+K_{L I} B_{L M} A_{j M}\right) B_{2 M}^{-1} B_{K M} L_{K N} F_{M p}\left(z_{M}\right)\right\}$.

Similarly, substituting Eqs. (48) and (26) into Eq. (18) gives
$\sigma_{l j}=-2 \sum_{M, N, K, p} \operatorname{Re}\left\{B_{I M} B_{2 M}^{-1} B_{K M} L_{K N} F_{M p}\left(z_{M}\right) B_{j M}\right\} \varepsilon_{N p}^{*}$
which delivers the eigenstiffness tensor $\boldsymbol{\Omega}^{\omega}$, defined by $\sigma_{l j}=\Omega_{l j N p}^{\omega} \varepsilon_{N p}^{*}$, as
$\Omega_{I j N p}^{\omega}=-2 \operatorname{Re}\left\{\sum_{M, K} B_{I M} B_{2 M}^{-1} B_{K M} L_{K N} F_{M p}\left(z_{M}\right) B_{j M}\right\}$.
Recall that the eigenstiffness tensor rather than the Eshelby tensor is directly involved in various micromechanics schemes for composites of inclusion-matrix types (e.g., Zheng and Du, 2001;

Zheng et al., 2006). Here, we can see that the expression for the eigenstiffness tensor $\boldsymbol{\Omega}^{\omega}$ is simpler than that for the Eshelby tensor $\boldsymbol{\Sigma}^{\omega}$.

## 4. An inclusion in an anisotropic multiferroic bimaterial plane

### 4.1. General integral expressions of eigenfunctions

The Eshelby's problem for an inclusion in a bimaterial full plane is of practical importance. For example, for buried strain semiconductor devices, the top barrier layer may be much thinner than the underlying substrate. Hence, the buried device can be modelled, more realistically, by an inclusion in a half-plane, rather than in an entire plane. In general, due to the presence of a free surface, analysis of the Eshelby's problem in a half-plane is more complicated.

We consider the more general case where two anisotropic multiferroic media $M_{-}, M_{+}$occupying the lower half-plane $S_{-}\left(x_{2}<0\right)$ and upper half-plane $S_{+}\left(x_{2}>0\right)$, respectively. Furthermore, the lower half-plane $S_{-}$contains an internal subdomain $\omega$ which undergoes the uniform eigenstrains, eigenelectric and eigenmagnetic fields. We let $\omega$ and $\varpi$ denote the subdomain and the remainder of $\omega$ in the lower half-plane, respectively, and $\Gamma$ the interface separating $\omega$ and $\varpi$ (see Fig. 1). Hereinafter, we indicate the quantities in $S_{-}$and $S_{+}$with the superscripts (or subscripts) '-' and '+', respectively. In virtue of the superposition model proposed by Zou et al. (in press), the solution of the Eshelby's inclusion problem in the bimaterial plane can be expressed as
$\mathbf{f}(z)=\mathbf{f}^{\infty}(z)+\mathbf{f}^{b}(z)$,
where $\mathbf{f}^{\infty}(z)$ is the corresponding Eshelby solution of inclusion in the full plane, as given in Section 3, and the complementary term $\mathbf{f}^{b}(z)$, which is piecewise analytic in the lower half-plane and the upper half-plane, will be solved below.

Across the interface $x_{2}=0$ the continuities of the extended traction and displacement are
$\sigma_{I 2}^{+}=\sigma_{I 2}^{-}, \quad u_{I}^{+}=u_{I}^{-}, \quad I=1,2,3,4,5$,
which can be expressed by

$$
\begin{equation*}
\mathbf{u}_{+}=\mathbf{u}_{-}, \quad \psi_{+}=\psi_{-}, \quad \text { on } x_{2}=0 \tag{56}
\end{equation*}
$$

On the other hand, similar to Eq. (33), the interface conditions along the curve $\Gamma$ in the lower half-plane are
$\mathbf{u}_{\text {out }}=\mathbf{u}_{\text {in }}+\mathbf{u}^{*}, \quad \psi_{\text {out }}=\psi_{\text {in }}, \quad$ when $x_{1}+i x_{2} \in \Gamma$.
It is noted that due to the analyticity of $\mathbf{f}^{b}(z)$ :
$\mathbf{f}_{\text {out }}^{b}(y)=\mathbf{f}_{\text {in }}^{b}(y), \quad y \in \Gamma$,
the second interface condition in Eq. (57) is satisfied naturally. Substitution of Eqs. (9) and (54) into Eq. (56) yields

$$
\left.\begin{array}{l}
\mathbf{A}_{+} \mathbf{f}_{+}^{b}\left(x_{1}\right)+\overline{\mathbf{A}}_{+} \overline{\mathbf{f}_{+}^{b}\left(x_{1}\right)}=\mathbf{A}_{-} \mathbf{f}_{-}^{b}\left(x_{1}\right)+\overline{\mathbf{A}}_{-} \overline{\mathbf{f}_{-}^{b}\left(x_{1}\right)}+\mathbf{F}_{A}^{\infty}\left(x_{1}\right), \\
\mathbf{B}_{+} \mathbf{f}_{+}^{b}\left(x_{1}\right)+\overline{\mathbf{B}}_{+} \overline{\mathbf{f}_{+}^{b}\left(x_{1}\right)}=\mathbf{B}_{-} \mathbf{f}_{-}^{b}\left(x_{1}\right)+\overline{\mathbf{B}}_{-} \overline{\mathbf{f}_{-}^{b}\left(x_{1}\right)}+\mathbf{F}_{B}^{\infty}\left(x_{1}\right) ; \tag{59}
\end{array}\right\}
$$

where the two real function vectors are defined by
$\left.\begin{array}{l}\mathbf{F}_{A}^{\infty}\left(x_{1}\right)=\left(\mathbf{A}_{-}-\mathbf{A}_{+}\right) \mathbf{f}^{\infty}\left(x_{1}\right)+\left(\overline{\mathbf{A}}_{-}-\overline{\mathbf{A}}_{+}\right) \overline{\mathbf{f}^{\infty}\left(x_{1}\right)}, \\ \mathbf{F}_{B}^{\infty}\left(x_{1}\right)=\left(\mathbf{B}_{-}-\mathbf{B}_{+}\right) \mathbf{f}^{\infty}\left(x_{1}\right)+\left(\overline{\mathbf{B}}_{-}-\overline{\mathbf{B}}_{+}\right) \overline{\mathbf{f}^{\infty}\left(x_{1}\right)} .\end{array}\right\}$
Utilizing the orthogonality relations

$$
\left[\begin{array}{cc}
\mathbf{B}_{+}^{T} & \mathbf{A}_{+}^{T}  \tag{61}\\
\overline{\mathbf{B}}_{+}^{T} & \overline{\mathbf{A}}_{+}^{T}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{A}_{+} & \overline{\mathbf{A}}_{+} \\
\mathbf{B}_{+} & \overline{\mathbf{B}}_{+}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{B}_{-}^{T} & \mathbf{A}_{-}^{T} \\
\overline{\mathbf{B}}_{-}^{T} & \overline{\mathbf{A}}_{-}^{T}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{A}_{-} & \overline{\mathbf{A}}_{-} \\
\mathbf{B}_{-} & \overline{\mathbf{B}}_{-}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{1}
\end{array}\right],
$$

and after some algebraic calculations, we have
$\mathbf{C}_{\mp} \mathbf{f}_{+}^{b}\left(x_{1}\right)+\mathbf{D}_{\mp} \overline{\mathbf{f}_{+}^{b}\left(x_{1}\right)}=\mathbf{f}_{-}^{b}\left(x_{1}\right)+\left(\mathbf{1}-\mathbf{C}_{\mp}\right) f^{\infty}\left(x_{1}\right)-\mathbf{D}_{\mp} \overline{\mathbf{f}^{\infty}\left(x_{1}\right)}$,
$\mathbf{f}_{+}^{b}\left(x_{1}\right)=\mathbf{C}_{ \pm} \mathbf{f}_{-}^{b}\left(x_{1}\right)+\mathbf{D}_{ \pm} \overline{\mathbf{f}_{-}^{b}\left(x_{1}\right)}+\left(\mathbf{C}_{ \pm}-1\right) \mathbf{f}^{\infty}\left(x_{1}\right)+\mathbf{D}_{ \pm} \overline{\mathbf{f}^{\infty}\left(x_{1}\right)}$,
where uses are made of
$\mathbf{C}_{ \pm / \mp}=\mathbf{B}_{ \pm}^{T} \mathbf{A}_{\mp}+\mathbf{A}_{ \pm}^{T} \mathbf{B}_{\mp}, \quad \mathbf{D}_{ \pm / \mp}=\mathbf{B}_{ \pm}^{T} \overline{\mathbf{A}}_{\mp}+\mathbf{A}_{ \pm}^{T} \overline{\mathbf{B}}_{\mp}$.
From the Cauchy formulae and the properties $\mathbf{f}_{-}^{b}(\infty)=$ $\mathbf{f}_{+}^{b}(\infty)=0$, we can derive that
$\left.\begin{array}{l}\mathbf{f}_{+}^{b}(z)=\frac{C_{-}^{-1}-1}{2 \pi i} \int_{-\infty}^{\infty} \frac{f^{\infty}(t)}{t-z} d t-\frac{C_{+}^{-1} D_{\mp}}{2 \pi i} \int_{-\infty}^{\infty} \frac{\overline{f^{\infty}(t)}}{t-z} d t, \quad z \in S_{+}, \\ \mathbf{f}_{-}^{b}(z)=\frac{1-c_{+}^{-1}}{2 \pi i} \int_{-\infty}^{\infty} \frac{f^{\infty}(t)}{t-z} d t+\frac{c_{+}^{-1} D_{ \pm}}{2 \pi i} \int_{-\infty}^{\infty} \frac{\overline{f \infty}(t)}{t-z} d t, z \in S_{-} .\end{array}\right\}$
Substituting the basic term on the real axis
$f_{I}^{\infty}(t)=\frac{d_{I}}{2 \pi i} \oint_{\Gamma_{I}} \frac{\bar{y}_{I}}{y_{I}-t} d y_{I}$
into Eq. (65), and using the integrals

$$
\begin{align*}
& \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{f_{I}^{\infty}(t)}{t-z} d t= \begin{cases}0, & z \in S_{-}, \\
\frac{d_{I}}{2 \pi i} \oint_{\Gamma_{I}} \frac{\bar{y}_{I}}{y_{I}-z} d y_{I}, & z \in S_{+}\end{cases}  \tag{66}\\
& \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\overline{f_{I}^{\infty}(t)}}{t-z} d t=\left\{\begin{array}{cc}
\frac{\bar{d}_{I}}{2 \pi i} \oint_{\Gamma_{I}} \frac{y_{I}}{y_{l}-z} d \bar{y}_{I}, & z \in S_{-}, \\
0, & z \in S_{+},
\end{array}\right. \tag{67}
\end{align*}
$$

we finally obtain
$f_{I}^{b+}\left(z_{I}\right)=\sum_{J}\left[\left(C_{\mp}^{-1}\right)_{I J}-\delta_{I J}\right] \frac{d_{J}}{2 \pi i} \oint_{\Gamma_{J}} \frac{\bar{y}_{I}}{y_{J}-z_{I}} d y_{J}, \quad z_{I} \in S_{+}$,
$f_{I}^{b-}\left(z_{I}\right)=\sum_{K, J}\left(C_{ \pm}^{-1}\right)_{I K}\left(D_{ \pm}\right)_{K J} \frac{d_{J}}{2 \pi i} \oint_{\Gamma_{J}} \frac{y_{J}}{\bar{y}_{J}-z_{I}} d \bar{y}_{J}, \quad z_{I} \in S_{-}$.
After careful observation, we find that the above formulae, as well as Eq. (42), are all associated with a kind of simple integral (Eq. (71)), and that the derivatives of the basic and complementary functions can further be expressed by this simple integral as
$f_{I}^{\prime}\left(z_{I}\right)= \begin{cases}\sum_{J}\left(C_{\mp}^{-1}\right)_{I J} d_{J} g\left(p_{J} ; z_{I}\right), & z_{I} \in S_{+}, \\ c_{I} \chi^{\omega}+d_{I} g\left(p_{I} ; z_{I}\right)+\sum_{K_{J} J}\left(C_{ \pm}^{-1}\right)_{I K}\left(D_{ \pm}\right)_{K J} \bar{d}_{J} g\left(\bar{p}_{J} ; z_{I}\right), & z_{I} \in S_{-},\end{cases}$
for $I=1,2,3,4,5$, where the essential eigenfunction $g\left(p_{J} ; z_{I}\right)$ is defined by
$g\left(p_{J} ; z_{I}\right)=\frac{1}{2 \pi i} \oint_{\Gamma_{J}} \frac{d \bar{y}_{J}}{y_{J}-z_{I}}$
with $y_{J}=x_{1}+p_{J} x_{2}$. It should be noticed that $z_{I}=x_{1}+p_{I}^{+} x_{2}$ if $x_{2}>0$ and $z_{I}=x_{1}+p_{I}^{-} x_{2}$ if $x_{2}<0$. Sometimes, we write $g\left(p_{J} ; z_{I}\right)$ as $g\left(p_{J} ; p_{I}, z, \bar{z}\right)$, taking the form

$$
\begin{align*}
g\left(p_{J} ; p_{I}, z, \bar{z}\right)= & \frac{1}{2 \pi i} \oint_{\Gamma} \\
& \times \frac{\left(1+i \bar{p}_{J}\right) d \bar{y}+\left(1-i \bar{p}_{J}\right) d y}{\left(1-i p_{J}\right)\left(y-\frac{1-i p_{I}}{1-i p_{J}} z\right)+\left(1+i p_{J}\right)\left(\bar{y}-\frac{1+i p_{I}}{1+i p_{J}}\right)} . \tag{72}
\end{align*}
$$

The essential eigenfunction $g$ in the Eshelby inclusion analysis can be very conveniently employed to derive the average of the extended eigenstrain-induced field, as presented below.

### 4.2. The average of the eigenfunction $g$

Utilizing the formula (e.g., Lavrentieff and Shabat, 2002) for any function $f(x, \bar{x})$
$\frac{1}{2 i} \oint_{\Gamma} f(\tau, \bar{\tau}) d \tau=\int_{\omega} \frac{\partial f(x, \bar{x})}{\partial \bar{x}} d x$,
we can calculate the average of $g$ over the inclusion $\omega$ by (with a hat on $g$ )
$\hat{g}\left(p_{J}, p_{I}\right)=\left\langle g\left(p_{J} ; z_{I}\right)\right\rangle=\frac{i}{2 \pi\left(p_{I}-\bar{p}_{I}\right)|\omega|} \oint_{\Gamma_{J}} \oint_{\Gamma_{I}} \frac{\bar{y}_{J}-\bar{z}_{I}}{y_{J}-z_{I}} d \bar{y}_{J} d z_{I}$,
or equivalently

$$
\begin{align*}
\hat{g}\left(p_{J} ; p_{I}\right)= & \frac{1+i \bar{p}_{J}}{4 \pi\left(1+i p_{I}\right)|\omega|} \oint_{\Gamma} \oint_{\Gamma} \ln \left(1+\varepsilon_{J} \frac{\bar{y}-\frac{1+i_{I}}{1+\bar{p}_{j}} \bar{z}}{y-\frac{1-p_{I}}{1-i \bar{p}_{J}} z}\right) \\
& \times\left(d \bar{y}+\bar{\varepsilon}_{J} d y\right) d z . \tag{75}
\end{align*}
$$

where $|\omega|$ stands for the area of the inclusion and
$\varepsilon_{J}=\frac{1+i p_{J}}{1-i p_{J}}$.
With the average of $g$, the average of $f_{I}^{\prime}\left(z_{I}\right)$ can also be determined by
$\left\langle f_{I}^{\prime}\left(z_{I}\right)\right\rangle=c_{I}+d_{I} \hat{g}\left(p_{I} ; p_{I}\right)+\sum_{K . J}\left(C_{ \pm}^{-1}\right)_{I K}\left(D_{ \pm}\right)_{K I} \bar{d}_{J} \hat{g}\left(\bar{p}_{I} ; p_{I}\right)$.
when the upper half-plane is empty, our bimaterial-plane solution is reduced to the half-plane solution with extended traction free on the surface of the half plane. The related expressions for this special case are given in Appendix A.

## 5. Analytical solutions for inclusions of various shapes

With the results derived in Section 4, we are now in the position to present the explicit analytical solutions for inclusions made of various geometric shapes. These include the ones described by the Laurent polynomial, the polygonal shapes, and the ones described by the Jordan curve. The details are presented below.

### 5.1. Inclusions described by the Laurent polynomial

By the Riemann mapping theorem (Henrici, 1986), the shape of any given inclusion $\omega$ can be approached by the Laurent polynomial of $w$ :
$y(w)=y_{0}+R\left(w+\sum_{k=1}^{N} b_{k} w^{-k}\right), \quad|w|=1$,
where $R>0$ and $y_{0}$ is a unique inner point of the domain $\omega$ bounded by $\Gamma$. Typically, the parameters $R$ and $y_{0}$ characterize the size and center of $\omega$. Some useful information on the shape expression (78) can be found in Zou et al. (2010).

Considering a circular inclusion described by $y(w)=y_{0}+R w$, $|w|=1$, then
$y_{J}=\frac{1-i p_{J}}{2} y+\frac{1+i p_{J}}{2} \bar{y}=\frac{1-i p_{J}}{2}\left[y_{0}+\varepsilon_{J} \bar{y}_{0}+R\left(w+\varepsilon_{J} w^{-1}\right)\right]$
defines a deformed elliptical inclusion, since $\left|\varepsilon_{j}\right|<1$ always holds, with its center at $y_{0}+\varepsilon_{1} \bar{y}_{0}$ and with the same size parameter $R$. The essential eigenfunction (71) can be written as
$g\left(p_{J} ; z_{I}\right)=\frac{1+i \bar{p}_{J}}{2 \pi i\left(1-i p_{J}\right)} \oint_{\Gamma_{J}} \frac{d \bar{y}^{\prime}}{y^{\prime}-z^{\prime}}, y^{\prime}=y_{0}^{\prime}+R\left(w+\varepsilon_{J} w^{-1}\right) \in \Gamma_{J}$,
with $z^{\prime}=\frac{z_{1}}{1-i p_{j}}$. If the point $z^{\prime}$ belongs to the domain surrounded by $\Gamma_{\mathrm{J}}$, then we have the following expansion
$\frac{1}{y^{\prime}-z^{\prime}}=R^{-1} w^{-1} \sum_{k}(-1)^{k}\left[R^{-1}\left(y_{0}^{\prime}-z^{\prime}\right) w^{-1}+\varepsilon_{j} w^{-2}\right]^{k}$.

Using the residual theorem and substituting Eq. (81) into Eq. (80) yields
$g\left(p_{J} ; z_{I}\right)=\frac{1+i \bar{p}_{J}}{2 \pi i\left(1-i p_{J}\right)} \oint_{c} \frac{\bar{\varepsilon}_{J}-w^{-2}}{y^{\prime}-z^{\prime}} d w=\frac{1+i \bar{p}_{J}}{1-i p_{J}} \bar{\varepsilon}_{J}=\frac{1-i \bar{p}_{J}}{1-i p_{J}}$.
However, the inequality of $p_{I}$ and $p_{J}$ largely reduces the solvability of (71) since the mismatch between the deformed domains from $p_{I}$ and $p_{J}$ makes the expansion like Eq. (81) generally impossible. Even though the solution such as the above can be obtained, the interior point $z^{\prime}$ of the deformed domain is still indecisive.

### 5.2. Polygonal inclusions

An elegant analysis was carried by Rodin (1996) for the polygonal inclusion problem in an elastic isotropic plane. Here we present the solution for the corresponding anisotropic and multiferroic plane. In this subsection, $p, y, s, \ldots$ can be freely replaced by $p_{J}, y_{J}, s_{j}, \ldots$ in a group. Let $\omega$ be an arbitrary polygonal inclusion with its boundary consisting of $N$ rectilinear sides $\partial \omega_{k}$ with $k=1,2, \ldots, N$. As illustrated in Fig. 2, denoting by $y_{k}$ and $y_{k+1}$ the two end points of the $k$ th side $\partial \omega_{k}$, we can parameterize all points of this side in the following form
$y=y_{k}+\left(y_{k+1}-y_{k}\right) t, \quad(0 \leqslant t \leqslant 1)$.
Then, it follows
$\int_{\partial \omega_{k}} \frac{d \bar{y}}{y-z_{I}}=\int_{0}^{1} \frac{\bar{s}_{k} d t}{w_{k}+s_{k} t}=\frac{\bar{s}_{k}}{s_{k}} \ln \frac{w_{k+1}}{w_{k}}$,
where $\quad w_{k}=y_{k}-z_{l}, s_{k}=y_{k+1}-y_{k}=w_{k+1}-w_{k}$ and $\ln z=\ln |z|+$ $\arg (z)$ with $-\pi<\arg (z)<\pi$ (Zill and Shanahan, 2003). We sum the integrals of all sides to obtain the explicit expression of the essential eigenfunction (71) as follows:
$g\left(p ; z_{I}\right)=\frac{1}{2 \pi i} \sum_{k=1}^{N} \frac{\bar{s}_{k}}{s_{k}} \ln \frac{w_{k+1}}{w_{k}}$.
We point out that care must be taken in using these solutions in which the logarithmic terms cannot be in general combined freely. To be able to operate the terms freely, the arguments $\theta_{k}$ of $w_{k}$ need to be prescribed as follows. Referring to Fig. 2, we first assign the range of $\theta_{1}$ to be $(-\pi, \pi]$ ]. If the direction of $w_{2}$ is counter-clockwise/clockwise rotated from the direction of $w_{1}$ through an angle less than $\pi$, then we assign $\theta_{2}$ to be larger/smaller than $\theta_{1}$. Analogously, we assign $\theta_{k+1}$ to be larger/smaller than $\theta_{k}$ when the coun-ter-clockwise/clockwise rotation from the direction of $w_{k}$ to that of $w_{k+1}$ is an angle less than $\pi$. For a simply connected polygonal inclusion with $N$ sides, the complex point $w_{N+1}$ can be superposed with $w_{1}$ but should possess an argument $2 \pi+\arg \left(w_{1}\right)$ if $z$ is an interior point. The ranges of $\varphi_{k}$ are defined in the same way, which will be crucial in calculating the average Eshelby tensor. By virtue of these prescriptions and the foregoing discussion, the general solutions can be written as
$g\left(p ; z_{I}\right)=\frac{1}{2 \pi i} \sum_{k=1}^{N} e^{-2 i \phi_{k}}\left[\ln \frac{R_{k+1}}{R_{k}}+i\left(\theta_{k+1}-\theta_{k}\right)\right]$,
where $R_{k}, L_{k}$ and $\theta_{k}, \varphi_{k}$ are the norms and arguments of $w_{k}$ and $s_{k}$ specified through
$w_{k}=R_{k} e^{i \theta_{k}}, \quad s_{k}=L_{k} e^{i \phi_{k}}$.
Furthermore, we parameterize point $z_{l}$ from the $j$ th side and point $y_{J}$ from the $k$ th side by
$z_{I}=y_{j}^{I}+s_{j}^{I} \tau, \quad y_{J}=y_{k}^{J}+s_{k}^{J} t$ with $\tau, t \in[0,1]$,
so that


Fig. 2. Prescriptions of the arguments for $s_{k}$ (i.e., the field point $\boldsymbol{x}$ is inside, left) and $z_{k}$ (i.e., the field point $\boldsymbol{x}$ is outside, right) of a polygonal inclusion.
$w^{I J}=y_{J}-z_{I}=s_{j, k}^{I J}+s_{k}^{J} t-s_{j}^{I} \tau$.
where, use is made of the notation
$s_{j, k}^{I J}=y_{k}^{J}-y_{j}^{I}, \quad s_{j}^{I}=s_{j, j+1}^{I I}$.
Then, introducing notations
$\Theta_{j k}^{ \pm}=\frac{e^{-i 2 \phi_{k}^{\prime}} \pm e^{-i 2 \phi_{j}^{l}}}{2}$,
starting from Eq. (74), and after some calculations, we can obtain the following simple formula for the average of $g$ :
$\hat{g}\left(p_{J} ; p_{I}\right)=\frac{i}{2 \pi\left(p_{I}-\bar{p}_{I}\right)|\omega|} \sum_{j, k} s_{j}^{I} \bar{s}_{k}\left[\Theta_{j k}^{+}+C_{j k}^{I J}\right]$,
where the expression of $C_{j k}$ is symmetric in $j$ and $k$, and takes the following form:
$C_{j k}^{I J}=\Theta_{j k}^{-}\left(\frac{s_{k}^{J}}{s_{j}^{I}} \ln \frac{s_{j, k+1}^{I J}}{s_{k}^{J}}+\frac{s_{j}^{I}}{s_{k}^{J}} \ln \frac{s_{j}^{I}}{s_{j, k+1}^{I J}}\right)$
if $k=j+1$, and

$$
\begin{align*}
C_{j k}^{I J}= & \frac{\bar{s}_{j, k}^{I J}-\Theta_{j k}^{+} s_{j, k}^{I J}}{s_{j}^{I} s_{k}^{J}} s_{j, k}^{I J} \ln \frac{s_{j, k+1}^{I J} s_{j+1, k}^{I J}}{s_{j, k}^{I J} s_{j+1, k+1}^{I J}}+\frac{\bar{s}_{j, k+1}^{I J}-\Theta_{j k}^{+} s_{j, k+1}^{I J}}{s_{j}^{I}} \ln \frac{s_{j, k+1}^{I J}}{s_{j+1, k+1}^{I I}} \\
& +\frac{\bar{s}_{j+1, k}^{I}-\Theta_{j k}^{+} s_{j+1, k}^{I J}}{s_{k}^{J}} \ln \frac{s_{j+1, k+1}^{J}}{s_{j+1, k}^{I J}} \\
& +\Theta_{j k}^{-} s_{j, k}^{I J}\left(\frac{1}{s_{j}^{I}} \ln \frac{s_{j, k+1}^{I J}}{s_{j+1, k+1}^{I I}}+\frac{1}{s_{k}^{J}} \ln \frac{s_{j+1, k}^{I J}}{s_{j+1, k+1}^{I J}}\right) \tag{94}
\end{align*}
$$

if $k>j+1$. Remark that, in the foregoing formulae, when $j$ (or $k$ ) is equal to $N$, we have to set $j+1=1$ (or $k+1=1$ ).

### 5.3. Inclusions bounded by the Jordan curve

We let $\Gamma$ be a simple closed curve, called a Jordan curve, composed of straight line segments and circular arcs which are one-by-one smoothly connected, say, ( $\overline{y_{1} y_{2}}, \widehat{y_{2} y_{3}}, \ldots, \overline{y_{2 M-1} y_{2 M}}, \widehat{y_{2 M} y_{1}}$ ), where $y_{1}, y_{2}, \ldots, y_{2 M}$ are $N(=2 M)$ end points. The phase angles of the straight line segments as prescribed in Section 5.2 are $\varphi_{k}$ which satisfy
$y_{2 k}-y_{2 k-1}=L_{k} e^{i \phi_{k}}, \quad k=1, \ldots, M ;$
The centers of circular arcs are
$c_{k}=y_{2 k}+r_{k} e^{i\left(\phi_{k}+\pi / 2\right)}=y_{2 k}-e^{i \phi_{k}} \frac{y_{2 k+1}-y_{2 k}}{e^{i \phi_{k+1}}-e^{i \phi_{k}}}$,
where $r_{k}=i \frac{y_{2 k+1}-y_{2 k}}{e^{i \phi_{k+1}}-e^{i \phi_{k}}}$ are the signed arc radii, namely

$$
\left.\begin{array}{ll}
r_{k}>0, & \text { if } \phi_{k+1}>\phi_{k} \\
r_{k}<0, & \text { if } \phi_{k+1}<\phi_{k} \tag{97}
\end{array}\right\}
$$

A Jordan curve with $2 M$ segments can be constructed by smoothing an $M$-sided polygon. We suppose that the vertices of the polygon are
$V_{k}=y_{2 k}+t_{k} e^{i \phi_{k}}=y_{2 k}+e^{i \phi_{k}} \frac{y_{2 k+1}-y_{2 k}}{e^{i \phi_{k+1}}+e^{i \phi_{k}}}$,
where $t_{k}(>0)$ are distances between $V_{k+1}$ and $y_{2 k}$ or $y_{2 k+1}$. We further assume that the vertices $V_{k}$ and the arc radii $r_{k}$ are given, then $y_{2 k}$, $y_{2 k+1}$, and $t_{k}$ can be calculated from
$\left.\begin{array}{l}y_{2 k}=V_{k+1}-r_{k} e^{i \phi_{k}} \tan \frac{\phi_{k+1}-\phi_{k}}{2}, \\ y_{2 k+1}=V_{k+1}+r_{k} e^{i \phi_{k+1}} \tan \frac{\phi_{k+1}-\phi_{k}}{2},\end{array}\right\}$
$t_{k}=\frac{y_{2 k+1}-y_{2 k}}{e^{i \phi_{k+1}}+e^{i \phi_{k}}}=r_{k} \tan \left(\phi_{k+1}-\phi_{k}\right)$.
It is natural that the inequality
$t_{k}+t_{k-1} \leqslant\left|V_{k+1}-V_{k}\right|$
is required for all $k$. When the radii of arcs are taken to be constant, say $\left|r_{k}\right|=r$, the above relation yields
$r \leqslant \min \left\{\frac{\left|V_{k+1}-V_{k}\right|}{\tan \left|\frac{\phi_{k+1}-\phi_{k}}{2}\right|+\tan \left|\frac{\phi_{k}-\phi_{k-1}}{2}\right|}, k=1,2, \cdots, M\right\}$.
Assume that $e^{k}$ is an elliptical domain deformed from a circular domain given by the arc $\partial \omega_{2 k}$ and parameter $p$, a distorted point $z$ $\left(z_{I}\right) \in e^{k}$ inside the arc means that the inequality
$\operatorname{sign}\left(r_{k}\right) \arg \frac{w_{2 k+1}}{w_{2 k}}<0$ and $z_{I} \in e^{k}$
must be satisfied, where $w_{i}=y_{i}-z$.
It is convenient to use the arc length coordinate to label a point on $\Gamma$. Letting the arc length coordinate at end $y_{j}$ be $l_{j}$, the arc length coordinate $l$ of point $y$ between two ends $y_{j}, y_{j+1}$ is calculated by
$l= \begin{cases}l_{j}+r_{k}\left[\arg \left(y-c_{k}\right)-\phi_{k}+\pi / 2\right], & \text { if } j=2 k, \\ l_{j}+\left|y-y_{j}\right|, & \text { if } j=2 k-1 .\end{cases}$
Inversely, a point with arc length coordinate $l$ has the following Cartesian coordinate:
$y= \begin{cases}c_{k}+r_{k} e^{i\left(\phi_{k}-\frac{\pi}{2}+\frac{l-l_{j}}{r}\right)}=c_{k}+r \zeta, & \text { if } j=2 k, \\ y_{j}+\frac{s_{k}}{\left|s_{k}\right|}\left(l-l_{j}\right), & \text { if } j=2 k-1 .\end{cases}$
After the foregoing preparation, we can now calculate the essential eigenfunction (71) of arc $\partial \omega_{2 k}$ through

$$
\begin{align*}
\int_{\partial \omega_{2 k}} \frac{d \bar{y}_{*}}{w_{*}} & =\int_{\partial \omega_{2 k}} \frac{\frac{1+i \bar{p}}{2} d \bar{y}+\frac{1-i \bar{p}}{2} d y}{2}\left(y-z^{a}\right)+\frac{1+i p}{2}\left(\bar{y}-\bar{z}^{b}\right) \\
& =\frac{1-i \bar{p}}{1-i p}\left[\ln \frac{w_{2 k+1}}{w_{2 k}}+2 \pi i \chi_{k}^{1} \operatorname{sign}\left(r_{k}\right)\right]-2 i \frac{p-\bar{p}}{(1-i p)^{2}} H \tag{105}
\end{align*}
$$

where
$H=\int_{\zeta_{k}}^{\zeta_{k+1}} \frac{d \zeta^{-1}}{\varepsilon \zeta^{-1}+\lambda+\zeta}$
with $\zeta_{k}=e^{i\left(\phi_{k}-\pi / 2\right)}, \zeta_{k+1}=e^{i\left(\phi_{k+1}-\pi / 2\right)}, z^{a}=\frac{1-i p_{I}}{1-i p_{J}} z, z^{b}=\frac{1-i \bar{p}_{I}}{1-i \bar{p}_{J}} z$, and
$\lambda=\frac{c_{k}-z^{a}}{r_{k}}+\varepsilon \frac{\bar{c}_{k}-\bar{z}^{b}}{r_{k}}=\frac{2}{1-i p} \frac{c_{j k}-z_{I}}{r_{k}}$.
The undetermined integral of $H$ in Eq. (106) has branches, it equals

$$
\begin{align*}
H= & \frac{\sqrt{\frac{1}{4} \lambda^{2}-\varepsilon}+\frac{\lambda}{2}}{2 \varepsilon \sqrt{\frac{1}{4} \lambda^{2}-\varepsilon}} \ln \frac{\zeta+\frac{1}{2} \lambda-\sqrt{\frac{1}{4} \lambda^{2}-\varepsilon}}{\zeta} \\
& +\frac{\sqrt{\frac{1}{4} \lambda^{2}-\varepsilon}-\frac{\lambda}{2}}{2 \varepsilon \sqrt{\frac{1}{4} \lambda^{2}-\varepsilon}} \ln \frac{\zeta+\frac{1}{2} \lambda+\sqrt{\frac{1}{4} \lambda^{2}-\varepsilon}}{\zeta} \tag{108}
\end{align*}
$$

if $\varepsilon \neq 0$, and
$H= \begin{cases}\frac{1}{\lambda^{2}} \ln \frac{\zeta}{\zeta+\lambda}+\frac{1}{\lambda \zeta}, & \text { if } \lambda \neq 0, \\ \frac{1}{2 \zeta^{2}}, & \text { if } \lambda=0 .\end{cases}$
if $\varepsilon=0$. Combining Eqs. (105)-(109) with the known integral of a straight segment
$g_{2 k-1}^{-}\left(p ; z_{I}\right)=\frac{1}{2 \pi i} \sum_{K=1}^{N} \frac{\bar{s}_{k}}{s_{k}} \ln \frac{w_{2 k}}{w_{2 k-1}}$,
we can rearrange the solution of $g$ as
$g\left(p ; z_{I}\right)=\chi^{\omega}+\sum_{k=1}^{N}\left[\frac{\bar{s}_{k}-s_{k}}{2 \pi i s_{k}} \ln \frac{w_{2 k}}{w_{2 k-1}}+\frac{1}{2 \pi i} I_{\hat{k}}\right]$,
where use is made of the property
$\frac{1}{2 \pi i} \sum_{k=1}^{N}\left[\ln \frac{w_{2 k}}{w_{2 k-1}}+\ln \frac{w_{2 k+1}}{w_{2 k}}\right]=\chi^{\omega}$,
and the integral $I_{\hat{k}}$ herein is specified below.
Introducing the notation
$h_{k}^{\mp}=c_{j k}-z_{I} \mp \sqrt{\left(c_{j k}-z_{I}\right)^{2}-\left(1+p^{2}\right) r_{k}^{2}}$,
$q_{k}=\frac{c_{j k}-z_{I}}{\sqrt{\left(c_{j k}-z_{I}\right)^{2}-\left(1+p^{2}\right) r_{k}^{2}}}$,
the integral $I_{\hat{k}}$ can be expressed as follows:
(i) if $\varepsilon=0$, then
$I_{\hat{k}}= \begin{cases}\frac{-r_{k}^{2}}{\left(c_{k}-z_{l}\right)^{2}}\left[\ln \frac{y_{2 k+1}-z_{l}}{y_{2 k}-z_{l}}-i\left(\phi_{k+1}-\phi_{k}-2 \pi \operatorname{sign}\left(r_{k}\right) \chi_{k}^{1}\right)\right] \\ +\frac{i r_{k}}{c_{k}-z_{l}}\left(e^{-i \phi_{k+1}}-e^{-i \phi_{k}}\right), & \text { if } \lambda \neq 0, \\ -\frac{1}{2}\left(e^{-2 i \phi_{k+1}}-e^{-2 i \phi_{k}}\right) ; & \text { if } \lambda=0 ;\end{cases}$
(ii) if $\varepsilon \neq 0$, then

$$
\begin{align*}
I_{\hat{k}}= & \frac{p-\bar{p}}{i\left(1+p^{2}\right)}\left(1+q_{k}\right)\left[\ln \frac{h_{k}^{-}-i(1-i p) r_{k} e^{i \phi_{k+1}}}{h_{k}^{-}-i(1-i p) r_{k} e^{i \phi_{k}}}-i\left(\phi_{k+1}-\phi_{k}-2 \pi \chi_{k}^{2}\right)\right] \\
& +\frac{p-\bar{p}}{i\left(1+p^{2}\right)}\left(1-q_{k}\right)\left[\ln \frac{h_{k}^{+}-i(1-i p) r_{k} e^{i \phi_{k+1}}}{h_{k}^{+}-i(1-i p) r_{k} e^{i \phi_{k}}}-i\left(\phi_{k+1}-\phi_{k}-2 \pi \chi_{k}^{3}\right)\right] \\
& -\frac{2 \pi i(p-\bar{p})}{1+p^{2}}\left[1-\operatorname{sign}\left(r_{k}\right)\right] \chi_{k}^{4} \tag{115}
\end{align*}
$$

or equivalently

$$
\begin{align*}
I_{\hat{k}}= & \frac{p-\bar{p}}{i\left(1+p^{2}\right)}\left[\ln \frac{y_{j, 2 k+1}-z_{I}}{y_{j, 2 k}-z_{I}}-i\left(\phi_{k+1}-\phi_{k}\right)\right] \\
& +2 \pi \frac{p-\bar{p}}{1+p^{2}}\left[\operatorname{sign}\left(r_{k}\right) \chi_{k}^{1}+q_{k}\left(\chi_{k}^{2}-\chi_{k}^{3}\right)\right] \\
& +\frac{p-\bar{p}}{i\left(1+p^{2}\right)} q_{k}\left[\ln \frac{h_{k}^{-}-i(1-i p) r_{k} e^{i \phi_{k+1}}}{h_{k}^{-}-i(1-i p) r_{k} e^{i \phi_{k}}}-\ln \frac{h_{k}^{+}-i(1-i p) r_{k} e^{i \phi_{k+1}}}{h_{k}^{+}-i(1-i p) r_{k} e^{i \phi_{k}}}\right] \tag{116}
\end{align*}
$$

where the indicator functions $\chi_{k}^{1}, \chi_{k}^{2}, \chi_{k}^{3}$ and $\chi_{k}^{4}$ are defined by
$\chi_{k}^{1}= \begin{cases}1, & \text { if } \operatorname{sign}\left(r_{k}\right) \arg \frac{w_{2 k+1}}{w_{2 k}}<0 \text { and } z_{I} \in e^{k}, \\ 0, & \text { else; }\end{cases}$
$\chi_{k}^{2} / \chi_{k}^{3}= \begin{cases}1, & \text { if } \arg \frac{h_{k}^{\mp}-i(1-i p) r_{k} e^{i \phi_{k+1}}}{h_{k}^{\mp}-i(1-i p) r_{k} e^{i \phi_{k}}}<0, z_{I} \in e^{k}, \\ 0, & \text { else }\end{cases}$
$\chi_{k}^{4}= \begin{cases}1, & z_{I} \in e^{k}, \\ 0, & \text { else }\end{cases}$
A special case of the Jordan curve is a semicircular arc $\partial \omega_{2 k}$ connecting two parallel sides such that
$c_{k}=\frac{1}{2}\left(y_{2 k}+y_{2 k+1}\right), \quad r_{k}=\frac{1}{2}\left|y_{2 k+1}-y_{2 k}\right|$,
which can be considered and will be shown as an example in the next Section. In this case, due to the abnormity of the branch structure, the conditions of the two middle indicator functions $\chi_{k}^{2}, \chi_{k}^{3}$ should be changed to
$\chi_{k}^{2} / \chi_{k}^{3}= \begin{cases}1, & \text { if } \arg \frac{h_{k}^{\mp}-i(1-i p) r_{k} e^{i \phi_{k+1}}}{h_{k}^{\mp}-i(1-i p) r_{k} e^{i \phi_{k}}}<-\pi / 2, \\ 0, & \text { else }\end{cases}$
Another extreme case, which would be interesting from a theoretical point of view, can be carried out if we take the sign definitions of arc radii in Eqs. (97) and (120) $)_{2}$ inversely.

In the case of a Jordan curve, the singularity analysis of the eigenfunction $f^{\prime}(z)$ around the end point $y_{j}$ is very intricate, but direct numerical calculations show that, other than the logarithmic singularity of the polygon around its vertices, there is no singularity at the boundary of an inclusion bounded by a Jordan curve.

## 6. Numerical examples and discussion

As applications, we present a couple of numerical results on the induced fields by the extended eigenstrain in the inclusion of different shapes. These include the polygonal inclusion in full and half planes, and an inclusion made of both curved and straight line segments in the bimaterial plane.

For the sake of easy demonstration, the dimensions of extended stress/strain are rescaled, based on the dimensions of elastic, electric displacement, and magnetic induction constants in $10^{11} \mathrm{~Pa}$, $10 \mathrm{C} \mathrm{m}^{-2}$, and $10^{3} \mathrm{~Wb} \mathrm{~m}^{-2}$. Thus the extended constitutive relation becomes

$$
\left[\begin{array}{c}
\sigma_{i j}\left(10^{11} \mathrm{~Pa}\right)  \tag{122}\\
D_{i}\left(10 \mathrm{Cm}^{-2}\right) \\
B_{i}\left(10^{3} \mathrm{Wbm}^{-2}\right)
\end{array}\right]=\left(\begin{array}{ccc}
C & e & q \\
e^{T} & -\kappa & -d \\
q^{T} & -d^{T} & -\mu
\end{array}\right)\left[\begin{array}{c}
\varepsilon_{i j} \\
\phi_{i}\left(10^{-10} \mathrm{Vm}^{-1}\right) \\
\varphi_{i}\left(10^{-8} \mathrm{Am}^{-1}\right)
\end{array}\right] .
$$

### 6.1. Some fundamental features

Before presenting the numerical results, we first analyze some of the important features associated with the problem.

In the frame of the extended Stroh formalism, the extended strain and stress are completely determined by Eqs. (43) and (44) where $\mathbf{A}, \mathbf{B}$ and $p_{I}(I=1,2,3,4,5)$ are related to the material properties in the reference plane. The spatial variances of the physical fields are simply controlled by the five eigenfunctions $f_{I}\left(z_{I}\right)$ ( $I=1,2,3,4,5$ ), which have different expressions for different problems, say, indicated by Eq. (45) for a full-plane problem, Eq. (A6) for a half-plane one, and Eq. (70) for a bimaterial one. Since $\chi^{\omega}$ is the characteristic function of the inclusion domain $\omega$ that equals to $1(0)$ if $z$ is an inner (outer) point of $\omega$, we find that, besides the two coefficients $c_{I}$ and $d_{I}$ and some other derived material parameters, say $\mathbf{B}$ in Eq. (A6) and $\mathbf{C}_{\mp}, \mathbf{C}_{ \pm}, \mathbf{D}_{ \pm}$in Eq. (70), which are all independent of the spatial coordinates, the five eigenfunctions $f_{I}\left(z_{I}\right)$ can be ascribed to a set of essential eigenfunctions $g$. It is easy to count the number of different essential eigenfunctions, namely $5\left(g\left(p_{I} ; z_{I}\right)\right.$, $I=1,2,3,4,5)$ for the full-plane case, $30\left(g\left(p_{I} ; z_{I}\right), g\left(\bar{p}_{J} ; z_{I}\right)\right.$, $I, J=1,2,3,4,5)$ for the half-plane one, and $50\left(g\left(p_{J} ; z_{I}\right), g\left(\bar{p}_{J} ; z_{I}\right)\right.$, $I, J=1,2,3,4,5)$ for the bimaterial case. From this analysis, we can observe that: (1) the effects of the inclusion shape and material interfaces are controlled by the essential eigenfunction set $\{g\}$, whilst the couplings between different physical fields by the two coefficients $c_{I}$ and $d_{l}$; (2) when the essential eigenfunction set $\{g\}$ and the coefficients $c_{I}$ and $d_{I}$ are fixed, the combinations to form different physical fields are stably controlled by the material matrices $\mathbf{A}$ and $\mathbf{B}$ as shown in Eqs. (43), (44), (A6), and (70); (3) according to the expression of $g$ in Eq. (71), only for the full-plane problem, the size and position of the inclusion have no influence on the induced field.

The two coefficients $c_{I}$ and $d_{I}$ are linearly related to the extended eigenstrain as below:
$c_{I}=-\sum_{J} \frac{B_{I I}^{T} \tilde{\varepsilon}_{J 2}^{*}-\bar{p}_{I} B_{I J}^{T} \tilde{\varepsilon}_{J 1}^{*}}{p_{I}-\bar{p}_{I}}, \quad d_{I}=-\sum_{J} \frac{p_{I} B_{I I}^{T} \tilde{\varepsilon}_{J 1}^{*}-B_{I I}^{T} \tilde{\varepsilon}_{J 2}^{*}}{p_{I}-\bar{p}_{I}}$,
where $\tilde{\varepsilon}_{j p}^{*}$ is defined by Eq. (26). For instance, a hydrostatic elastic eigenstrain $\varepsilon_{11}=\varepsilon_{22}=0.5$ results in
$c_{I}=-0.5 \frac{B_{I 2}^{T}-\bar{p}_{I} B_{I 1}^{T}}{p_{I}-\bar{p}_{I}}, \quad d_{I}=-0.5 \frac{p_{I} B_{I 1}^{T}-B_{I 2}^{T}}{p_{I}-\bar{p}_{I}}$.
Using property Eq. (17), the results can be further simplified to
$c_{I}=-0.5 \frac{1+p_{I} \bar{p}_{I}}{p_{I}-\bar{p}_{I}} B_{I 2}^{T}, \quad d_{I}=0.5 \frac{1+p_{I} p_{I}}{p_{I}-\bar{p}_{I}} B_{I 2}^{T}$,
where, for the material considered in Section 6.2 below, we have
$\left(B_{2 I}\right)=\left(\begin{array}{r}-0.3661682+0.4807980 i \\ -0.7544860+0.1823175 i \\ -0.7574741+0.1825447 i \\ -0.3379321-0.8959268 i \\ 0.6047747+0.3394896 i\end{array}\right)$.
Since there are different essential eigenfunctions $g$ corresponding to different $p_{I}$ and $p_{J}$ for the given problem, we prefer to present the induced elastic strain and $E$ - $/ H$-fields instead of the set of $g$ though the latter may be more fundamentally associated with


Fig. 3. Regular $N$-side polygons inside a unit circle in a full plane or a half plane. For the half-plane case, the center distance of the circle to the surface of the half plane equals 2 .
the geometry of the problem. Therefore, in the following numerical examples, we calculate the invariants of the extended strains as below: the hydrostatic strain $\left(\varepsilon_{h}=\varepsilon_{11}+\varepsilon_{22}\right)$, the deviatoric strain $\varepsilon_{d}=\sqrt{\left(\varepsilon_{11}-\varepsilon_{22}\right)^{2}+4 \varepsilon_{12}^{2}}$, the anti-plane strain magnitude $\varepsilon_{a}=$ $2 \sqrt{\varepsilon_{31}^{2}+\varepsilon_{32}^{2}}$, and the $E$-/H-fields $E_{h}=\sqrt{E_{1}^{2}+E_{2}^{2}}$ and $H_{h}=\sqrt{H_{1}^{2}+H_{2}^{2}}$.

### 6.2. An $N$-side polygon in a full and half multiferroic plane

The inclusion is a regular $N$-side polygon inside a circle with unit radius (Fig. 3), which is in a full plane or a half plane, as in Pan and Jiang (2006). In the half plane case, the center distance of the inclusion to the surface is 2 . The multiferroic composite material properties ( $50 \%$ of $\mathrm{BaTiO}_{3}$ and $50 \%$ of $\mathrm{CoFe}_{2} \mathrm{O}_{4}$ ) are listed in Property column of Table B1 in Appendix B (Xue et al., 2011). The rescaled material properties are listed in the Rescaled column of Table B1 in Appendix B. This composite, as well as its corresponding decoupled cases (piezoelectric with $q_{i j}=0$ and piezomagnetic with $e_{i j}=0$ ) are all analyzed to investigate the effect of different couplings on the eigenstrain-induced fields.

For a circular inclusion, we have a constant essential eigenfunction $g$ inside the inclusion
$g\left(p_{I} ; z_{I}\right)=\frac{1-i \bar{p}_{I}}{1-i p_{I}}$.
For regular polygons with $N$ vertices defined by
$y_{k}=e^{2 \pi i^{\frac{k-1}{N}}}, \quad k=1,2, \ldots, N$,

Table 1a
Invariants of extended strains under eigenstrain $\varepsilon_{11}^{*}=1$ in $N$-side polygonal inclusion embedded in a multiferroic composite full plane.

| $N$ | Area | $\varepsilon_{h}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{h}\left(10^{-1}\right)$ | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $E_{h}\left(10^{-2}\right)$ | $\hat{E}_{h}\left(10^{-2}\right)$ | $H_{h}\left(10^{-4}\right)$ |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 1.30 | -2.27 | -2.26 | 3.57 | 3.48 | 4.24 | 4.23 | 9.55 |
| 4 | 2.00 | -2.29 | -2.26 | 4.63 | 3.05 | 4.60 | 4.08 | 17.2 |
| 5 | 2.38 | -2.27 | -2.27 | 3.52 | 3.52 | 4.23 | 4.23 | 2.38 |
| 6 | 2.60 | -2.27 | -2.27 | 3.56 | 3.53 | 4.24 | 4.23 | 7.42 |
| 10 | 2.94 | -2.27 | -2.27 | 3.52 | 3.52 | 4.23 | 4.23 | 2.34 |
| $\geqslant 20$ | 3.14 | -2.27 | -2.27 | 3.52 | 3.52 | 4.23 | 4.23 | 3.81 |
|  |  |  |  |  | 1.99 |  |  |  |

Table 1b
Invariants of extended strains under eigenstrain $\varepsilon_{12}^{*}=1$ in $N$-side polygonal inclusion embedded in a multiferroic composite full plane.

| $N$ | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $E_{h}\left(10^{-2}\right)$ | $\hat{E}_{h}\left(10^{-2}\right)$ | $H_{h}\left(10^{-3}\right)$ | $\hat{H}_{h}\left(10^{-3}\right)$ |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 7.91 | 7.99 | 3.01 | 2.94 | 3.67 | 3.05 |
| 4 | 5.80 | 8.86 | 2.17 | 3.31 | 2.33 | 3.79 |
| 5 | 7.96 | 7.96 | 2.97 | 2.97 | 3.32 | 3.29 |
| 6 | 7.92 | 7.95 | 3.00 | 2.98 | 3.57 | 3.38 |
| 10 | 7.96 | 7.96 | 2.97 | 2.97 | 3.32 | 3.31 |
| $\geqslant 20$ | 7.96 | 7.96 | 2.97 | 2.97 | 3.30 | 3.30 |

Table 1c
Invariants of extended strains under eigenstrain $E_{1}^{*}=1$ in $N$-side polygonal inclusion embedded in a multiferroic composite full plane.

| $N$ | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $E_{h}\left(10^{-2}\right)$ | $\hat{E}_{h}\left(10^{-2}\right)$ | $H_{h}\left(10^{-4}\right)$ | $\hat{H}_{h}\left(10^{-4}\right)$ |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 3.64 | 3.53 | 5.25 | 5.25 | 4.53 | 8.84 |
| 4 | 4.39 | 3.24 | 5.18 | 5.28 | 5.70 | 7.11 |
| 5 | 3.58 | 3.58 | 5.25 | 5.25 | 6.93 | 7.15 |
| 6 | 3.62 | 3.59 | 5.25 | 5.25 | 5.25 | 6.57 |
| 10 | 3.58 | 3.58 | 5.25 | 5.25 | 6.95 | 7.06 |
| $\geqslant 20$ | 3.58 | 3.58 | 5.25 | 5.25 | 7.08 | 7.08 |

Table 1d
Invariants of extended strains under eigenstrain $H_{1}^{*}=1$ in N -side polygonal inclusion embedded in a multiferroic composite full plane.

| $N$ | $\varepsilon_{d}\left(10^{-2}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-2}\right)$ | $E_{h}\left(10^{-3}\right)$ | $\hat{E}_{h}\left(10^{-3}\right)$ | $H_{h}\left(10^{-1}\right)$ | $\hat{H}_{h}\left(10^{-1}\right)$ |
| ---: | :---: | :--- | :--- | :--- | :--- | :--- |
| 3 | 10.6 | 7.22 | 4.44 | 5.15 | 3.57 | 3.40 |
| 4 | 14.4 | 6.54 | 3.61 | 5.31 | 3.87 | 3.32 |
| 5 | 8.72 | 8.57 | 4.84 | 4.87 | 3.48 | 3.47 |
| 6 | 10.1 | 9.02 | 4.56 | 4.77 | 3.55 | 3.49 |
| 10 | 8.70 | 8.63 | 4.84 | 4.86 | 3.47 | 3.47 |
| $\geqslant 20$ | 8.61 | 8.61 | 4.86 | 4.86 | 3.47 | 3.47 |

Table 2a
Invariants of extended strains under eigenstrain $\varepsilon_{11}^{*}=1$ in $N$-side polygonal inclusion embedded in a piezomagnetic full plane.

| $N$ | $\varepsilon_{h}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{h}\left(10^{-1}\right)$ | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $H_{h}\left(10^{-4}\right)$ | $\hat{H}_{h}\left(10^{-4}\right)$ |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | -2.09 | -2.08 | 3.86 | 3.78 | 3.71 | 8.40 |
| 4 | -2.08 | -2.09 | 4.92 | 3.35 | 10.6 | 8.03 |
| 5 | -2.08 | -2.08 | 3.82 | 3.81 | 3.07 | 3.61 |
| 6 | -2.09 | -2.09 | 3.85 | 3.82 | 1.70 | 2.00 |
| 10 | -2.08 | -2.08 | 3.82 | 3.81 | 3.11 | 3.39 |
| $\geqslant 20$ | -2.08 | -2.08 | 3.81 | 3.81 | 3.44 | 3.44 |

the center and average values of $g\left(p_{\mathrm{I}} ; z_{\mathrm{I}}\right)$ can be calculated using Eqs. (85) and (92) in Section 4.2 with the circular inclusion case $(N \rightarrow \infty)$ being calculated using Eq. (126). For the non-zero eigenstrain $\varepsilon_{11}^{*}=1, \varepsilon_{12}^{*}=1, E_{1}^{*}=1$, and $H_{1}^{*}=1$ respectively, we calculate the following invariants of the extended strains: the hydrostatic strain

Table 2b
Invariants of extended strains under eigenstrain $\varepsilon_{12}^{*}=1$ in $N$-side polygonal inclusion embedded in a piezomagnetic full plane.

| $N$ | 3 | 4 | 5 | 6 | 10 | $\geqslant 20$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\varepsilon_{d}\left(10^{-1}\right)$ | 8.08 | 5.90 | 8.10 | 8.08 | 8.10 | 8.10 |
| $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | 8.11 | 9.02 | 8.10 | 8.09 | 8.10 | 8.10 |
| $H_{h}\left(10^{-3}\right)$ | 3.96 | 2.54 | 3.60 | 3.85 | 3.60 | 3.58 |
| $\hat{H}_{h}\left(10^{-3}\right)$ | 3.32 | 4.09 | 3.57 | 3.66 | 3.58 | 3.58 |

Table 2c
Invariants of extended strains under eigenstrain $E_{1}^{*}=1$ in $N$-side polygonal inclusion embedded in a piezomagnetic full plane.

| $N$ | 3 | 4 | $\geqslant 5$ |
| :--- | :--- | :--- | :--- |
| $E_{h}\left(10^{-2}\right)$ | 5.15 | 5.11 | 5.15 |
| $\hat{E}_{h}\left(10^{-2}\right)$ | 5.15 | 5.17 | 5.15 |

Table 2d
Invariants of extended strains under eigenstrain $H_{1}^{*}=1$ in N -side polygonal inclusion embedded in a piezomagnetic full plane.

| $N$ | 3 | 4 | 5 | 6 | 10 | $\geqslant 20$ |
| :--- | :---: | :---: | :--- | :--- | :--- | :--- |
| $\varepsilon_{d}\left(10^{-2}\right)$ | 10.5 | 14.3 | 8.58 | 9.90 | 8.57 | 8.48 |
| $\hat{\varepsilon}_{d}\left(10^{-2}\right)$ | 7.10 | 6.37 | 8.43 | 8.88 | 8.49 | 8.48 |
| $H_{\mathrm{h}}\left(10^{-1}\right)$ | 3.58 | 3.87 | 3.48 | 3.55 | 3.48 | 3.47 |
| $\hat{H}_{h}\left(10^{-1}\right)$ | 3.40 | 3.32 | 3.47 | 3.50 | 3.47 | 3.47 |

Table 3a
Invariants of extended strains under eigenstrain $\varepsilon_{11}^{*}=1$ in $N$-side polygonal inclusion embedded in a multiferroic composite half plane.

| $N$ | Area | $\varepsilon_{h}$ <br> $\left(10^{-1}\right)$ | $\hat{\varepsilon}_{h}$ <br> $\left(10^{-1}\right)$ | $\varepsilon_{d}$ <br> $\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}$ <br> $\left(10^{-1}\right)$ | $E_{h}$ <br> $\left(10^{-2}\right)$ | $\hat{E}_{h}$ <br> $\left(10^{-2}\right)$ | $H_{h}$ <br> $\left(10^{-4}\right)$ | $\hat{H}_{h}$ <br> $\left(10^{-4}\right)$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 1.30 | -2.33 | -2.14 | 3.40 | 2.27 | 4.26 | 3.32 | 6.84 | 16.4 |
| 4 | 2.00 | -2.38 | -2.26 | 4.38 | 2.71 | 4.64 | 3.98 | 13.0 | 4.24 |
| 5 | 2.38 | -2.37 | -2.16 | 3.22 | 3.41 | 4.28 | 4.08 | 2.62 | 0.701 |
| 6 | 2.60 | -2.38 | -2.14 | 3.24 | 2.68 | 4.29 | 3.79 | 1.95 | 1.00 |
| 10 | 2.94 | -2.39 | -2.13 | 3.17 | 3.26 | 4.30 | 3.98 | 3.85 | 0.899 |
| 20 | 3.09 | -2.40 | -2.12 | 3.16 | 3.22 | 4.31 | 3.94 | 4.53 | 1.46 |
| 30 | 3.12 | -2.40 | -2.11 | 3.16 | 3.21 | 4.31 | 3.93 | 4.59 | 2.08 |
| $\geqslant 100$ | 3.14 | -2.40 | -2.11 | 3.16 | 3.21 | 4.31 | 3.91 | 4.64 | 2.96 |

$\varepsilon_{h}$, deviatoric strain $\varepsilon_{d}$, anti-plane strain magnitude $\varepsilon_{a}$, and the $E$-/ $H$-fields $E_{h}$ and $H_{h}$.

It is observed that for the present eigenstrain problem, $\varepsilon_{a}$ is identically zero, and that some of the other quantities are zero under different eigenstrain and different coupling conditions. Furthermore, when the polygon side number $N>20$, the results for these quantities approach the same as those of a circular inclusion.

Listed in Tables 1a-1d are the center and average (with a hat) values of the induced fields in the $N$-side polygonal inclusion

Table 3b
Invariants of extended strains under eigenstrain $\varepsilon_{12}^{*}=1$ in $N$-side polygonal inclusion embedded in a multiferroic composite half plane.

| $N$ |  | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $E_{h}\left(10^{-2}\right)$ | $\hat{E}_{h}\left(10^{-2}\right)$ | $H_{h}\left(10^{-3}\right)$ |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 7.97 | 15.2 | 3.04 | 5.82 | 3.74 | 7.07 |
| 4 | 5.95 | 9.28 | 2.22 | 3.55 | 2.45 | 4.34 |
| 5 | 8.17 | 5.46 | 3.05 | 2.02 | 3.48 | 2.04 |
| 6 | 8.18 | 4.49 | 3.10 | 1.64 | 3.75 | 1.54 |
| 10 | 8.30 | 4.32 | 3.10 | 1.55 | 3.54 | 1.37 |
| 20 | 8.34 | 5.09 | 3.11 | 1.86 | 3.53 | 1.79 |
| 30 | 8.34 | 5.52 | 3.11 | 2.03 | 3.54 | 2.04 |
| $\geqslant 100$ | 8.35 | 6.20 | 3.11 | 2.30 | 3.54 | 2.44 |

( $N=3,4,5,6,10$, and 20 ), by the nonzero eigenstrains $\varepsilon_{11}^{*}=1$, $\varepsilon_{12}^{*}=1, E_{1}^{*}=1$, and $H_{1}^{*}=1$, respectively. The area of each polygon is also listed, and the matrix is assumed to be a full plane made of fully coupled multiferroic composite (Table B1). It is observed that, under $\varepsilon_{11}^{*}=1$, the induced hydrostatic strain and its average are all the same $(=-2.27)$ for different $N$, except for $N=3$ and 4 . However, the deviatoric strain, the $E$ - and $H$-fields are different for different $N$, and that for a given $N$, their center and average values are also different. This is particularly true for the $H$-field, where the difference between its center and average values can be over 5times (for $N=4$ ). Under eigenstrains $\varepsilon_{12}^{*}=1, E_{1}^{*}=1$, and $H_{1}^{*}=1$, the center and average values of the induced hydrostatic strain are identically zero. The center and average values of the other induced fields depend slightly on $N$, and for fixed $N$, small difference between the center and average values can be observed (Tables 1b-1d).

We have also calculated the center and average values of the induced fields in the N -side polygonal inclusion in the corresponding piezoelectric full plane. For this case, since there is no coupling between the piezoelectric and magnetic fields, it is obvious that under eigenstrains $\varepsilon_{11}^{*}=1, \varepsilon_{12}^{*}=1$ and $E_{1}^{*}=1$, the induced H -field is identically zero, and under $H_{1}^{*}=1$, the only induced nonzero field is the $H$-field. It is interesting, however, that the center and average values of the induced nonzero fields in the piezoelectric full plane are nearly the same as those in the corresponding multiferroic composite full plane. In other words, the effect of the magnetic coupling coefficients $q_{i j}$ on the induced fields is very small.


Fig. 4. An inclusion with its boundary made of straight and curved lines in material \#2 $\left(x_{2}<0\right)$ of the bimaterial plane (a square with side length 2 plus two half disks with radius 1 ). The center of the inclusion is at $\left(x_{1}, x_{2}\right)=(0,-2)$, and the domain for the numerical calculation is $-3<x_{1}<3,-4<x_{2}<1$.


Fig. 5a. Contour of $\varepsilon_{h}$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{11}^{*}=1$ in the inclusion made of a square plus two half circles (dash lines denote negative values).

Table 3c
Invariants of extended strains under eigenstrain $E_{1}^{*}=1$ in $N$-side polygonal inclusion embedded in a multiferroic composite half plane.

| $N$ | $\varepsilon_{h}\left(10^{-2}\right)$ | $\hat{\varepsilon}_{h}\left(10^{-2}\right)$ | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $E_{h}\left(10^{-1}\right)$ | $\hat{E}_{h}\left(10^{-1}\right)$ | $H_{h}\left(10^{-4}\right)$ | $\hat{H}_{h}\left(10^{-4}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | -0.0129 | -1.31 | 3.14 | 4.00 | 4.98 | 4.49 | 2.93 | 8.92 |
| 4 | 0.00 | 0.00 | 4.04 | 3.14 | 4.98 | 5.30 | 3.94 | 5.82 |
| 5 | -0.00136 | -0.276 | 3.15 | 3.82 | 5.01 | 4.66 | 4.82 | 7.15 |
| 6 | 0.00 | 0.00 | 3.14 | 4.00 | 4.98 | 4.49 | 2.93 | 8.92 |
| 10 | 0.00 | 0.00 | 3.02 | 4.00 | 4.95 | 4.45 | 4.29 | 8.27 |
| 20 | 0.00 | 0.00 | 2.98 | 3.92 | 4.93 | 4.63 | 4.27 | 8.32 |
| 30 | 0.00 | 0.00 | 2.97 | 3.86 | 4.93 | 4.71 | 4.24 | 8.24 |
| $\geqslant 100$ | 0.00 | 0.00 | 2.97 | 3.77 | 4.93 | 4.84 | 4.22 | 8.04 |

Table 3d
Invariants of extended strains under eigenstrain $H_{1}^{*}=1$ in $N$-side polygonal inclusion embedded in a multiferroic composite half plane.

| $N$ | $\varepsilon_{h}\left(10^{-2}\right)$ | $\hat{\varepsilon}_{h}\left(10^{-2}\right)$ | $\varepsilon_{d}\left(10^{-2}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-2}\right)$ | $E_{h}\left(10^{-3}\right)$ | $\hat{E}_{h}\left(10^{-3}\right)$ | $H_{h}\left(10^{-1}\right)$ | $\hat{H}_{h}\left(10^{-1}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 0.00 | -1.94 | 9.73 | 7.15 | 4.59 | 6.56 | 3.51 | 3.34 |
| 4 | 0.00 | 0.00 | 13.0 | 7.08 | 3.86 | 5.13 | 3.76 | 3.28 |
| 5 | 0.00 | $-0.0366$ | 7.01 | 10.8 | 5.14 | 3.68 | 3.35 | 3.31 |
| 6 | 0.00 | 0.00 | 8.18 | 12.1 | 4.89 | 2.85 | 3.40 | 3.17 |
| 10 | 0.00 | 0.00 | 6.55 | 11.2 | 5.23 | 3.31 | 3.32 | 3.23 |
| 20 | 0.00 | 0.00 | 6.33 | 10.7 | 5.27 | 3.58 | 3.30 | 3.26 |
| 30 | 0.00 | 0.00 | 6.31 | 10.4 | 5.28 | 3.93 | 3.30 | 3.28 |
| $\geqslant 100$ | 0.00 | 0.00 | 6.29 | 9.84 | 5.28 | 3.98 | 3.30 | 3.28 |



Fig. 5b. Contour of $\varepsilon_{d}$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{11}^{*}=1$ in the inclusion made of a square plus two half circles.


Fig. 5c. Contour of $\varepsilon_{a}$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{11}^{*}=1$ in the inclusion made of a square plus two half circles.


Fig. 5d. Contour of $E_{\mathrm{h}}\left(\times 10^{-10} \mathrm{~V} / \mathrm{m}\right)$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{11}^{*}=1$ in the inclusion made of a square plus two half circles.

Listed in Tables 2a-2d are the center and average (with a hat) values of the induced fields in the N -side polygonal inclusion in the corresponding piezomagnetic full plane. Similar to the piezoelectric half plane case, that under eigenstrains $\varepsilon_{11}^{*}=1, \varepsilon_{12}^{*}=1$ and $H_{1}^{*}=1$, the induced $E$-field is identically zero, and under $E_{1}^{*}=1$, the only induced nonzero field is the $E$-field. However, the center and average values of the induced nonzero fields in the piezomagnetic full plane are substantially different to those in the corresponding multiferroic composite full plane (Tables 2a and 2 b vs. Tables 1 a and 1 b ), or apparent difference between them can be observed (Tables 2c and 2d vs. Tables 1c and 1d). In other words, the effect of the piezoelectric coupling coefficients $e_{i j}$ on the induced fields is important.

Listed in Tables 3a-3d are the center and average (with a hat) values of the induced fields in the $N$-side polygonal inclusion ( $N=3,4,5,6,10,20,30$, and 100 ), respectively, by the nonzero eigenstrains $\varepsilon_{11}^{*}=1, \varepsilon_{12}^{*}=1, E_{1}^{*}=1$, and $H_{1}^{*}=1$. The matrix is now assumed to be a half plane made of fully coupled multiferroic composite (Table B1 in Appendix B), and the distance of the center of the $N$-side polygon to the surface is 2 . On the surface of the half plane, the extended traction is assumed to be zero. From Tables 3a-3d, the following features can be observed: (1) Due to the effect of the free surface, the center and average values are both more sensitive to $N$ when $N$ is large. (2) In general, the center and average values are much different in the half-plane case as compared to the full-plane case (these values could be increased or decreased). (3) For example, comparing Tables 3a to 1 a , it is observed that un$\operatorname{der} \varepsilon_{11}^{*}=1$, the magnitude of the hydrostatic strain at the center is increased in the half-plane case, whilst its average is decreased. (4) Under $E_{1}^{*}=1$ or $H_{1}^{*}=1$, the center and average value of the hydrostatic strain is nonzero when $N=3$ and 5 (Tables 3c and 3d), a feature again different to the corresponding full-plane case.

### 6.3. Inclusions bounded by both straight and curved lines in a bimaterial plane

In strain energy band engineering, the self-assembled or selforganized quantum-wires can be in various shapes (Faux et al., 1997; Pan and Jiang, 2004; Maranganti and Sharma, 2007), bounded by straight and curved line segments. Therefore, as a new example, we consider now that there is an inclusion bounded by both straight and curved lines in the lower half plane of the bimaterial system as shown in Fig. 4. The inclusion is actually composed of a square with side length 2 , and two half circles of unit radius on both sides of the square. The distance of the center of the inclusion to the interface is equal to 2 . The material properties in both half planes are taken from Pan (2002), and the corresponding rescaled values are listed in Table B2 and B3 in Appendix B.

Again, we are interested in the behavior of the combined Eshelby tensor components, that is, the hydrostatic, deviatoric and antiplane strains $\left(\varepsilon_{h}, \varepsilon_{d}, \varepsilon_{a}\right)$, and the $E$ - $/ H$-fields $\left(E_{h}, H_{h}\right)$. Under the applied non-zero eigenstrain $\varepsilon_{11}^{*}=1$ and $\varepsilon_{12}^{*}=1$ respectively, the induced fields are calculated in the region $-3<x_{1}<3,-4<x_{2}<1$ (as shown by the dashed rectangle in Fig. 4). This region covers both inside and outside of the inclusion, as well as both half planes. Before the numerical calculation, the solution was checked to satisfy the continuity conditions along the interface of the bimaterial plane.

First, under the nonzero eigenstrain $\varepsilon_{11}^{*}=1$, the contours of $\varepsilon_{h}$, $\varepsilon_{d}, \varepsilon_{a}, E_{h}$, and $H_{h}$ are shown, respectively, in Figs. 5a-5e. Fig. 5a shows that there is a large compressive hydrostatic strain inside the inclusion; however, outside, this field is positive, particularly near the interface in the lower half plane. Strain concentrations can be also observed near the left and right ends of the inclusion. Fig. 5b shows that the deviatoric strain is much larger inside than outside. Compared to the in-plane strain field in Figs. 5a and 5b,

Table 4.1a
Center and average values of the field quantities in the inclusion induced by the nonzero eigenstrain $\varepsilon_{11}^{*}=1$ in a multiferroic bimaterial plane (Fig. 4 ).

| Shape | Area | $\varepsilon_{h}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{h}\left(10^{-1}\right)$ | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $\varepsilon_{a}\left(10^{-2}\right)$ | $\hat{\varepsilon}_{a}\left(10^{-2}\right)$ | $E_{h}\left(10^{-2}\right)$ | $\hat{E}_{h}\left(10^{-2}\right)$ | $H_{h}\left(10^{-2}\right)$ | $\hat{H}_{h}\left(10^{-2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Square | 4.000 | -3.717 | -3.591 | 3.934 | 4.904 | 3.681 | 5.445 | 6.714 | 4.817 | 10.04 | 10.26 |
|  |  |  |  | -3.908 | -4.898 | 3.337 | 2.046 | 6.106 | 4.787 | 0.3151 | -0.1009 |
|  |  |  |  | 0.4441 | 0.2364 | -1.555 | -5.047 | 2.792 | -0.5404 | 10.03 | 10.26 |
| L-half | 1.571 | -0.2226 | $-2.776$ | 2.264 | 2.295 | 1.347 | 3.376 | 0.9110 | 3.530 | 3.953 | 5.311 |
|  |  |  |  | -2.264 | -2.293 | -0.2798 | -0.1313 | -0.7783 | 3.530 | -0.7074 | 0.3630 |
|  |  |  |  | 0.005177 | 0.08315 | -1.318 | -3.373 | 0.4734 | 0.02099 | 3.889 | 5.299 |
| R-half | 1.571 | $-0.3847$ | -2.769 | 1.997 | 2.414 | 3.231 | 3.349 | 0.9833 | 3.750 | 3.893 | 5.371 |
|  |  |  |  | -1.973 | -2.312 | 1.744 | -3.348 | 0.7160 | 3.749 | 1.096 | 0.3535 |
|  |  |  |  | 0.3087 | 0.08267 | -2.721 | -0.07260 | 0.6739 | 0.09506 | 3.736 | 5.359 |
| Sum | 7.142 | -4.324 | -4.437 | 8.181 | 7.397 | 7.371 | 9.157 | 7.214 | 5.435 | 17.67 | 14.32 |
|  |  |  |  | -8.145 | -7.389 | 4.801 | 0.8638 | 6.044 | 5.420 | 0.7039 | 0.7541 |
|  |  |  |  | 0.7580 | 0.3231 | -5.594 | -9.1116 | 3.939 | 0.3965 | 17.66 | 14.30 |



Fig. 5e. Contour of $H_{\mathrm{h}}\left(\times 10^{-8} \mathrm{~A} / \mathrm{m}\right)$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{11}^{*}=1$ in the inclusion made of a square plus two half circles.


Fig. 6a. Contour of $\varepsilon_{h}$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{12}^{*}=1$ in the inclusion made of a square plus two half circles (dash lines denote negative values).
the anti-plane strain is much smaller as shown in Fig. 5c. The contour of the $E$-field magnitude $E_{h}\left(\times 10^{-10} \mathrm{~V} / \mathrm{m}\right)$ is shown in Fig. 5 d . In contrary to the strain feature, a large $E_{h}$-field is observed in the matrix, instead of in the inclusion. More interestingly, there are


Fig. 6b. Contour of $\varepsilon_{d}$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{12}^{*}=1$ in the inclusion made of a square plus two half circles.


Fig. 6c. Contour of $\varepsilon_{a}$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{12}^{*}=1$ in the inclusion made of a square plus two half circles.
four concentrations around points A, B, C, and D marked in Fig. 4. As for the magnetic $H_{\mathrm{h}}\left(\times 10^{-8} \mathrm{~A} / \mathrm{m}\right)$-field, large values are all located inside the inclusion, just as for the strain distribution. In all these figures, the values in the upper half plane are very small as compared to those in the lower half plane.


Fig. 6d. Contour of $E_{\mathrm{h}}\left(\times 10^{-10} \mathrm{~V} / \mathrm{m}\right)$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{12}^{*}=1$ in the inclusion made of a square plus two half circles.


Fig. 6e. Contour of $H_{\mathrm{h}}\left(\times 10^{-8} \mathrm{~A} / \mathrm{m}\right)$ in the multiferroic bimaterial plane due to the eigenstrain $\varepsilon_{12}^{*}=1$ in the inclusion made of a square plus two half circles.

The center and average (with a hat) values in the inclusion are listed in Table 4.1a, separately for the square, the two half circles, as well as for the entire inclusion. Also in this table, for the deviatoric strain $\varepsilon_{d}$, anti-plane strain magnitude $\varepsilon_{a}, E$-field $E_{h}$, and $H$-field $H_{h}$, the upper and lower subrows are, respectively, their corresponding components ( $\varepsilon_{11}-\varepsilon_{22}$ ) and $2 \varepsilon_{12}, 2 \varepsilon_{31}$ and $2 \varepsilon_{32}, E_{1}$ and $E_{2}$,
and $H_{1}$ and $H_{2}$. It is interesting to note that the values of $\varepsilon_{h} \varepsilon_{\mathrm{d}}, \varepsilon_{a}$, $E_{h}$, and $\mathrm{H}_{h}$ at the center of the inclusion equals, respectively, the summation of its individual center values; however, the corresponding average values have no such superposition property.

Similarly, under the nonzero eigenstrain $\varepsilon_{12}^{*}=1$, the contours of $\varepsilon_{h}, \varepsilon_{d}, \varepsilon_{a}, E_{h}$, and $H_{h}$ are shown, respectively, in Figs. 6a-6e. It is interesting to observe from Fig. 6a that the distribution of the hydrostatic strain $\varepsilon_{h}$ is anti-symmetric with respect to the two symmetry lines of the inclusion (horizontal and vertical lines), and that again there are four concentrations around points A, B, C, and D as marked in Fig. 4. Also, under this eigenstrain, the induced hydrostatic strain is much larger outside! The induced deviatoric strain (Fig. 6b) shows concentrations along the interface between the inclusion and matrix, and also concentrations completely within the matrix. Two large concentrations can be observed near points B and C. Four concentrations at points A, B, C, and D are also observed for the contours of $\varepsilon_{a}$ (Fig. 6c), $E_{h}$ (Fig. 6d), and $H_{h}$ (Fig. 6e).

The center and average (with a hat) values in the inclusion are listed in Table 4.1b, separately for the square, the two half circles, as well as for the entire inclusion. Similarly, it is observed that the values of $\varepsilon_{h}, \varepsilon_{d}, \varepsilon_{a}, E_{h}$, and $H_{h}$ at the center of the inclusion equals, respectively, the summation of its individual center values, whilst the corresponding average values have no such superposition property.

## 7. Concluding remarks

In this paper, we have proposed a comprehensive and unified approach to solve the Eshelby's inclusion problem in an anisotropic multiferroic bimaterial plane. The solutions are not only general but also in explicit analytical forms. The inclusion can be of an arbitrary shape, described by a Laurent polynomial, a polygon, or the one bounded by a Jordan curve. Our solutions contain further the results in the corresponding half plane and full plane. We have also identified the essential eigenfunctions by which the induced fields can be simply determined. Numerical results are presented to investigate the features of these eigenfunctions as well as the strain, electric and magnetic fields (components of the extended Eshelby tensor). Particularly, we presented the values of these fields at the center of the $N$-side regular polygonal inclusion and also the average values of these fields over the inclusion. The effect of the half-plane traction-free surface condition and the effect of various couplings on the induced fields are discussed in detail. For the $N$-side regular polygonal inclusion, we found that, when the inclusion is in the full plane, both the center and average values of the induced fields are independent of $N$, except for $N=4$. We also showed that the piezoelectric and piezomagnetic coupling coefficients could significantly affect the Eshelby tensor. This feature

Table 4.1b
Center and average values of the field quantities in the inclusion induced by the nonzero eigenstrain $\varepsilon_{12}^{*}=1$ in a mulitferroic bimaterial plane (Fig. 4).

| Shape | Area | $\varepsilon_{h}\left(10^{-2}\right)$ | $\hat{\varepsilon}_{h}\left(10^{-2}\right)$ | $\varepsilon_{d}\left(10^{-1}\right)$ | $\hat{\varepsilon}_{d}\left(10^{-1}\right)$ | $\varepsilon_{a}\left(10^{-2}\right)$ | $\hat{\varepsilon}_{a}\left(10^{-2}\right)$ | $E_{h}\left(10^{-2}\right)$ | $\hat{E}_{h}\left(10^{-2}\right)$ | $H_{\mathrm{h}}\left(10^{-2}\right)$ | $\hat{H}_{h}\left(10^{-2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Square | 4.000 | 4.729 | 0.3400 | 7.055 | 3.968 | 4.977 | 1.998 | 14.95 | 7.880 | 8.695 | 2.359 |
|  |  |  |  | -1.195 | -0.1782 | -4.395 | 1.282 | -3.896 | 0.8725 | -8.495 | -1.908 |
|  |  |  |  | -6.953 | -3.964 | 2.336 | -1.533 | -14.43 | -7.832 | 1.852 | 1.387 |
| L-half | 1.571 | $-0.9870$ | $-0.7842$ | 0.2364 | 3.905 | 1.276 | 2.198 | 2.476 | 7.180 | 1.863 | 0.7398 |
|  |  |  |  | -0.05897 | 0.01428 | -0.9933 | 1.128 | -0.1445 | 0.06666 | -1.706 | -0.5611 |
|  |  |  |  | 0.2290 | -3.902 | -0.8013 | -1.886 | -2.471 | -7.179 | -0.7489 | 0.4822 |
| R-half | 1.571 | 2.692 | 0.7893 | 0.9423 | 3.926 | 1.740 | 2.140 | 1.087 | 6.863 | 2.271 | 0.7925 |
|  |  |  |  | -0.6911 | 0.1455 | 1.375 | 1.053 | -0.5009 | $-0.03369$ | -0.8665 | -0.6416 |
|  |  |  |  | 0.6406 | -3.923 | 1.067 | -1.862 | 0.9646 | -6.863 | 2.099 | 0.4652 |
| Sum | 7.142 | 6.435 | 1.220 | 6.387 | 3.816 | 4.783 | 6.733 | 16.57 | 10.04 | 11.52 | 2.219 |
|  |  |  |  | -1.945 | 0.2440 | -4.014 | 5.310 | -4.542 | -0.2271 | -11.07 | -1.682 |
|  |  |  |  | -6.083 | -3.808 | 2.601 | -4.139 | -15.94 | -10.03 | 3.203 | 1.448 |

should be useful in controlling the Eshelby tensor for the design of better multiferroic composites. Typical contours of the field quantities in and around the inclusion bounded by both straight and curved line segments in a multiferroic bimaterial plane are also presented.

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## Appendix A. Solution of the Eshelby's inclusion problem in a multiferroic half-plane

From the bimaterial-plane solutions given in Section 4, we can obtain the solution in the corresponding half plane. We assume that the inclusion is in the lower half-plane, and that its surface is (extended) traction free. In other words, the conditions in Eq. (56) are changed to
$\psi=0, \quad$ on $x_{2}=0$.
We can derive the half-plane solution with the extended trac-tion-free boundary condition as
$\mathbf{f}(z)=\mathbf{f}^{\infty}(z)+\mathbf{f}^{b}(z)$,
with the relation
$\mathbf{B f}^{b}\left(x_{1}\right)+\overline{\mathbf{B f}^{b}\left(x_{1}\right)}=-\mathbf{B f}^{\infty}\left(x_{1}\right)-\overline{\mathbf{B f}^{\infty}\left(x_{1}\right)}$.
From the Cauchy formulae and the property $f_{-}^{b}(\infty)=0$, we can find that
$f^{b}(z)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{f^{\infty}(t)}{\overline{t-z}} d t+\frac{\mathbf{B}^{-1} \overline{\mathbf{B}}}{2 \pi i} \int_{-\infty}^{\infty} \frac{\overline{f^{\infty}(t)}}{\overline{t-z}} d t$,
namely
$f_{I}^{b}\left(z_{I}\right)=\sum_{K, J} \frac{B_{I K}^{-1} \bar{B}_{K} \bar{d}_{J}}{2 \pi i} \oint_{\Gamma_{J}} \frac{y_{J}}{\bar{y}_{J}-z_{I}} d \bar{y}_{J}$,
$f_{I}^{b \prime}\left(z_{I}\right)=\sum_{K, J} \frac{B_{I K}^{-1} \bar{B}_{K J} \bar{d}_{J}}{2 \pi i} \oint_{\Gamma_{J}} \frac{d y_{J}}{\bar{y}_{J}-z_{I}}$,
Combining the basic part in Eq. (45) with Eq. (A5) $)_{2}$ and making use of Eq. (71), we finally have

$$
\begin{align*}
f_{I}^{\prime}\left(z_{I}\right) & =f_{I}^{\infty \prime}\left(z_{l}\right)+f_{I}^{b \prime}\left(z_{l}\right) \\
& =c_{I} \chi^{\omega}+d_{l} g\left(p_{I} ; z_{I}\right)+\sum_{K_{J} J} B_{I K}^{-1} \bar{B}_{K J} \bar{d}_{J} g\left(\bar{p}_{J} ; z_{I}\right) . \tag{A6}
\end{align*}
$$

## Appendix B. Multiferroic material properties in full, half, and bimaterial planes

## Table B1

Material coefficients for the $\mathrm{BaTiO}_{3}-\mathrm{CoFe}_{2} \mathrm{O}_{4}$ multiferroic composite, $50 \%$ of $\mathrm{BaTiO}_{3}$ and $50 \%$ of $\mathrm{CoFe}_{2} \mathrm{O}_{4}$ (Xue et al., 2011). In the Property column, the units are: elastic constants $C_{i j}$ in $10^{9} \mathrm{~N} / \mathrm{m}^{2}$, piezoelectric constants $e_{i j}$ in $\mathrm{C} / \mathrm{m}^{2}$, piezomagnetic constants $q_{i j}$ in $\mathrm{N} / \mathrm{A} \mathrm{m}$, dielectric constants $\kappa_{i j}$ in $10^{-9} \mathrm{C}^{2} / \mathrm{N} \mathrm{m}^{2}$, magnetic constants $\mu_{i j}$ in $10^{-4} \mathrm{~N} \mathrm{~s}^{2} / \mathrm{C}^{2}$. In the rescaled column, the dimensions of the elastic stress, electric displacement, and magnetic induction fields are, respectively, in $10^{11} \mathrm{~Pa}, 10 \mathrm{C} \mathrm{m}^{-2}$, and $10^{3} \mathrm{~Wb} \mathrm{~m}^{-2}$, and the electric and magnetic fields are in $10^{-10} \mathrm{~V} \mathrm{~m}^{-1}$ and $10^{-8} \mathrm{~A} \mathrm{~m}^{-1}$.

|  | Property | Rescaled |  | Property | Rescaled |
| :--- | :--- | :---: | :--- | :--- | :---: |
| $C_{11}$ | 225 | 2.25 | $q_{31}=q_{32}$ | 290.2 | 0.2902 |
| $C_{12}$ | 125 | 1.25 | $q_{33}$ | 350 | 0.35 |
| $C_{13}$ | 124 | 1.24 | $q_{15}=q_{24}$ | 275 | 0.275 |
| $C_{33}$ | 216 | 2.16 | $\kappa_{11}=\kappa_{22}$ | 5.64 | 5.64 |
| $C_{44}$ | 44 | 0.44 | $\kappa_{33}$ | 6.35 | 6.35 |
| $e_{31}=e_{32}$ | -2.2 | -0.22 | $\mu_{11}=\mu_{22}$ | 2.97 | 29.7 |
| $e_{33}$ | 9.3 | 0.93 | $\mu_{33}$ | 0.835 | 8.35 |
| $e_{15}=e_{24}$ | 5.8 | 0.58 |  |  |  |

Table B2
The rescaled extended material property matrix in the upper half plane of the bimaterial case (the dimensions of the elastic stress, electric displacement, and magnetic induction fields are, respectively, in $10^{11} \mathrm{~Pa}, 10 \mathrm{C} \mathrm{m}^{-2}$, and $10^{3} \mathrm{~Wb} \mathrm{~m}^{-2}$, and the electric and magnetic fields are in $10^{-10} \mathrm{~V} \mathrm{~m}^{-1}$ and $10^{-8} \mathrm{~A} \mathrm{~m}^{-1}$ ).
$\left(\begin{array}{cccccccccccc}1.66 & 0.77 & 0.78 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0 & 0 & 0.5803 \\ 0.77 & 1.66 & 0.78 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0 & 0 & 0.5803 \\ 0.78 & 0.78 & 1.62 & 0 & 0 & 0 & 0 & 0 & 1.86 & 0 & 0 & 0.6997 \\ 0 & 0 & 0 & 0.43 & 0 & 0 & 0 & 1.16 & 0 & 0 & 0.55 & 0 \\ 0 & 0 & 0 & 0 & 0.43 & 0 & 1.16 & 0 & 0 & 0.55 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.445 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.16 & 0 & -11.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.16 & 0 & 0 & 0 & -11.2 & 0 & 0 & 0 & 0 \\ -0.44 & -0.44 & 1.86 & 0 & 0 & 0 & 0 & 0 & -12.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.55 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.55 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0.5803 & 0.5803 & 0.6997 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\end{array}\right)$

Table B3
The rescaled extended material property matrix in the lower half plane of the bimaterial case (The dimensions of the elastic stress, electric displacement, and magnetic induction fields are, respectively, in $10^{11} \mathrm{~Pa}, 10 \mathrm{C} \mathrm{m}^{-2}$, and $10^{3} \mathrm{~Wb} \mathrm{~m}^{-2}$, and the electric and magnetic fields are in $10^{-10} \mathrm{~V} \mathrm{~m}^{-1}$ and $10^{-8} \mathrm{~A} \mathrm{~m}^{-1}$ ).

| ( 0.8674 | -0.0825 | 0.2715 | -0.0366 | 0 | 0 | $1.71 \times 10^{-2}$ | 0 | 0 | 0 | 0 | 0.5803 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| -0.0825 | 1.2977 | -0.0742 | 0.057 | 0 | 0 | $-1.52 \times 10^{-2}$ | 0 | 0 | 0 | 0 | 0.5803 |
| 0.2715 | -0.0742 | 1.0283 | 0.0992 | 0 | 0 | $-1.87 \times 10^{-3}$ | 0 | 0 | 0 | 0 | 0.6997 |
| -0.0366 | 0.057 | 0.0992 | 0.3881 | 0 | 0 | $6.7 \times 10^{-3}$ | 0 | 0 | 0 | 0.55 | 0 |
| 0 | 0 | 0 | 0 | 0.6881 | 0.0253 | 0 | $1.08 \times 10^{-2}$ | $-7.61 \times 10^{-3}$ | 0.55 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0.0253 | 0.2901 | 0 | $-9.5 \times 10^{-3}$ | $6.7 \times 10^{-3}$ | 0 | 0 | 0 |
| $1.71 \times 10^{-2}$ | $-1.52 \times 10^{-2}$ | $-1.87 \times 10^{-3}$ | $6.7 \times 10^{-3}$ | 0 | 0 | $-3.921 \times 10^{-2}$ | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | $1.08 \times 10^{-2}$ | $-9.5 \times 10^{-3}$ | 0 | $-3.962 \times 10^{-2}$ | $-8.6 \times 10^{-4}$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | $-7.61 \times 10^{-3}$ | $6.7 \times 10^{-3}$ | 0 | $-8.6 \times 10^{-4}$ | $-4.042 \times 10^{-2}$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0.55 | 0 | 0 | 0 | 0 | -0.5 | 0 | 0 |
| 0 | 0 | 0 | 0.55 | 0 | 0 | 0 | 0 | 0 | 0 | -0.5 | 0 |
| ( 0.5803 | 0.5803 | 0.6997 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |

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