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# Circular loadings on the surface of an anisotropic and magnetoelectroelastic half-space 

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#### Abstract

In this paper, we derive the analytical solutions for two types of surface loadings over an anisotropic magnetoelectroelastic half-space: a uniform and an indentation-type load. Our solutions in terms of the simple line integral over $[0, \pi]$ contain various decoupled materials and materials with high symmetry as special cases. Furthermore, the surface results of the solutions of the half-space are also provided. It is shown that on the surface: (1) for the uniform loading, the integrands for the extended displacements are continuous whilst those for the extended stresses have two weak singularities of order $1 / r^{1 / 2}$ along the integral interval $[0, \pi]$, which are integrable in the sense of Cauchy principal value and (2) for the indentation-type loading, the order of the singularity in the extended stress is exactly Cauchy type of $1 / r$, which can also be integrated via the Cauchy principal value. For a general anisotropic magnetoelectroelastic half-space, numerical examples in the vertical plane are presented for a uniform horizontal/vertical load and a vertical indentation-type load. The physical quantities presented are the magnitude of the elastic displacement vector, the electric and magnetic potentials, the hydrostatic and effective stresses, and the magnitudes of the electric and magnetic field vectors. These numerical results not only show some distinguished features associated with the loadings, but also can serve as benchmarks for future numerical endeavors in this field.


(Some figures may appear in colour only in the online journal)

## 1. Introduction

Solutions to the problem of surface loading over an elastic half-space are important in various engineering and science fields. Since the exact closed-form solutions of a vertical and horizontal point-force loading on the elastic isotropic half-space by Boussinesq and Cerruti (see Love 1944), a variety of extensions have been pursued. On the one hand, these point-force solutions can be applied to find
more complicated surface loading distributions by simple integration where solutions to the uniform load within a circle and even a triangle can be exactly found (i.e. Li and Berger 2001). On the other hand, the half-space materials were extended to the case of transverse isotropy with different loadings and even with material layering (see Wang et al 2005, Hanson 1999, Ding et al 2006, Yue et al 2005).

Since the work by Willis $(1966,1967)$ and Barnett and Lothe (1975), surface loading over a general anisotropic
elastic half-space has received a great deal of attention in the mechanics community due to its connection to indentation (Yu 2001) and inversion of material properties of solids. Representative works in this new direction are those by Vlassak and Nix (1993, 1994), Gao and Pharr (2007), Wang et al (2008), Chen et al (2010), Chu et al (2011).

The magnetoelectroelastic (MEE) material belongs to the category of multi-phase materials where different phases within the materials can interact with each other for the best output (Nan et al 2008). It is particularly important for the energy conversion among the mechanical, electric and magnetic ones, and thus it has potential application in energy harvesting using this type of material (e.g. Pan and Wang 2009, Wang et al 2010).

This paper is organized as follows: in section 2, we present the basic equations and describe the problem. In section 3, we derive the solution in the Fourier-transformed domain. The physical-domain solutions are given in section 4. Numerical results are presented in section 5, and conclusions are given in section 6 .

## 2. Basic equations and problem description

Under the static deformation, the governing equations for a linear, anisotropic MEE solid can be summarized in terms of the extended notations below (Pan 2002).

The equilibrium equations without internal source can be recast into

$$
\begin{equation*}
\sigma_{i J, i}=0 \tag{1}
\end{equation*}
$$

In this and other equations, a subscript comma denotes the partial differentiation and a repeated lower-case (uppercase) index takes the summation from 1 to 3 (5). Also, in equation (1) the extended stresses are defined as

$$
\sigma_{i J}= \begin{cases}\sigma_{i j} & J=j=1,2,3  \tag{2}\\ D_{i} & J=4 \\ B_{i} & J=5\end{cases}
$$

with $\sigma_{i j}, D_{i}$ and $B_{i}$ being, respectively, the stress, electric displacement and magnetic induction.

The individual constitutive relation as well as the extended one can be written as

$$
\begin{gather*}
\sigma_{i j}=c_{i j l m} \gamma_{l m}-e_{k i j} E_{k}-q_{k i j} H_{k} \\
D_{i}=e_{i j k} \gamma_{j k}+\varepsilon_{i j} E_{j}+\alpha_{i j} H_{j}  \tag{3a}\\
B_{i}=q_{i j k} \gamma_{j k}+\alpha_{j i} E_{j}+\mu_{i j} H_{j} \\
\sigma_{i J}=c_{i J K l} \gamma_{K l} \tag{3b}
\end{gather*}
$$

where $c_{i j l m}, e_{i j k}, q_{i j k}$ and $\alpha_{i j}$ are, respectively, the elastic, piezoelectric, piezomagnetic and magnetoelectric coefficients, and $\varepsilon_{i j}$ and $\mu_{i j}$ are, respectively, the dielectric permittivities and magnetic permeabilities. It should be noted that, in general, the magnetoelectric coefficients $\alpha_{i j}$ are not symmetric. The extended strains in equation (3b) are defined as

$$
\gamma_{I j}= \begin{cases}\gamma_{i j} \equiv \frac{1}{2}\left(u_{i, j}+u_{j, i}\right) & I=i=1,2,3 \\ -E_{j} \equiv \phi_{, j} & I=4 \\ -H_{j} \equiv \psi_{, j} & I=5\end{cases}
$$



Figure 1. Sketch of an anisotropic MEE half-space subjected to uniform surface loads within a circular area. Two types of loadings are considered: uniform and indentation-type loads.
with $\gamma_{i j}, E_{i}$ and $H_{i}$ being, respectively, the elastic strain, electric field and magnetic field, and $u_{i}, \phi$ and $\psi$ being, respectively, the elastic displacement, electric potential and magnetic potential.

Comparing equations (3b)-(3a), we find that the extended material coefficients in equation (3b) have the following expression:

$$
c_{i J K l}= \begin{cases}c_{i j k l} & J, K=j, k=1,2,3  \tag{5}\\ e_{l i j} & J=j=1,2,3 ; \quad K=4 \\ e_{i k l} & J=4 ; \quad K=k=1,2,3 \\ q_{l i j} & J=j=1,2,3 ; \quad K=5 \\ q_{i k l} & J=5 ; \quad K=k=1,2,3 \\ -\alpha_{i l} / \alpha_{l i} & J=4, \quad K=5 / K=4, \quad J=5 \\ -\varepsilon_{i l} & J, K=4 \\ -\mu_{i l} & J, K=5 .\end{cases}
$$

We also introduce the extended displacement as

$$
u_{I}= \begin{cases}u_{i} & I=i=1,2,3  \tag{6}\\ \phi & I=4 \\ \psi & I=5\end{cases}
$$

and the extended traction as.

$$
t_{J}=\sigma_{i J} n_{i}= \begin{cases}\sigma_{i j} n_{i} & J=j=1,2,3  \tag{7}\\ D_{i} n_{i} & J=4 \\ B_{i} n_{i} & J=5\end{cases}
$$

where $n_{i}$ is the normal vector of a prescribed plane.
We assume that the anisotropic MEE half-space is subjected to a general traction on the surface. The loading is applied over a circular area with a radius $r=R$. The center of the circular area is denoted as $O$. The Cartesian coordinate system is introduced to describe the problem. We let the $x_{1} O x_{2}$ plane be the surface of the half-space. The half-space occupies the domain $x_{3} \geq 0$, see figure 1 . We consider two types of surface loadings as listed below. That is, on the surface $x_{3}=0$,
we have (for $q=0$ and $1 / 2$ )

$$
t_{J}= \begin{cases}t_{J}^{0} /\left[1-(r / R)^{2}\right]^{q} ; & r \leq R  \tag{8}\\ 0 ; & r>R .\end{cases}
$$

It is clear that $q=0$ corresponds to a uniform load and the case $q=1 / 2$ corresponds to the common indentation load when the traction has only a vertical component (Gao and Pharr 2007).

## 3. Fundamental solutions in the transformed domain

We first briefly review the Fourier-domain solutions as they are needed in the subsequent analysis. While the process is straightforward, more details can be found in Ting (1996) for the purely elastic solid and in Pan (2002) for the general magnetoelectroelastic solid.

We define the following two-dimensional (2D) Fourier transforms:

$$
\begin{equation*}
\tilde{f}\left(k_{1}, k_{2}, x_{3}\right)=\iint f\left(x_{1}, x_{2}, x_{3}\right) \mathrm{e}^{\mathrm{i}\left(k_{1} x_{1}+k_{2} x_{2}\right)} \mathrm{d} x_{1} \mathrm{~d} x_{2} \tag{9}
\end{equation*}
$$

where $\mathrm{i}=\sqrt{-1}$ is the imaginary unit. Therefore in the transformed domain, equation (1) in terms of the extended displacement becomes

$$
\begin{align*}
& C_{\alpha I K \beta} k_{\alpha} k_{\beta} \tilde{u}_{K}+\mathrm{i}\left(C_{\alpha I K 3}+C_{3 I K \alpha}\right) k_{\alpha} \tilde{u}_{K, 3} \\
& \quad-C_{3 I K 3} \tilde{u}_{K, 33}=0 \tag{10}
\end{align*}
$$

where $\alpha, \beta=1,2$. We now introduce the polar coordinates $(\eta, \theta)$ which are related to the Fourier variables $\left(k_{1}, k_{2}\right)$ as

$$
\begin{gather*}
{\left[\begin{array}{l}
k_{1} \\
k_{2} \\
0
\end{array}\right]=\eta \boldsymbol{m}, \quad \boldsymbol{m}=\left[\begin{array}{l}
m_{1} \\
m_{2} \\
0
\end{array}\right]=\left[\begin{array}{l}
\cos \theta \\
\sin \theta \\
0
\end{array}\right],}  \tag{11}\\
\boldsymbol{n}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]
\end{gather*}
$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ are the two normal vectors which enter into the Stroh formalism. The general solution of equation (10) can then be expressed as

$$
\begin{equation*}
\tilde{\boldsymbol{u}}\left(k_{1}, k_{2}, x_{3}\right)=\boldsymbol{a} \mathrm{e}^{-\mathrm{i} p \eta z} \tag{12}
\end{equation*}
$$

with $p$ and $\boldsymbol{a}$ satisfying the following extended Stroh eigenrelation:

$$
\begin{equation*}
\left[\boldsymbol{Q}+p\left(\boldsymbol{R}+\boldsymbol{R}^{\mathrm{t}}\right)+p^{2} \boldsymbol{T}\right] \boldsymbol{a}=0 \tag{13}
\end{equation*}
$$

where the superscript $t$ denotes the matrix transpose and

$$
\begin{gather*}
Q_{I K}=C_{j I K s} m_{j} m_{s}, \quad R_{I K}=C_{j I K s} m_{j} n_{s},  \tag{14}\\
T_{I K}=C_{j I K s} n_{j} n_{s} .
\end{gather*}
$$

We introduce (Ting 1996)

$$
\begin{equation*}
\boldsymbol{b}=\left(\boldsymbol{R}^{\mathrm{t}}+p \boldsymbol{T}\right) \boldsymbol{a}=-\frac{1}{p}(\boldsymbol{Q}+p \boldsymbol{R}) \boldsymbol{a} \tag{15}
\end{equation*}
$$

where the new Stroh vector $\boldsymbol{b}$ is actually related to the extended traction (as in equation (20) below). Then the quadratic Stroh eigenrelation (13) can be changed into the following linear Stroh eigenrelation:

$$
\left[\begin{array}{ll}
\boldsymbol{N}_{1} & \boldsymbol{N}_{2}  \tag{16}\\
\boldsymbol{N}_{3} & \boldsymbol{N}_{1}^{\mathrm{T}}
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{a} \\
\boldsymbol{b}
\end{array}\right]=p\left[\begin{array}{l}
\boldsymbol{a} \\
\boldsymbol{b}
\end{array}\right]
$$

where

$$
\begin{gather*}
\boldsymbol{N}_{1}=-\boldsymbol{T}^{-1} \boldsymbol{R}^{\mathrm{t}}, \quad \boldsymbol{N}_{2}=\boldsymbol{T}^{-1}, \\
\boldsymbol{N}_{3}=\boldsymbol{R} \boldsymbol{T}^{-1} \boldsymbol{R}^{\mathrm{t}}-\boldsymbol{Q} . \tag{17}
\end{gather*}
$$

Equation (16) is the extended MEE Stroh eigenrelation in the oblique plane spanned by $\boldsymbol{m}$ and $\boldsymbol{n}$ defined in equations (11). We point out that the eigenvalues of equation (16) are either complex or purely imaginary (Ting 1996). Once the eigenvalue problem is solved, the extended displacements in the Fourier-transformed domain are then obtained from equation (12).

In order to find the extended stresses in the Fouriertransformed domain, we start with the physical-domain relation. In the physical domain, the extended traction vector $t$ on the $x_{3}=$ constant plane and the extended in-plane stress vector $\boldsymbol{s}$ are related to the extended displacements as

$$
\begin{align*}
\boldsymbol{t} & =\left(\sigma_{31}, \sigma_{32}, \sigma_{33}, D_{3}, B_{3}\right)^{\mathrm{t}} \\
& =\left(C_{31 K l} u_{K, l}, C_{32 K l} u_{K, l}, C_{33 K l} u_{K, l}, C_{34 K l} u_{K, l},\right. \\
& \left.C_{35 K l} u_{K, l}\right)^{\mathrm{t}} \tag{18}
\end{align*}
$$

$\boldsymbol{s} \equiv\left(\sigma_{11}, \sigma_{12}, \sigma_{22}, D_{1}, D_{2}, B_{1}, B_{2}\right)^{\mathrm{t}}$
$=\left(C_{11 K l} u_{K, l}, C_{12 K l} u_{K, l}, C_{22 K l} u_{K, l}, C_{14 K l} u_{K, l}\right.$,

$$
\begin{equation*}
\left.C_{24 K l} u_{K, l}, C_{15 K l} u_{K, l}, C_{25 K l} u_{K, l}\right)^{\mathrm{t}} . \tag{19}
\end{equation*}
$$

Taking the Fourier transform, we then find that the transformed extended traction and in-plane stress vectors can be expressed as

$$
\begin{align*}
& \tilde{\boldsymbol{t}}=-\mathrm{i} \eta \boldsymbol{b} \mathrm{e}^{-\mathrm{i} p \eta x_{3}}  \tag{20}\\
& \tilde{\boldsymbol{s}}=-\mathrm{i} \eta \boldsymbol{c} \mathrm{e}^{-\mathrm{i} p \eta x_{3}} \tag{21}
\end{align*}
$$

with

$$
\begin{equation*}
c=H a \tag{22}
\end{equation*}
$$

where the matrix $\boldsymbol{H}$ is defined by

$$
\left.\begin{array}{c}
C_{113 \alpha} m_{\alpha}+p C_{1133} C_{114 \alpha} m_{\alpha}+p C_{1143} \\
C_{123 \alpha} m_{\alpha}+p C_{1233} C_{124 \alpha} m_{\alpha}+p C_{1243} \\
C_{223 \alpha} m_{\alpha}+p C_{2233} C_{224 \alpha} m_{\alpha}+p C_{2243} \\
C_{143 \alpha} m_{\alpha}+p C_{1433} C_{144 \alpha} m_{\alpha}+p C_{1443} \\
C_{243 \alpha} m_{\alpha}+p C_{2433} C_{244 \alpha} m_{\alpha}+p C_{2443} \\
C_{153 \alpha} m_{\alpha}+p C_{1533} C_{154 \alpha} m_{\alpha}+p C_{1543} \\
C_{253 \alpha} m_{\alpha}+p C_{2533} C_{254 \alpha} m_{\alpha}+p C_{2543} \\
C_{115 \alpha} m_{\alpha}+p C_{1153} \\
C_{125 \alpha} m_{\alpha}+p C_{1253} \\
C_{225 \alpha} m_{\alpha}+p C_{2253} \\
C_{145 \alpha} m_{\alpha}+p C_{1453} \\
C_{245 \alpha} m_{\alpha}+p C_{2453} \\
C_{155 \alpha} m_{\alpha}+p C_{1553} \\
C_{255 \alpha} m_{\alpha}+p C_{2553}
\end{array}\right]
$$

with $\alpha=1,2$.
Since the eigenvalues cannot be real (Ting 1996), we let $p_{I}, \boldsymbol{a}_{I}$, and $\boldsymbol{b}_{I}(I=1-5)$ be the eigenvalues and the associated eigenvectors, with the following arrangement:

$$
\begin{gather*}
\operatorname{Im}\left(p_{J}\right)>0, \quad p_{J+5}=\bar{p}_{J}, \quad \boldsymbol{a}_{J+5}=\overline{\boldsymbol{a}}_{J}, \\
\boldsymbol{b}_{J+5}=\overline{\boldsymbol{b}}_{J} \\
(J=1,2,3,4,5)  \tag{24}\\
\boldsymbol{A}=\left[\boldsymbol{a}_{1}, \boldsymbol{a}_{2}, \boldsymbol{a}_{3}, \boldsymbol{a}_{4}, \boldsymbol{a}_{5}\right], \quad \boldsymbol{B}=\left[\boldsymbol{b}_{1}, \boldsymbol{b}_{2}, \boldsymbol{b}_{3}, \boldsymbol{b}_{4}, \boldsymbol{b}_{5}\right] \\
\boldsymbol{C}=\left[\boldsymbol{c}_{1}, \boldsymbol{c}_{2}, \boldsymbol{c}_{3}, \boldsymbol{c}_{4}, \boldsymbol{c}_{5}, \boldsymbol{c}_{6}, \boldsymbol{c}_{7}\right] .
\end{gather*}
$$

In equations (24), Im stands for the imaginary part and the overbar denotes the complex conjugate. The general solutions in the transformed domain can be obtained by superposing the ten eigensolutions of equation (16), that is

$$
\begin{gather*}
\tilde{\boldsymbol{u}}\left(k_{1}, k_{2}, x_{3}\right)=\mathrm{i} \eta^{-1} \overline{\boldsymbol{A}}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} \eta x_{3}}\right\rangle \overline{\boldsymbol{q}}+\mathrm{i} \eta^{-1} \boldsymbol{A}\left\langle\mathrm{e}^{-\mathrm{i} p_{*} \eta x_{3}}\right\rangle \boldsymbol{q}^{\prime} \\
\tilde{\boldsymbol{t}}\left(k_{1}, k_{2}, x_{3}\right)=\overline{\boldsymbol{B}}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} \eta x_{3}}\right\rangle \overline{\boldsymbol{q}}+\boldsymbol{B}\left\langle\mathrm{e}^{-\mathrm{i} p_{*} \eta x_{3}}\right\rangle \boldsymbol{q}^{\prime}  \tag{25}\\
\tilde{\boldsymbol{s}}\left(k_{1}, k_{2}, x_{3}\right)=\overline{\boldsymbol{C}}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} \eta x_{3}}\right\rangle \overline{\boldsymbol{q}}+\boldsymbol{C}\left\langle\mathrm{e}^{-\mathrm{i} p_{*} * \eta x_{3}}\right\rangle \boldsymbol{q}^{\prime}
\end{gather*}
$$

where $\overline{\boldsymbol{q}}$ and $\boldsymbol{q}^{\prime}$ are two arbitrary complex vectors which can be determined by the boundary conditions on the surface and the radiation condition at infinity. Also in equations (25), the diagonal matrix is defined as

$$
\begin{align*}
& \left\langle\mathrm{e}^{-\mathrm{i} p_{*} \eta x_{3}}\right\rangle \\
& \quad=\operatorname{diag}\left[\mathrm{e}^{-\mathrm{i} p_{1} \eta x_{3}}, \mathrm{e}^{-\mathrm{i} p_{2} \eta x_{3}}, \mathrm{e}^{-\mathrm{i} p_{3} \eta x_{3}}, \mathrm{e}^{-\mathrm{i} p_{4} \eta x_{3}}, \mathrm{e}^{-\mathrm{i} p_{5} \eta x_{3}}\right] . \tag{26}
\end{align*}
$$

It is noteworthy that, besides their obvious dependence on the material properties, the vectors $\overline{\boldsymbol{q}}$ and $\boldsymbol{q}^{\prime}$, and the Stroh eigenvalues $p_{J}$ and matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are also functions of the unit vector $\boldsymbol{m}$. It is further noted that, due to the special arrangement on the eigenvalues $p_{J}$ in equations (24), the general solutions associated with the first terms on the right-hand side of (25) are finite in the half-space $x_{3}>0$ and the second terms are finite in the half-space $x_{3}<0$. Thus, in the present half-space occupying $x_{3}>0$, only the first terms on the right-hand side of (25) should be included. Specifically,
we have

$$
\begin{gather*}
\tilde{\boldsymbol{u}}\left(k_{1}, k_{2}, x_{3}\right)=\mathrm{i} \eta^{-1} \overline{\boldsymbol{A}}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} * \eta x_{3}}\right\rangle \overline{\boldsymbol{q}} \\
\tilde{\boldsymbol{t}}\left(k_{1}, k_{2}, x_{3}\right)=\overline{\boldsymbol{B}}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} * x_{3}}\right\rangle \overline{\boldsymbol{q}}  \tag{27}\\
\tilde{\boldsymbol{s}}\left(k_{1}, k_{2}, x_{3}\right)=\overline{\boldsymbol{C}}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} * \eta x_{3}}\right\rangle \overline{\boldsymbol{q}} .
\end{gather*}
$$

In the following, the boundary conditions are introduced to determine the unknown vector $\overline{\boldsymbol{q}}$. On the surface $x_{3}=0$, the expression for the transformed extended vector becomes

$$
\begin{equation*}
\tilde{\boldsymbol{t}}\left(k_{1}, k_{2}, 0\right)=\overline{\boldsymbol{B}} \overline{\boldsymbol{q}} \tag{28}
\end{equation*}
$$

Thus, the unknown vector can be solved as

$$
\begin{equation*}
\overline{\boldsymbol{q}}=(\overline{\boldsymbol{B}})^{-1} \tilde{\boldsymbol{t}}\left(k_{1}, k_{2}, 0\right) . \tag{29}
\end{equation*}
$$

We assume that the region of the traction loadings is a circle. To handle the double infinite integral, the polar coordinate transform will be employed so that the infinite integral with respect to the radial variable can be performed exactly and the solution in the physical domain can be obtained in terms of the regular line integrals over $[0, \pi]$. Below, we first present the general relation between the Fourier and Hankel transforms, and then, for the two loading cases, we derive the exact closed-form expressions in the transformed domain.

### 3.1. A general relation between the Fourier and Hankel transforms

For a point $(r, \psi)$ in the polar coordinate system within the horizontal $\left(x_{1}, x_{2}\right)$ plane, it can be transformed to the Cartesian coordinates as

$$
\begin{equation*}
x_{1}=r \cos \psi, \quad x_{2}=r \sin \psi \tag{30}
\end{equation*}
$$

Also from equations (11), we have

$$
\begin{equation*}
k_{1}=\eta \cos \theta, \quad k_{2}=\eta \sin \theta \tag{31}
\end{equation*}
$$

We now look at a 2 D function $f\left(x_{1}, x_{2}\right)$ in the horizontal ( $x_{1}, x_{2}$ ) plane. Utilizing equations (30) and (31), we find that its 2D Fourier transform can be written as

$$
\begin{equation*}
\tilde{f}\left(k_{1}, k_{2}\right)=\int_{0}^{\infty} r \mathrm{~d} r \int_{0}^{2 \pi} f(r, \psi) \mathrm{e}^{\mathrm{i} \eta r \cos (\psi-\theta)} \mathrm{d} \psi \tag{32}
\end{equation*}
$$

We further assume that this function $f\left(x_{1}, x_{2}\right)$ can be expanded in terms of a multipole series as

$$
\begin{equation*}
f\left(x_{1}, x_{2}\right) \equiv f(r, \psi)=\sum_{m=-\infty}^{\infty} f_{m}(r) \mathrm{e}^{\mathrm{i} m \psi} \tag{33}
\end{equation*}
$$

Let $\phi=\psi-\theta$, we then have

$$
\begin{align*}
& \tilde{f}\left(k_{1}, k_{2}\right)=\sum_{m} \int_{0}^{\infty} r \mathrm{~d} r \int_{0}^{2 \pi} f_{m}(r) \mathrm{e}^{\mathrm{i} m \psi} \mathrm{e}^{\mathrm{i} \eta r \cos (\psi-\theta)} \mathrm{d} \psi \\
& \quad=\sum_{m} \mathrm{e}^{\mathrm{i} m \theta} \int_{0}^{\infty} r \mathrm{~d} r f_{m}(r) \int_{0}^{2 \pi} \mathrm{e}^{\mathrm{i} m \phi} \mathrm{e}^{\mathrm{i} \eta r \cos \phi} \mathrm{~d} \phi \\
& \quad=2 \pi \sum_{m} \mathrm{i}^{m} \mathrm{e}^{\mathrm{i} m \theta} \int_{0}^{\infty} f_{m}(r) J_{m}(\eta r) r \mathrm{~d} r \tag{34}
\end{align*}
$$

where $J_{m}(\eta r)$ is the first-kind Bessel function of order $m$. Equation ((34)) provides the 2D Fourier transform result when the function $f\left(x_{1}, x_{2}\right)$ is expanded as a multipole series equation (33).

### 3.2. Double Fourier transforms of the two types of surface loadings

Applying equation (34) to the uniform and indentation loadings, we then have the following exact closed-form expressions for the extended traction vectors applied on the surface.
(1) When $q=0$ (uniform loading case)

$$
\begin{align*}
& \tilde{\boldsymbol{t}}\left(k_{1}, k_{2}, 0\right)=2 \pi \int_{0}^{\infty} t(r) J_{0}(\eta r) r \mathrm{~d} r \\
& \quad=2 \pi t^{0} \int_{0}^{R} J_{0}(\eta r) r \mathrm{~d} r=2 \pi R \frac{J_{1}(\eta R)}{\eta} \boldsymbol{t}^{0} . \tag{35}
\end{align*}
$$

(2) When $q=1 / 2$ (indentation loading case)

$$
\begin{align*}
& \tilde{\boldsymbol{t}}\left(k_{1}, k_{2}, 0\right)=2 \pi \int_{0}^{\infty} \boldsymbol{t}(r) J_{0}(\eta r) r \mathrm{~d} r \\
& \quad=2 \pi t^{0} \int_{0}^{R} \frac{J_{0}(\eta r) r}{\left[1-(r / R)^{2}\right]^{1 / 2}} \mathrm{~d} r=2 \pi R \frac{\sin (\eta R)}{\eta} \boldsymbol{t}^{0} \tag{36}
\end{align*}
$$

## 4. Fundamental solutions in the physical domain

Having derived the general solution in the transformed domain, we can apply the inverse Fourier transform to equation (27) to obtain the solutions in the physical domain. We discuss the two types of loadings separately.

### 4.1. Uniform traction

For a uniform traction $t^{0}$ applied within the circle $r=R$ on the surface of the half-space, the solution in the physical domain can be expressed as

$$
\begin{align*}
& {\left[\begin{array}{l}
\boldsymbol{u}\left(x_{1}, x_{2}, x_{3}\right) \\
\boldsymbol{t}\left(x_{1}, x_{2}, x_{3}\right) \\
\boldsymbol{s}\left(x_{1}, x_{2}, x_{3}\right)
\end{array}\right]=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \theta \int_{0}^{\infty}\left[\begin{array}{l}
\mathrm{i} \eta^{-1} \overline{\boldsymbol{A}} \\
\overline{\boldsymbol{B}} \\
\overline{\boldsymbol{C}}
\end{array}\right]\left\langle\mathrm{e}^{\left.-\mathrm{i} \bar{p}_{*} \eta x_{3}\right\rangle}\right.} \\
& \quad \times J_{1}(\eta R) \mathrm{e}^{-\mathrm{i} \eta\left(x_{1} \cos \theta+x_{2} \sin \theta\right)} R(\overline{\boldsymbol{B}})^{-1} \boldsymbol{t}^{0} \mathrm{~d} \eta \tag{37}
\end{align*}
$$

Or, equivalently, equation (37) can be rewritten as

$$
\begin{align*}
& \boldsymbol{u}\left(x_{1}, x_{2}, x_{3}\right)=\frac{\mathrm{i}}{\pi} \int_{0}^{\pi} \overline{\boldsymbol{A}}\left\langle Q_{1}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle(\overline{\boldsymbol{B}})^{-1} \boldsymbol{t}^{0} \mathrm{~d} \theta \\
& \boldsymbol{t}\left(x_{1}, x_{2}, x_{3}\right)=\frac{1}{\pi} \int_{0}^{\pi} \overline{\boldsymbol{B}}\left\langle Q_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle(\overline{\boldsymbol{B}})^{-1} \boldsymbol{t}^{0} \mathrm{~d} \theta  \tag{38}\\
& \boldsymbol{s}\left(x_{1}, x_{2}, x_{3}\right)=\frac{1}{\pi} \int_{0}^{\pi} \overline{\boldsymbol{C}}\left\langle Q_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle(\overline{\boldsymbol{B}})^{-1} \boldsymbol{t}^{0} \mathrm{~d} \theta
\end{align*}
$$

where the periodic condition has been used to reduce the integral in the interval from $[0,2 \pi]$ to $[0, \pi]$, and the two
diagonal matrices are defined as

$$
\begin{align*}
& \left\langle Q_{1}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle=\int_{0}^{\infty}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} \eta x_{3}}\right\rangle \mathrm{e}^{-\mathrm{i} \eta\left(x_{1} \cos \theta+x_{2} \sin \theta\right)} \\
& \quad \times R J_{1}(\eta R) \eta^{-1} \mathrm{~d} \eta \\
& \left\langle Q_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle=\int_{0}^{\infty}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} \neq \eta x_{3}}\right\rangle \mathrm{e}^{-\mathrm{i} \eta\left(x_{1} \cos \theta+x_{2} \sin \theta\right)}  \tag{39}\\
& \quad \times R J_{1}(\eta R) \mathrm{d} \eta .
\end{align*}
$$

Using the following results (Watson 1996)

$$
\begin{align*}
\int_{0}^{\infty} & \mathrm{e}^{-a t} J_{1}(b t) t^{-1} \mathrm{~d} t=\frac{\sqrt{a^{2}+b^{2}}-a}{b} \\
& {[\operatorname{Re}(a)>0, \operatorname{Im}(b)=0] } \tag{40}
\end{align*}
$$

$\int_{0}^{\infty} \mathrm{e}^{-a t} J_{1}(b t) \mathrm{d} t=\frac{\sqrt{a^{2}+b^{2}}-a}{b \sqrt{a^{2}+b^{2}}}$

$$
[\operatorname{Re}(a)>0, \operatorname{Im}(b)=0]
$$

exact closed-form expressions of the diagonal matrices in equation (39) can be obtained as

$$
\begin{align*}
& \left\langle Q_{1}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle=\left\langle\sqrt{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta+x_{3} \bar{p}_{*}\right)^{2}}\right. \\
& \left.\quad-\mathrm{i}\left(x_{1} \cos \theta+x_{2} \sin \theta+x_{3} \bar{p}_{*}\right)\right\rangle \\
& \left\langle Q_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle=\left\langle\left(\sqrt{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta+x_{3} \bar{p}_{*}\right)^{2}}\right.\right. \\
& \left.\quad-\mathrm{i}\left(x_{1} \cos \theta+x_{2} \sin \theta+x_{3} \bar{p}_{*}\right)\right) \\
& \left.\quad \times\left(\sqrt{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta+x_{3} \bar{p}_{*}\right)^{2}}\right)^{-1}\right\rangle \tag{41}
\end{align*}
$$

On the surface $x_{3}=0$ of the half-space, these two diagonal matrices are matrices with a proportional factor, as given below:

$$
\begin{align*}
& \left\langle Q_{1}\left(x_{1}, x_{2}\right)\right\rangle=\left[\sqrt{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta\right)^{2}}\right. \\
& \left.\quad-\mathrm{i}\left(x_{1} \cos \theta+x_{2} \sin \theta\right)\right] \boldsymbol{I} \\
& \left\langle Q_{2}\left(x_{1}, x_{2}\right)\right\rangle=\left[\left(\sqrt{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta\right)^{2}}\right.\right.  \tag{42}\\
& \left.\quad-\mathrm{i}\left(x_{1} \cos \theta+x_{2} \sin \theta\right)\right) \\
& \left.\quad \times\left(\sqrt{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta\right)^{2}}\right)^{-1}\right] \boldsymbol{I}
\end{align*}
$$

where $\boldsymbol{I}$ is the $5 \times 5$ identity matrix.
It is observed from equations (38) and (42) that, while the integrand for the extended displacements are regular, the one for the extended stresses on the surface has two weak and integrable singularities located at $R \pm\left(x_{1} \cos \theta+x_{2} \sin \theta\right)=0$.

### 4.2. Indentation-type traction

Similarly, for the indentation-type traction within the circle $r=R$ on the surface of the half-space, the solution in the physical domain is expressed as

$$
\begin{align*}
& {\left[\begin{array}{l}
\boldsymbol{u}\left(x_{1}, x_{2}, x_{3}\right) \\
\boldsymbol{t}\left(x_{1}, x_{2}, x_{3}\right) \\
\boldsymbol{s}\left(x_{1}, x_{2}, x_{3}\right)
\end{array}\right]=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \theta \int_{0}^{\infty}\left[\begin{array}{l}
\mathrm{i} \eta^{-1} \overline{\boldsymbol{A}} \\
\overline{\boldsymbol{B}} \\
\overline{\boldsymbol{C}}
\end{array}\right]} \\
& \quad \times\left\langle\mathrm{e}^{\left.-\mathrm{i} \bar{p}_{*} \eta x_{3}\right\rangle} \sin (\eta R) \mathrm{e}^{-\mathrm{i} \eta\left(x_{1} \cos \theta+x_{2} \sin \theta\right)} R(\overline{\boldsymbol{B}})^{-1} \boldsymbol{t}^{0} \mathrm{~d} \eta .\right. \tag{43}
\end{align*}
$$

Equation (43) can be recast as

$$
\begin{align*}
& \boldsymbol{u}\left(x_{1}, x_{2}, x_{3}\right)=\frac{\mathrm{i}}{\pi} \int_{0}^{\pi} \overline{\boldsymbol{A}}\left\langle P_{1}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle(\overline{\boldsymbol{B}})^{-1} \boldsymbol{t}^{0} \mathrm{~d} \theta \\
& \boldsymbol{t}\left(x_{1}, x_{2}, x_{3}\right)=\frac{1}{\pi} \int_{0}^{\pi} \overline{\boldsymbol{B}}\left\langle P_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle(\overline{\boldsymbol{B}})^{-1} t^{0} \mathrm{~d} \theta  \tag{44}\\
& \boldsymbol{s}\left(x_{1}, x_{2}, x_{3}\right)=\frac{1}{\pi} \int_{0}^{\pi} \overline{\boldsymbol{C}}\left\langle P_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle(\overline{\boldsymbol{B}})^{-1} t^{0} \mathrm{~d} \theta
\end{align*}
$$

where the two diagonal matrices are defined as

$$
\begin{align*}
& \left\langle P_{1}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle=\int_{0}^{\infty}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} \eta x_{3}}\right\rangle \\
& \quad \times \mathrm{e}^{-\mathrm{i} \eta\left(x_{1} \cos \theta+x_{2} \sin \theta\right)} R \sin (\eta R) \eta^{-1} \mathrm{~d} \eta \\
& \left\langle P_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle=\int_{0}^{\infty}\left\langle\mathrm{e}^{-\mathrm{i} \bar{p}_{*} \eta x_{3}}\right\rangle  \tag{45}\\
& \quad \times \mathrm{e}^{-\mathrm{i} \eta\left(x_{1} \cos \theta+x_{2} \sin \theta\right)} R \sin (\eta R) \mathrm{d} \eta
\end{align*}
$$

Noticing that (Watson 1996)

$$
\begin{gather*}
\int_{0}^{\infty} \mathrm{e}^{-a t} \sin (b t) t^{-1} \mathrm{~d} t=\operatorname{arc} \cot \left(\frac{a}{b}\right) \\
{[\operatorname{Re}(a)>0, \operatorname{Im}(b)=0]} \\
\int_{0}^{\infty} \mathrm{e}^{-a t} \sin (b t) \mathrm{d} t=\frac{b}{a^{2}+b^{2}}  \tag{46}\\
{[\operatorname{Re}(a)>0, \operatorname{Im}(b)=0]}
\end{gather*}
$$

equation (45) can be rewritten simply as

$$
\begin{align*}
\left\langle P_{1}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle & =\left\langle R \operatorname{arccot} \frac{\mathrm{i}\left(x_{1} \cos \theta+x_{2} \sin \theta+x_{3} \bar{p}_{*}\right)}{R}\right\rangle \\
\left\langle P_{2}\left(x_{1}, x_{2}, x_{3}\right)\right\rangle & =\left\langle\frac{R^{2}}{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta+x_{3} \bar{p}_{*}\right)^{2}}\right\rangle . \tag{47}
\end{align*}
$$

On the surface $x_{3}=0$, these two diagonal matrices are reduced to

$$
\begin{align*}
\left\langle P_{1}\left(x_{1}, x_{2}\right)\right\rangle & =\left[R \operatorname{arccot} \frac{\mathrm{i}\left(x_{1} \cos \theta+x_{2} \sin \theta\right)}{R}\right] \boldsymbol{I} \\
\left\langle P_{2}\left(x_{1}, x_{2}\right)\right\rangle & =\left[\frac{R^{2}}{R^{2}-\left(x_{1} \cos \theta+x_{2} \sin \theta\right)^{2}}\right] \boldsymbol{I} . \tag{48}
\end{align*}
$$

Similar to the uniform loading case, it is observed that the integrand for the extended displacements due to the indentation-type load is regular. However, the integrand associated with the extended stresses on the surface due to the indentation-type load has a high-order singularity, actually the Cauchy-type singularity, at $R \pm\left(x_{1} \cos \theta+x_{2} \sin \theta\right)=0$. Such singularities can be easily treated using various existing integral approaches based on the Cauchy principal value.

## 5. Numerical examples

We have first checked our formulation for the transversely isotropic $\mathrm{BaTiO}_{3}$ case with the poling axis in the vertical direction of the half-space and compared our numerical results to those in Chu et al (2011). It shows that our results are the same as those in Chu et al (2011) under a uniform vertical load within a circle over the half-space.


Figure 2. Geometric relation between the global $\left(x_{1}, x_{2}, x_{3}\right)$ and local $\left(y_{1}, y_{2}, y_{3}\right)$ coordinates.

We now use the $50 \%$ MEE transversely isotropic material data by Xue et al (2011) (see the appendix) and rotate this local coordinate system $\left(\left(y_{1} y_{2} y_{3}\right)\right.$ with $y_{3}$ axis parallel to the poling axis) with respect to the global ( $x_{1} x_{2} x_{3}$ ) coordinates as shown in figure 2 by the following transform:

$$
\left[\begin{array}{l}
x_{1}  \tag{49}\\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{lll}
-\sin \alpha \cos \beta & \cos \alpha & \sin \alpha \sin \beta \\
-\cos \alpha \cos \beta & -\sin \alpha & \cos \alpha \sin \beta \\
\sin \beta & 0 & \cos \beta
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3}
\end{array}\right] .
$$

The transform matrix between these two systems and the corresponding global material matrices in $\left(x_{1} x_{2} x_{3}\right)$ coordinates for $\alpha=60^{\circ}$ and $\beta=45^{\circ}$ degrees are given in the appendix. For easy presentation of the numerical results below, the global coordinates $\left(x_{1} x_{2} x_{3}\right)$ are denoted by $(x, y, z)$.

For loading within the circle of $r=R(=1 \mathrm{~m})$, numerical results are presented for the following examples: a uniform horizontal load in the $x$ direction ( $q=0$ and $t=(1,0,0,0,0) \mathrm{MPa})$, a vertical load $(q=0$ and $t=$ $(0,0,1,0,0) \mathrm{MPa})$ and an indentation-type vertical load ( $q=$ $1 / 2$ and $t_{3}^{0}=1 \mathrm{MPa}$ ). The following quantities are used to show the response: the magnitude of the whole elastic displacement $u$, the electric potential $\phi$ and magnetic potential $\psi$, the hydrostatic stress $\sigma_{\mathrm{h}}$, the effective stress $\sigma_{\mathrm{e}}$, the magnitude of the electric displacement $D$ and the magnitude of the magnetic induction $B$. The definitions for $u, \sigma_{\mathrm{h}}, \sigma_{\mathrm{e}}, D$ and $B$ are
$u=\sqrt{u_{x}^{2}+u_{y}^{2}+u_{z}^{2}} ; \quad \sigma_{\mathrm{h}}=\frac{\sigma_{x x}+\sigma_{y y}+\sigma_{z z}}{3} ;$
$\sigma_{\mathrm{e}}=\left(\frac{1}{2}\left[\left(\sigma_{x x}-\sigma_{y y}\right)^{2}+\left(\sigma_{y y}-\sigma_{z z}\right)^{2}+\left(\sigma_{z z}-\sigma_{x x}\right)^{2}\right]\right.$
$\left.+3\left(\sigma_{x y}^{2}+\sigma_{y z}^{2}+\sigma_{x z}^{2}\right)\right)^{1 / 2}$
$D=\sqrt{D_{x}^{2}+D_{y}^{2}+D_{z}^{2}} ; \quad B=\sqrt{B_{x}^{2}+B_{y}^{2}+B_{z}^{2}}$.
Under the uniform horizontal load in the $x$ direction within the circle of $r=R(R=1 \mathrm{~m})$, figures 3(a)-(c) show


Figure 3. Contours of the elastic displacement $u$ (in $\mu \mathrm{m}=10^{-6} \mathrm{~m}$ ) in (a), electric potential $\phi$ (in $\mathrm{kV}=10^{3} \mathrm{~V}$ ) in (b) and magnetic potential $\psi$ (in $\mathrm{V} \mathrm{s} \mathrm{m}^{-1}=A$ ) in (c), in the plane $y=0$, induced by a uniform horizontal load in the $x$ direction (with density equals $1 \mathrm{MPa}=10^{6} \mathrm{~Pa}$ ) within the circle of $R=1 \mathrm{~m}$ on the surface.


Figure 4. Contours of the hydrostatic stress $\sigma_{\mathrm{h}}$ (in MPa) in (a) and effective stress $\sigma_{\mathrm{e}}$ (in MPa) in (b), in the plane $y=0$, induced by a uniform horizontal load in the $x$ direction (with density equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface.
the magnitude of the elastic displacement, and the electric and magnetic potential in the vertical $y=0$ plane within the domain of $x<2.5 R$ and $z<2.5 R$. While figures 4(a) and (b) show the corresponding results of the hydrostatic and effective stresses in this region, figures 5(a) and (b) are the magnitude of the electric displacement and magnetic induction. It is noticed that the pattern of the elastic displacement and electric/magnetic potentials are very similar: all reach their maximum magnitude at the center of the surface (the center of the circle), with their maximum values being $16.6 \mu \mathrm{~m}$, -9.3 kV and $-6.6 \mathrm{~V} \mathrm{~s} \mathrm{~m}^{-1}$, respectively, for the elastic displacement, electric potential and magnetic potential. While not shown here, we noticed that, for this given half-space material and surface loading, the magnitude of $u_{x}$ is about one order larger than $u_{z}$, and $u_{z}$ in turn is about one order larger than $u_{y}$. As such, the magnitude of the whole displacement is nearly the same as that of its component $u_{x}$. It


Figure 5. Contours of the electric displacement $D$ (in $\mu \mathrm{C} \mathrm{m}^{-2}$ ) in (a) and magnetic induction $B$ (in $\mathrm{mT}=10^{-3} \mathrm{~T}$ ) in (b), in the plane $y=0$, induced by a uniform horizontal load in the $x$ direction (with density equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface.
is observed from figures 4(a) and (b) that the distributions of the hydrostatic and effective stresses are different than those of the elastic displacement and electric/magnetic potential. More specifically, the field concentration for the former case is now at $x=R$, i.e. on the edge of the circular loading, with the concentration being 1.3 and 2.8 MPa , respectively, for the hydrostatic and effective stresses. The field concentration is also at the same edge location for both electric displacement and magnetic induction as shown in figures 5(a) and (b), with their maxima being respectively $141 \mu \mathrm{C} \mathrm{m}^{-2}$ and 6.2 mT . Furthermore, the contour shapes of the electric displacement and magnetic induction are very similar to each other.

Figures 6(a)-(c) are the contours of the magnitude of the total elastic displacement, and electric and magnetic potentials under a uniform vertical load. Compared to the corresponding horizontal loading case, it is noticed that not


Figure 6. Contours of the elastic displacement $u$ (in $\mu \mathrm{m}$ ) in (a), electric potential $\phi$ (in kV ) in (b) and magnetic potential $\psi$ (in $\mathrm{V} \mathrm{s} \mathrm{m}{ }^{-1}$ ) in (c), in the plane $y=0$, induced by a uniform vertical load in the $z$ direction (with density equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface.


Figure 7. Contours of the hydrostatic stress $\sigma_{\mathrm{h}}$ (in MPa) in (a) and effective stress $\sigma_{\mathrm{e}}$ (in MPa) in (b), in the plane $y=0$, induced by a uniform vertical load in the $z$ direction (with density equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface.
only the contour shapes are different but also the field magnitudes are different. For instance, the magnitude of the elastic displacement under vertical loading is smaller than that under the horizontal loading ( $13.0 \mu \mathrm{~m}$ for vertical load versus $16.6 \mu \mathrm{~m}$ for horizontal load). It is further noticed that, under a uniform vertical load, the magnitude of $u_{z}$ is about one order larger than $u_{x}$, and $u_{x}$ in turn is about one order larger than $u_{y}$. Compared with figures 6(b) and (c), we further notice that the contours of electric and magnetic potentials are different: while the electric potential reaches its maximum value -8.6 kV at $x=0.5 \mathrm{~m}$ on the surface, the maximum value for the magnetic potential $9.4 \mathrm{~V} \mathrm{~s} \mathrm{~m}^{-1}$ is located at $x=0.2 \mathrm{~m}$ on the surface. While figures 7(a) and (b) show the hydrostatic and effective stresses, the corresponding electric displacement and magnetic induction are shown in figures 8(a) and (b). It is noted from figure 7(a) that the


Figure 8. Contours of the electric displacement $D$ (in $\mu \mathrm{C} \mathrm{m}^{-2}$ ) in (a) and magnetic induction $B$ (in mT ) in (b), in the plane $y=0$, induced by a uniform vertical load in the $z$ direction (with density equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface.
hydrostatic stress is constant immediately under the circular loading, implying that the three normal stresses are all equal to 1 MPa there. (It has been checked to be true by looking at these three components.) It is also observed that the contours of the effective stress are different to those of the hydrostatic stress, where the former has a singularity on the edge $x=R$ and a concentration of 0.6 MPa in the center at $z=0.65 \mathrm{~m}$. The electric displacement and magnetic induction in figures 8 (a) and (b) are all very different. For instance, the electric displacement has two concentrations, at 9.8 and $73.8 \mu \mathrm{C} \mathrm{m}^{-2}$ located at $(x, z)=(0.5,0.05)$ and $(x, z)=$ $(1.05,0)$, and the magnetic induction has concentrations at $2.0,1.26,1.23$ and 0.8 mT located at $(x, z)=(0,0),(x, z)=$ $(1,0),(x, z)=(0.6,0.5)$ and $(x, z)=(0.8,0.1)$.

The field distributions by an indentation-type surface load are shown in figures $9-11$. It is observed that, similar to




Figure 9. Contours of the elastic displacement $u$ (in $\mu \mathrm{m}$ ) in (a), electric potential $\phi$ (in kV ) in (b) and magnetic potential $\psi$ (in $\mathrm{V} \mathrm{s} \mathrm{m}^{-1}$ ) in (c), in the plane $y=0$, induced by an indentation-type vertical load in the $z$ direction (with $t_{3}^{0}$ equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface.


Figure 10. Contours of the hydrostatic stress $\sigma_{\mathrm{h}}$ (in MPa) in (a) and effective stress $\sigma_{\mathrm{e}}$ (in MPa) in (b), in the plane $y=0$, induced by an indentation-type vertical load in the $z$ direction (with $t_{3}^{0}$ equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface
the uniform vertical loading case, the magnitude of the three elastic displacement components also follows about the same trend: the magnitude of $u_{z}$ is about one order larger than $u_{x}$, and $u_{x}$ in turn is about one order larger than $u_{y}$. However, for the indentation case, the magnitude of the displacement is larger than that for the uniform vertical load case, and also the displacement component $u_{z}$ is observed to be constant immediately under the circular indenter, a well-known feature in flat-end circular indentation. Figures 9(b) and (c) show that the electric and magnetic potentials reach their maximum values on the surface at $x=1 \mathrm{~m}(-16.5 \mathrm{kV}$ for electric potential and $-16.3 \mathrm{~V} \mathrm{~s} \mathrm{~m}{ }^{-1}$ for magnetic potential). Comparing figures 10 (a) and (b), we observe their complete different contour features. In particular, while the hydrostatic stress shows only one concentration at $x=R(R=1 \mathrm{~m})$, the effective stress has two concentrations: one minimum at $x=0$ with value 0.13 MPa and one maximum at $x=R$ with value


Figure 11. Contours of the electric displacement $D$ (in $\mu \mathrm{C} \mathrm{m}^{-2}$ ) in (a) and magnetic induction $B$ (in mT$)$ in (b), in the plane $y=0$, induced by an indentation-type vertical load in the $z$ direction (with $t_{3}^{0}$ equals 1 MPa ) within the circle of $R=1 \mathrm{~m}$ on the surface.
2.6 MPa. The electric displacement and magnetic induction under indentation load also show interesting features, which are different than the uniform vertical loading case. In particular, the magnitudes of the electric displacement under indentation are about three times larger than those due to the uniform vertical load.

## 6. Conclusions

An analytical and rigorous solution is presented for the circular surface loading over an anisotropic magnetoelectroelastic half-space. The solution is expressed in terms of the line integral over $[0, \pi]$, with the integrand being the Stroh eigenvalues and eigenvectors. Under uniform horizontal and vertical loads and vertical indentation, the surface response is also derived where the involved singularity within the
integrand is identified as integrable. The numerical results not only show some interesting features associated with different surface loads, but also could serve as benchmarks for future numerical methods where indentation research is involved.

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## Appendix

(1) The local material coefficients for the $\mathrm{BaTiO}_{3}-\mathrm{CoFe}_{2} \mathrm{O}_{4}$ MEE composite with $50 \%$ of $\mathrm{BaTiO}_{3}$, or called $50 \% \mathrm{MEE}$ where the poling axis is along the third axis.
(1) Elastic constants

$$
[c]=\left[\begin{array}{rrrrrr}
225 & 125 & 124 & 0 & 0 & 0 \\
& 225 & 124 & 0 & 0 & 0 \\
& & 216 & 0 & 0 & 0 \\
& & 44 & 0 & 0 \\
& \text { symm. } & & & 44 & 0 \\
& & & & 50
\end{array}\right]\left(10^{9} \mathrm{~N} \mathrm{~m}^{-2}\right) .
$$

(2) Piezoelectric constants

$$
[e]=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 5.8 & 0 \\
0 & 0 & 0 & 5.8 & 0 & 0 \\
-2.2 & -2.2 & 9.3 & 0 & 0 & 0
\end{array}\right]\left(\mathrm{C} \mathrm{~m}^{-2}\right)
$$

(3) Dielectric permeability coefficients

$$
[\varepsilon]=\left[\begin{array}{lll}
5.64 & 0 & 0 \\
0 & 5.64 & 0 \\
0 & 0 & 6.35
\end{array}\right]\left(10^{-9} \mathrm{C} \mathrm{~V}^{-1} \mathrm{~m}^{-1}\right)
$$

(4) Piezomagnetic constants

$$
[q]=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 275 & 0 \\
0 & 0 & 0 & 275 & 0 & 0 \\
290.2 & 290.2 & 350 & 0 & 0 & 0
\end{array}\right]\left(\mathrm{N} \mathrm{~A}^{-1} \mathrm{~m}^{-1}\right)
$$

(5) Magnetoelectric coefficients $\alpha(i, j)=0$ (for $i, j=1,3$ ) (in $\mathrm{Ns} \mathrm{V}^{-1} \mathrm{C}^{-1}$ )
(6) Magnetic permeability coefficients

$$
[\mu]=\left[\begin{array}{lll}
297 & 0 & 0 \\
0 & 297 & 0 \\
0 & 0 & 83.5
\end{array}\right]\left(10^{-6} \mathrm{~N} \mathrm{~s}^{2} \mathrm{C}^{-2}\right)
$$

(2) The global material property of $50 \%$ MEE (with rotation angles $\alpha=60^{\circ}, \beta=45^{\circ}$ ).

For this case, the global and local coordinate transform is given by the two orientation angles in equation (49). For
fixed $\alpha=60^{\circ}$ and $\beta=45^{\circ}$, we have the following coordinate transform matrix and global material matrices.
(1) Coordinate transform matrix

$$
\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{lll}
-0.61 & 0.50 & 0.61 \\
-0.35 & -0.87 & 0.35 \\
0.71 & 0 & 0.71
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3}
\end{array}\right] .
$$

(2) Elastic constants

$$
[c]=\left[\begin{array}{cccccc}
217.64 & 125.3 & 127.31 & 1.34 & -2.87 & -1.43 \\
& 222.02 & 125.44 & -2.72 & 0.49 & -2.35 \\
& & 216.25 & -1.13 & -1.95 & 1.62 \\
& & & 47.31 & 0.54 & -1.68 \\
& \text { symm. } & & 47.94 & 0.09 \\
& & & & & 47.8
\end{array}\right]
$$

$$
\times\left(10^{9} \mathrm{Nm}^{-2}\right)
$$

(3) Piezoelectric constants

$$
\begin{aligned}
{[e]=} & {\left[\begin{array}{llllll}
5.73 & -1.35 & -1.38 & -0.02 & 4.07 & 2.04 \\
-0.79 & 3.32 & -0.80 & 4.09 & -0.02 & 3.54 \\
-1.58 & -1.56 & 6.61 & 2.03 & 3.52 & -0.02
\end{array}\right] } \\
& \times\left(\mathrm{C} \mathrm{~m}^{-2}\right) .
\end{aligned}
$$

(4) Dielectric permeability coefficients

$$
[\varepsilon]=\left[\begin{array}{lll}
6.00 & 0 & 0.35 \\
0 & 5.64 & 0 \\
0.35 & 0 & 6.00
\end{array}\right]\left(10^{-9} \mathrm{C} \mathrm{~V}^{-1} \mathrm{~m}^{-1}\right)
$$

(5) Piezomagnetic constants

$$
\begin{aligned}
{[q] } & =\left[\begin{array}{llllll}
401.95 & 140.19 & 27.62 & -75.05 & 64.47 & 32.24 \\
37.61 & 275.39 & 15.95 & 151.13 & -75.05 & 130.88 \\
75.22 & 161.87 & 420.8 & 10.57 & 18.31 & -75.05
\end{array}\right] \\
& \times\left(\mathrm{N} \mathrm{~A}^{-1} \mathrm{~m}^{-1}\right)
\end{aligned}
$$

(6) Magnetoelectric coefficients $\alpha(i, j)=0$ (for $i, j=1,3$ ) (in $\mathrm{Ns} \mathrm{V}^{-1} \mathrm{C}^{-1}$ ).
(7) Magnetic permeability coefficients

$$
[\mu]=\left[\begin{array}{lll}
190.25 & 0 & -106.75 \\
0 & 297 & 0 \\
-106.75 & 0 & 190.25
\end{array}\right]\left(10^{-6} \mathrm{~N} \mathrm{~s}^{2} \mathrm{C}^{-2}\right)
$$

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