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### Electric and magnetic polarization saturation and breakdown models for penny shaped cracks in 3D magnetoelectroelastic media

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#### ABSTRACT

Two nonlinear fracture models, i.e., the electric and magnetic polarization saturation (EMPS) and electric and magnetic breakdown (EMBD) models for penny shaped cracks in three-dimensional magnetoelectroelastic media are studied via the extended displacement discontinuity integral equation method. In the EMPS model, the electric displacement and magnetic induction are constant respectively in the planar electric yielding zone and magnetic yielding zone on the original crack plane, whilst in the EMBD model, the electric and magnetic strengths are constant respectively in their planar yielding zones. Under the electrically and magnetically impermeable conditions and uniformly applied mechanical–electric–magnetic loadings on the faces of the penny shaped crack, analytical solutions are derived for the sizes of electric and magnetic yielding zones and the field intensity factors. It is demonstrated that the fracture behaviors can be predicted equivalently by both EMPS and EMBD models even though they are built on two different physical grounds.

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#### 1. Introduction

Defects in materials, such as cracks, avoids, etc., can greatly influence the safety and life of structures and systems. Analysis of cracks is one of the main tasks in fracture mechanics. Various nonlinear models have been proposed in dealing with nonlinear behaviors of cracks in elastic–plastic materials and structures. Among these models, the Dugdale model (Dugdale, 1960) is one of the most famous ones in nonlinear fractures.

In 1960, Dugdale proposed a model to analyze yielding of steel sheets containing slits, in which the stress at the crack tip should not be infinite, thus the coefficient of the singular term must vanish. The strip yielding zone size was also determined from the non-singularity condition. Up to now, Dugdale model has been receiving intensive study and it is now widely utilized in fracture mechanics. For example, Janson (1977) analyzed the influence of continuous damage on the stress ahead of the crack and on the plastic zone length using the Dugdale model. Wu et al. (1992) established a model for fracture crack growth based on the Dugdale model along with the damage accumulation concept introduced by Budiansky and Hutchinson (1978). Mou and Han (1994) derived the size of the damage zone via the Dugdale model by considering the interaction between the macrocracks and

0020-7683/\$ - see front matter @ 2013 Elsevier Ltd. All rights reserved. http://dx.doi.org/10.1016/j.ijsolstr.2013.02.003 microcracks. Collins and Cartwright (2001) derived an analytical solution for two equal-length collinear strip-yield cracks based on the Dugdale model. Crapps and Daniewicz (2010) derived the weight function, also based on the Dugdale model for mixed-mode crack problems with arbitrary crack surface tractions. Xu and Wu (2012) derived the weight functions to analyze three collinear cracks with the strip-yield model. Chang and Kotousov (2012) studied analytically and numerically the strip yield model for two collinear cracks.

The nonlinear fracture behaviors of multiferroic materials are attracting more and more attentions. Gao and Barnett (1996) and Gao et al. (1997) extended the classical Dugdale model and proposed a strip polarization saturation (PS) model in piezoelectricity by assuming that the electric displacement is constant on a strip adjacent to a crack tip. The PS model treats piezoelectric ceramics as mechanically brittle and electrically ductile, and takes advantage of the fact that the constitutive relations between electric displacement and electric field strength are similar to that between stress and strain in the Dugdale model. Prediction of the piezoelectric fields and fracture features based on the PS model are in general agreement with experimental observations. Based on the complete constitutive equation and the PS model, extended analysis was conducted by Ru and Mao (1999), Ru (1999), Wang (2000), Zhang et al. (2002), Jeong et al. (2004), Beom et al. (2006a,b), and Loboda et al. (2008). Li (2003) re-examined the saturation-strip model for a piezoelectric crack in a permeable environment. Lapusta and Loboda

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(2009) and Loboda et al. (2010) considered a plane problem for a limited (or semi-) permeable crack in a thin interlayer between two piezoelectric semi-infinite spaces. Very recently, Fan et al. (2012) derived a semi-analytical solution for a semi-permeable crack in a 2D piezoelectric medium based on the PS model.

McMeeking (2001) reconsidered the relation among the electric displacement, electric field and the polarization, and their effect on the fracture mechanics of the brittle piezoelectric material. It was found that the PS model corresponds to a mechanical Dugdale (1960) model in which the strain remains a constant value as the stress increases. For this reason, Zhang and Gao (2004), Zhang (2004) and Zhang et al. (2005) proposed the strip dielectric breakdown (DB) model, which is exactly analogous to the classical Dugdale model from the energy point of view. The electric field strength on a strip adjacent to a crack tip is taken to be the dielectric breakdown strength. Except for the slight difference between the values of the J-integrals derived from the PS and DB models, these two models are qualitatively consistent in the predication of the effect of electric fields on the fracture behavior. Later, Gao et al. (2006) extended the DB model to a conductive crack. Fan et al. (2009) developed a non-linear hybrid extended displacement discontinuity-fundamental solution (NLHEDD-FS) method for the numerical analysis of 2D finite piezoelectric cracks based on both the PS and DB models where the crack surface is under impermeable and semi-permeable electric boundary conditions. Zhang and Gao (2012) studied the strip DB model for a conductive crack in an infinite electrostrictive material by means of the complex variable method.

Based on the similarities between the electric and magnetic quantities, the existing electric yielding models (e.g., the PS and DB model) and the magnetic yielding models, Zhao and Fan (2008) proposed the strip electric-magnetic breakdown (EMBD) model to study the non-linear fracture behaviors of an electrically and magnetically impermeable crack in magnetoelectroelastic (MEE) materials. In the strip EMBD model, the electric field in the strip of the electric breakdown zone ahead of the crack tip is equal to the electric breakdown strength, while the magnetic field in the strip of the magnetic breakdown zone is equal to the magnetic breakdown strength. Fan and Zhao (2011) developed a strip electric-magnetic polarization saturation (EMPS) model to study the electric and magnetic yielding effects on a crack in an infinite or finite MEE medium. The analytical solution of this strip EMPS model for a crack in the infinite MEE medium was obtained using an integral equation approach. The equivalence between the proposed strip EMPS model and the existing strip EMBD model was demonstrated. The NLHEDD-FS method was extended to analyze the nonlinear fracture problem in the finite MEE medium. Recently, Bhargava and Gupta (2012) developed a magnetic, electric and mechanical yield model for a cracked piezoelectromagnetic ceramic narrow strip subjected to anti-plane mechanical and in-plane electric and magnetic loads.

For the 3D elastic-plastic problems, Danyluck et al. (1995) estimated the plastic zone for a penny-shaped crack in a thick transversely isotropic layer due to a radial shear employing the Dugdale hypothesis. Chaiyat et al. (2008) used analytical and numerical approaches to solve an axisymmetric crack problem with a refined Barenblatt–Dugdale approach, where the original Dugdale's condition, Tresca yield condition and von Mises criterion were imposed. Li et al. (2012) studied the penny-shaped Dugdale crack embedded in a power-law graded elastic infinite medium. The validity of the solutions was examined both analytically and numerically. Zhao et al. (1999) used the displacement discontinuity boundary integral equation method to study the PS model of a penny-shaped crack in a 3D transversely isotropic piezoelectric medium. So far, however, the two nonlinear fracture models are still limited to 2D MEE materials. Would be the corresponding 3D nonlinear fracture behaviors different than those based on the 2D model? What are the difference and similarity between the EMPS and EMBD models in 3D? Answering these questions is important for fracture analysis in 3D MEE materials. Motivated by this, in this paper, we study a penny shaped crack in 3D MEE media based on both the EMPS and EMBD models via the extended displacement discontinuity integral equation method we introduced previously. The paper is organized as follows: In Section 2, we discuss the EMPS model; In Section 3, the EMBD model is presented. Both models are also discussed and compared to each other. Conclusions are drawn in Section 4.

## 2. Electric and magnetic polarization saturation model for a penny shaped crack in a 3D magnetoelectroelastic medium

Consider a penny shaped crack S of radius *a* in the *oxy* plane centered at the origin *o* in a 3D MEE medium where the *oxy* plane coincides with the plane of isotropy and the polarization direction is along the *z*-direction, as schematically shown in Fig. 1. Uniformly distributed mechanical, electric and magnetic loadings are applied on the crack faces  $S \equiv \{0 \le r \le a; 0 \le \theta \le 2\pi\}$  and they have the same value but opposite directions on the upper (S<sup>+</sup>) and lower (S<sup>-</sup>) crack face (see Fig. 2a), which read

$$p(r) \equiv -\sigma_z(r, \mathbf{0}^+) = p_0,$$
  

$$\omega(r) \equiv -D_z(r, \mathbf{0}^+) = \omega_0, \quad \mathbf{0} \le r \le a,$$
  

$$\gamma(r) \equiv -B_z(r, \mathbf{0}^+) = \gamma_0,$$
(1)

where  $\sigma_z(r, 0^+)$ ,  $D_z(r, 0^+)$  and  $B_z(r, 0^+)$  are the components of stress, electric displacement and magnetic induction in the *z*-direction, respectively.

It is observed that due to the given material and crack orientations and the applied loads, the problem to be solved is obviously axisymmetric. Based on the EMPS model for MEE media (Fan and Zhao, 2011), we assume that the piezoelectric displacement and the magnetic induction reach their saturated values  $D_s$  and  $B_s$ , respectively, in the annular electric polarization saturation region  $\{a \le r \le c; \ 0 \le \theta \le 2\pi\}$  and magnetic induction saturation region  $\{a \le r \le b; \ 0 \le \theta \le 2\pi\}$  in front of the crack tip on the crack plane (Fig. 2). Thus, the electric loadings on the equivalent electric crack  $S_c \equiv \{0 \le r \le c; \ 0 \le \theta \le 2\pi\}$  can be rewritten as



**Fig. 1.** A penny-shaped crack in the isotropic plane of an infinite solid of a transversely isotropic MEE material.



Fig. 2. (a) The EMPS model of a penny shaped crack under applied loadings on its surface where  $b \ge c \ge a$ . (b) The EMPS model of a penny shaped crack with the annular yielding zones on the crack plane.

$$\omega(r) = \begin{cases} \omega_0, & 0 \le r \le a, \\ -D_s, & a < r \le c, \end{cases}$$
(2)

and the magnetic loading on the equivalent magnetic crack  $S_b \equiv \{0 \le r \le b; \ 0 \le \theta \le 2\pi\}$  are

$$\gamma(r) = \begin{cases} \gamma_0, & 0 \le r \le a, \\ -B_s, & a < r \le b. \end{cases}$$
(3)

By using the extended displacement discontinuity Green's function and the extended Somigliana relation, the integral equations that govern the extended displacement discontinuities are derived as (Zhao et al., 2007)

$$\begin{split} \int_{S} L_{31} \|w\| \frac{1}{R^3} \mathrm{d}S(\xi,\eta) + \int_{S_c} L_{32} \|\varphi\| \frac{1}{R^3} \mathrm{d}S(\xi,\eta) + \int_{S_b} L_{33} \|\psi\| \frac{1}{R^3} \mathrm{d}S(\xi,\eta) \\ &= -p(x,y), \, (x,y) \in \mathbf{S}, \end{split}$$
(4)

$$\begin{split} &\int_{S} L_{41} \|w\| \frac{1}{R^3} dS(\xi, \eta) + \int_{S_c} L_{42} \|\varphi\| \frac{1}{R^3} dS(\xi, \eta) + \int_{S_b} L_{43} \|\psi\| \frac{1}{R^3} dS(\xi, \eta) \\ &= -\omega(x, y), \, (x, y) \in S_c \end{split}$$
(5)

$$\begin{split} &\int_{S} L_{51} \|w\| \frac{1}{R^3} dS(\xi, \eta) + \int_{S_c} L_{52} \|\varphi\| \frac{1}{R^3} dS(\xi, \eta) + \int_{S_b} L_{53} \|\psi\| \frac{1}{R^3} dS(\xi, \eta) \\ &= -\gamma(\mathbf{x}, \mathbf{y}), \, (\mathbf{x}, \mathbf{y}) \in \mathsf{S}_b, \end{split}$$
(6)

where the material-related constants  $L_{ij}$  are given in Appendix A. The extended displacement discontinuities are the elastic displacement discontinuity in the *z*-direction, the electric potential and the magnetic potential discontinuities across the crack faces, namely

$$\begin{split} \|w(r)\| &= w(r, 0^+) - w(r, 0^-), \\ \|\varphi(r)\| &= \varphi(r, 0^+) - \varphi(r, 0^-), \\ \|\psi(r)\| &= \psi(r, 0^+) - \psi(r, 0^-), \end{split}$$
(7)

and

$$R = \sqrt{(x - \xi)^2 + (y - \eta)^2}.$$
 (8)

Depending upon the relation between *b* and *c*, we solve the EMPS model for the following two different cases.

2.1. Case 1: 
$$b \ge c$$

Equation (6) can be rewritten as the Abel type integral equation (Chen and Tang, 1997; Zhao et al., 1999)

$$4\int_{0}^{r} \frac{t^{2}}{(r^{2}+t^{2})^{3/2}} \left[ \int_{t}^{b} \frac{\rho(L_{51}\|w\|+L_{52}\|\phi\|+L_{53}\|\psi\|)}{(\rho^{2}-t^{2})^{3/2}} \mathrm{d}\rho \right] \mathrm{d}t$$
  
=  $\gamma(r), \quad 0 \le r \le b,$  (9)

with its solution being found as (Chen and Tang, 1997; Zhao et al., 1999)

$$L_{51}\|w\| + L_{52}\|\phi\| + L_{53}\|\psi\| = L_{53}f_{\psi}(r), \quad 0 \le r \le b,$$
(10)

where

$$\begin{split} L_{53}f_{\psi}(r) &= \frac{1}{\pi^2} \int_{r}^{b} \frac{1}{\sqrt{t^2 - r^2}} \left[ \int_{0}^{t} \frac{\rho \gamma(\rho)}{\sqrt{t^2 - \rho^2}} d\rho \right] dt \\ &= \frac{1}{\pi^2} \left[ \gamma_0 \sqrt{b^2 - r^2} - (\gamma_0 + B_s) \int_{\max(r,a)}^{b} \sqrt{\frac{t^2 - a^2}{t^2 - r^2}} dt \right]. \end{split}$$
(11)

From the EMPS model, we have

||w|| = 0, for a < r < c, (12)

 $\|w\| = 0, \quad \|\varphi\| = 0$ for  $c \leq r \leq b$ , (13)

Then, we obtain

 $L_{52}\|\phi\| + L_{53}\|\psi\| = L_{53}f_{\psi}(r), \quad a \leq r \leq c,$ (14)

$$\|\psi\| = f_{\psi}(r), \quad c \le r \le b, \tag{15}$$

Introducing the dimensionless parameters

$$x = -\frac{r}{a}, \quad x_1 = -\frac{b}{a}, \quad y_1 = -\frac{\gamma_0}{B_s},$$
 (16)

$$I(x, x_1, y_1) = \frac{1}{\pi^2} \left[ y_1 \sqrt{x_1^2 - x^2} - (y_1 + 1) I_0(x, x_1) \right],$$
(17)

$$I_0(x,x_1) = \int_{\max(x,1)}^{x_1} \sqrt{\frac{\xi^2 - 1}{\xi^2 - x^2}} d\xi.$$
 (18)

Eq. (11) can be rewritten as L<sub>53</sub>

$$f_{\psi}(\mathbf{x}) = aB_{\rm s}I(\mathbf{x}, \mathbf{x}_1, \mathbf{y}_1). \tag{19}$$

Based on the expression for the magnetic induction intensity factor (Zhao et al., 2007)

$$K_{I}^{B} = \sqrt{2\pi}\pi \lim_{r \to b} [L_{51} \|w\| + L_{52} \|\phi\| + L_{53} \|\psi\|] / \sqrt{b - r},$$
(20)

and the condition that the magnetic induction intensity factor at r = b should be zero, we obtain the magnetic yielding zone size as

$$x_1 = \frac{1 + y_1}{\sqrt{1 + 2y_1}}.$$
(21)

Equation (21) indicates that the magnetic yielding zone size is only dependent on the applied magnetic loading and the magnetic yield-ing property.

Now, we combine Eqs. (5) and (6) to yield

$$\int_{S} L_{41}^{*} \|w\| \frac{1}{R^{3}} dS + \int_{S_{c}} L_{42}^{*} \|\varphi\| \frac{1}{R^{3}} dS = -\omega(r) + (L_{43}/L_{53})\gamma(r),$$
  
  $0 \le r \le c,$  (22)

where

$$L_{4i}^* = L_{4i} - (L_{43}/L_{53})L_{5i}, \quad i = 1, 2, \tag{23}$$

and the extended loading is

$$\omega(r) - (L_{43}/L_{53})\gamma(r) = \begin{cases} \omega_0 - (L_{43}/L_{53})\gamma_0, & 0 \le r \le a, \\ -D_s + (L_{43}/L_{53})B_s, & a \le r \le c. \end{cases}$$
(24)

Following the similar procedure above, we obtain

$$L_{41}^* \| w \| + L_{42}^* \| \varphi \| = (D_s - (L_{43}/L_{53})B_s)al(x, x_2, y_2), \quad 0 \le r \le c,$$
(25)

where the dimensionless parameters are defined as

$$x_2 = c/a, \quad y_2 = (\omega_0 - (L_{43}/L_{53})\gamma_0)/(D_s - (L_{43}/L_{53})B_s).$$
 (26)  
From Eq. (12), we have

$$L_{43}^* \|\phi\| = (D_5 - (L_{43}/L_{53})B_5)aI(x, x_2, y_2), \quad a \le r \le c.$$
(27)

It can be proven that function  $f_{\psi}(x)$  in Eq. (14) induces no singularity at  $x = x_2$ . Then from Eq. (14), we have

$$\|\psi\| - \|\psi_c\| \approx -\frac{L_{52}}{L_{53}} \|\phi\|, \quad r \to c,$$
 (28)

where  $\|\psi_{c}\| = \|\psi(c)\|$ .

Based on the electric displacement intensity factor expressed in terms of the extended displacement discontinuities (Zhao et al., 2007)

$$K_{I}^{D} = \sqrt{2\pi\pi} \lim_{r \to c} [L_{41} \|w\| + L_{42} \|\phi\| + L_{43} (\|\psi\| - \|\psi_{c}\|)] / \sqrt{c - r}, \quad (29)$$

and the condition that  $K_I^D = 0$  at r = c, we obtain the size of the electric yielding zone as

$$x_2 = \frac{1 + y_2}{\sqrt{1 + 2y_2}}.$$
(30)

Equation (30) shows that the size of the electric polarization zone depends on both the electric and magnetic loadings as well as on the electric and magnetic saturated properties.

Finally, substituting Eq. (6) into Eq. (4) yields

$$\int_{S} L_{31}^{*} \|w\| \frac{1}{R^{3}} dS + \int_{S_{c}} L_{32}^{*} \|\varphi\| \frac{1}{R^{3}} dS = -p(r) + (L_{33}/L_{53})\gamma(r), \quad 0 \le r \le a,$$
(31)

where

$$L_{3i}^* = L_{3i} - (L_{33}/L_{53})L_{5i} \quad i = 1, 2,$$
(32)

From Eq. (22), we have

$$\int_{S} L_{41}^{*} \|w\| \frac{1}{R^{3}} dS + \int_{S_{c}} L_{42}^{*} \|\varphi\| \frac{1}{R^{3}} dS$$
  
=  $-\omega(r) + (L_{43}/L_{53})\gamma(r), \quad 0 \le r \le a,$  (33)

From Eqs. (31) and (33), we have

$$\int_{S} \frac{\|w\|}{R^{3}} dS = \frac{-L_{42}^{*}p(r) + L_{32}^{*}\omega(r) + [L_{42}^{*}(L_{33}/L_{53}) - L_{32}^{*}(L_{43}/L_{53})]\gamma(r)}{L_{31}^{*}L_{42}^{*} - L_{32}^{*}L_{41}^{*}},$$

$$0 \le r \le a,$$

$$(34)$$

Thus, we obtain

$$\|w\| = \frac{L_{42}^* p_0 - L_{32}^* \omega_0 - [L_{42}^* (L_{33}/L_{53}) - L_{32}^* (L_{43}/L_{53})] \gamma_0}{L_{31}^* L_{42}^* - L_{32}^* L_{41}^*} \frac{a}{\pi^2}$$

$$\sqrt{1 - x^2}, \quad 0 \le x \le 1.$$
(35)

It can be proven that  $\sqrt{x_2^2 - x^2}$  and  $I_0(x, x_2)$  induce no singularity at x = 1. Then from Eq. (25), we have

$$\|\varphi\| - \|\varphi_a\| \approx -\frac{L_{41}^*}{L_{42}^*} \|w\|, \quad x \to 1,$$
(36)

where  $\|\varphi_a\| = \|\varphi(a)\|$ .It can also be proven that function  $f_{\psi}(x)$  in Eq. (10) induces no singularity at x = 1. Substituting Eq. (36) into Eq. (10) leads to

$$\|\psi\| - \|\psi_a\| \approx -\left(L_{51} - L_{52} \frac{L_{41}^*}{L_{42}^*}\right) \|w\|/L_{53}, \quad x \to 1,$$
(37)

where  $\|\psi_a\| = \|\psi(a)\|$ .

Substituting Eqs. (35)–(37) into the expression of the stress intensity factor at the mechanical crack tip (Zhao et al., 2007)

$$K_{I}^{F} = \sqrt{2\pi\pi} \lim_{r \to a} [L_{31}||w|| + L_{32}(||\varphi|| - ||\varphi_{a}||) + L_{33}(||\psi|| - ||\psi_{a}||)]\sqrt{a - r},$$
(38)

gives the local stress intensity factor

$$K_{I}^{F} = 2\sqrt{\frac{a}{\pi}}(p_{0} + L_{F2}\omega_{0} + L_{F3}\gamma_{0}),$$
(39)

where the constants are

$$L_{F2} = -\frac{L_{32}^*}{L_{42}^*}, \quad L_{F3} = \frac{L_{32}L_{43} - L_{33}L_{42}}{L_{42}L_{53} - L_{43}L_{52}}.$$
 (40)

This result demonstrates that the local stress intensity factor at the crack tip is related to the applied mechanical, electric and magnetic loadings, but independent of the electric and magnetic yielding zones, similar to the corresponding 2D results (Ru, 1999). Furthermore, the electric displacement and magnetic induction intensity factors are zero.

#### 2.2. Case 2: c ≥ b

Based on the similar solution procedure for Case 1, we obtain the dimensionless sizes of the electric and magnetic yielding zones

$$x_2 = \frac{1 + y_3}{\sqrt{1 + 2y_3}},\tag{41}$$

$$x_1 = \frac{1 + y_4}{\sqrt{1 + 2y_4}},\tag{42}$$

where

$$x_2 = c/a, \quad y_3 = \omega_0/D_s,$$
 (43)

$$x_1 = b/a, \quad y_4 = (\omega_0 - (L_{42}/L_{52})\gamma_0)/(D_s - (L_{42}/L_{52})B_s).$$
 (44)

In this case, the electric yielding zone is dependent only on the electric loading, while magnetic yielding zone is dependent both on the electric and magnetic loadings. Furthermore, it is interesting to find that the local intensity factor at the mechanical crack tip is still given by Eq. (39).

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# 3. Electric and magnetic breakdown model for a penny shaped crack in a 3D magnetoelectroelastic medium

In the electric and magnetic breakdown model of the penny shaped crack (Zhao and Fan, 2008), the electric breakdown region is also annular denoted by  $a \le r \le c$  and the annular magnetic breakdown region denoted by  $a \le r \le b$ , in which

$$E_z = E_b, \quad a < r < c, \tag{45}$$

$$H_z = H_b, \quad a < r < b, \tag{46}$$

where  $E_b$  and  $H_b$  are the electric and magnetic breakdown strengths, respectively. By using the extended displacement discontinuity Green's functions (Zhao et al., 2007), the extended displacement discontinuity boundary integral equations are derived

$$\int_{S} L_{31} \|w\| \frac{1}{R^{3}} dS + \int_{S_{c}} L_{32} \|\varphi\| \frac{1}{R^{3}} dS + \int_{S_{b}} L_{33} \|\psi\| \frac{1}{R^{3}} dS = -p_{0},$$
  

$$0 \le r \le a,$$
(47)

$$\int_{S} L_{41} \|w\| \frac{1}{R^{3}} dS + \int_{S_{c}} L_{42} \|\varphi\| \frac{1}{R^{3}} dS + \int_{S_{b}} L_{43} \|\psi\| \frac{1}{R^{3}} dS = -\omega_{0},$$
  

$$0 \le r \le a,,$$
(48)

$$\int_{S} L_{51} \|w\| \frac{1}{R^3} dS + \int_{S_c} L_{52} \|\varphi\| \frac{1}{R^3} dS + \int_{S_b} L_{53} \|\psi\| \frac{1}{R^3} dS = -\gamma_0,$$
  

$$0 \le r \le a,$$
(49)

$$\int_{S} K_{41} \|w\| \frac{1}{R^{3}} dS + \int_{S_{c}} K_{42} \|\varphi\| \frac{1}{R^{3}} dS + \int_{S_{b}} K_{43} \|\psi\| \frac{1}{R^{3}} dS = E_{b},$$
  
$$a \le r \le c,$$
 (50)

$$\int_{S} K_{51} \|w\| \frac{1}{R^3} dS + \int_{S_c} K_{52} \|\varphi\| \frac{1}{R^3} dS + \int_{S_b} K_{53} \|\psi\| \frac{1}{R^3} dS = H_b,$$
  
$$a \le r \le b,$$
 (51)

where

$$K_{4j} = -\sum_{i=1}^{4} \vartheta_{ij} B_i s_i, \quad K_{5j} = -\sum_{i=1}^{4} \vartheta_{ij} C_i s_i, \ j = 1, 2, 3,$$
 (52)

with  $s_i, B_i, C_i$  and  $\vartheta_{ij}$  being all material-related constants given in Zhao et al. (2007).

Solving Eqs. (47)–(49) gives

$$\int_{S} K_{51} \|w\| \frac{1}{R^3} dS + \int_{S_c} K_{52} \|\varphi\| \frac{1}{R^3} dS + \int_{S_b} K_{53} \|\psi\| \frac{1}{R^3} dS = -H_0,$$
  
$$0 \le r \le a,$$
 (56)

where

$$E_0 = -(K_{41}P + K_{42}Q + K_{43}T),$$
  

$$H_0 = -(K_{51}P + K_{52}Q + K_{53}T).$$
(57)

Similar to the EMPS model in Section 2, we discuss the following two cases for the EMBD model.

3.1. Case 1:  $b \ge c$ 

Form Eqs. (51) and (56), we obtain

$$K_{51} \|w\| + K_{52} \|\phi\| + K_{53} \|\psi\| = K_{53} g_{\psi}(r), \quad 0 \le r \le b,$$
(58)

where

$$K_{53}g_{\psi}(x) = aH_bI(x, x_1, y_5), \tag{59}$$

$$y_5 = H_0/H_b.$$
 (60)

Then we have

$$K_{52}\|\varphi\| + K_{53}\|\psi\| = K_{53}g_{\psi}(r), \quad a \le r \le c,$$
(61)

$$\|\psi\| = g_{\psi}(r), \quad c \le r \le b, \tag{62}$$

Applying the condition  $K_{l}^{H} = 0$  at r = b to the above equation gives

$$x_1 = \frac{1+y_5}{\sqrt{1+2y_5}}.$$
(63)

Since the loading  $H_0$  is dependent on  $p_0$ ,  $\omega_0$ ,  $\gamma_0$ , the magnetic yielding zone size is dependent on the applied mechanical, electrical and magnetic loadings and the magnetic yielding property.

We now combine Eqs. (50), (51), (55), and (56) to obtain

$$\int_{S_{c}} (K_{41}^{*} \| w \| + K_{42}^{*} \| \varphi \|) \frac{1}{R^{3}} dS = -E(r) + (K_{43}/K_{53})H(r), \quad 0 \le r \le c, \quad (64)$$

where

$$K_{4i}^* = K_{4i} - (K_{43}/K_{53})K_{5i}, \ i = 1, 2, \tag{65a}$$

$$E(r) = \begin{cases} E_0, & 0 \le r \le a, \\ -E_b, & a < r \le c, \end{cases} \quad H(r) = \begin{cases} H_0, & 0 \le r \le a, \\ -H_b, & a < r \le b. \end{cases}$$
(65b)

$$\begin{split} & \int_{S} \|w\| \frac{1}{R^{3}} dS = \frac{-(L_{42}L_{53} - L_{43}L_{52})p_{0} + (L_{32}L_{53} - L_{33}L_{52})\omega_{0} - (L_{32}L_{43} - L_{33}L_{42})\gamma_{0}}{\Delta} = P, \\ & \int_{S_{e}} \|\varphi\| \frac{1}{R^{3}} dS = \frac{(L_{41}L_{53} - L_{43}L_{51})p_{0} - (L_{31}L_{53} - L_{33}L_{51})\omega_{0} + (L_{31}L_{43} - L_{33}L_{41})\gamma_{0}}{\Delta} = Q, \\ & \int_{S_{b}} \|\psi\| \frac{1}{R^{3}} dS = \frac{-(L_{41}L_{52} - L_{42}L_{51})p_{0} + (L_{31}L_{52} - L_{32}L_{51})\omega_{0} - (L_{31}L_{42} - L_{32}L_{41})\gamma_{0}}{\Delta} = T, \quad 0 < r < a \end{split}$$

where

$$\Delta = \begin{vmatrix} L_{31} & L_{32} & L_{33} \\ L_{41} & L_{42} & L_{43} \\ L_{51} & L_{52} & L_{53} \end{vmatrix}.$$
(54)

Making use of Eq. (53), we obtain the following dual boundary equations from Eqs. (50) and (51)

$$\int_{S} K_{41} \|w\| \frac{1}{R^{3}} dS + \int_{S_{c}} K_{42} \|\varphi\| \frac{1}{R^{3}} dS + \int_{S_{b}} K_{43} \|\psi\| \frac{1}{R^{3}} dS = -E_{0},$$
  

$$0 \le r \le a ,$$
(55)

Following the similar procedure as above, we obtain

$$K_{41}^* \|w\| + K_{42}^* \|\varphi\| = (E_b - (K_{43}/K_{53})H_b)aI(x, x_2, y_6), \quad 0 \le r$$
  
$$\le c, \tag{66}$$

where the dimensionless parameters are defined as

$$y_6 = (E_0 - (K_{43}/K_{53})H_0)/(E_b - (K_{43}/K_{53})H_b).$$
 (67)  
Then, we have

$$K_{42}^* \|\varphi\| = (E_b - (K_{43}/K_{53})H_b)aI(x, x_2, y_6), \quad a \le r \le c.$$
(68)

It can be proven that function  $g_{\psi}(x)$  in Eq. (61) induces no singularity at  $x = x_2$ . Then from Eq. (61), we have

$$\|\psi\| - \|\psi_c\| \approx -\frac{K_{52}}{K_{53}} \|\varphi\|, \quad r \to c.$$
 (69)

Based on the electric field intensity factor expressed in terms of the extended displacement discontinuities

$$K_{I}^{E} = \sqrt{2\pi\pi} \lim_{r \to c} (K_{41} \|w\| + K_{42} \|\varphi\| + K_{43} (\|\psi\| - \|\psi_{c}\|)) / \sqrt{c - r},$$
(70)

and the condition that  $K_I^E = 0$ , the size of the electric yielding zone is derived as

$$x_2 = \frac{1 + y_6}{\sqrt{1 + 2y_6}}.$$
(71)

From Eq. (53), we obtain

.

$$\begin{split} \|w\| &= \frac{L_{42}^* p_0 - L_{32}^* \omega_0 - \left[L_{42}^* (L_{33}/L_{53}) - L_{32}^* (L_{43}/L_{53})\right] \gamma_0}{L_{31}^* L_{42}^* - L_{32}^* L_{41}^*} \frac{a}{\pi^2} \sqrt{1 - x^2}, \\ 0 &\leq r \leq a. \end{split}$$
(72)

The displacement discontinuity ||w|| is the same as that predicted based on the EMPS model as given in Eq. (35). It can be proven that  $\sqrt{x_2^2 - x^2}$  and  $I_0(x, x_2)$  induce no singularity at x = 1. Then from Eq. (66), we have

$$\|\varphi\| - \|\varphi_a\| \approx -\frac{K_{41}^*}{K_{42}^*} \|w\|, \quad x \to 1,$$
(73)

Similarly, from Eq. (58), we have

$$\|\psi\| - \|\psi_a\| \approx -\left(K_{51} - K_{52}\frac{K_{41}^*}{K_{42}^*}\right)\|w\|/K_{53}, \quad x \to 1,$$
(74)

Finally, the local extended factors at the mechanical crack tip are derived as

$$K_{I}^{F} = 2\sqrt{\frac{a}{\pi}}(G_{F1}p_{0} + G_{F2}\omega_{0} + G_{F3}\gamma_{0}),$$
(75a)

$$K_{I}^{D} = 2\sqrt{\frac{a}{\pi}}(G_{D1}p_{0} + G_{D2}\omega_{0} + G_{D3}\gamma_{0}),$$
(75b)

$$K_{I}^{B} = 2\sqrt{\frac{a}{\pi}}(G_{B1}p_{0} + G_{B2}\omega_{0} + G_{B3}\gamma_{0}),$$
(75c)

where the constants are

$$\begin{aligned} G_{F1} &= \frac{L_{42}^{*}}{(L_{31}^{*}L_{42}^{*} - L_{32}^{*}L_{41}^{*})K_{53}K_{42}^{*}} \left[ K_{53}(L_{31}K_{42}^{*} - L_{32}K_{41}^{*}) - L_{33}(K_{51}K_{42}^{*} - K_{52}K_{41}^{*}) \right], \\ G_{F2} &= -\frac{L_{32}^{*}}{L_{42}^{*}}G_{F1}, \\ G_{F3} &= -\frac{1}{L_{53}}(L_{33} - L_{43}\frac{L_{32}^{*}}{L_{42}^{*}})G_{F1}, \end{aligned}$$

$$(76)$$

$$\begin{aligned} G_{D1} &= \frac{L_{42}^*}{(L_{31}^*L_{42}^* - L_{32}^*L_{41}^*)K_{53}K_{42}^*} [K_{53}(L_{41}K_{42}^* - L_{42}K_{41}^*) - L_{43}(K_{51}K_{42}^* - K_{52}K_{41}^*)], \\ G_{D2} &= -\frac{L_{32}^*}{L_{42}^*}G_{D1}, \\ G_{D3} &= -\frac{1}{L_{53}} \left( L_{33} - L_{43}\frac{L_{32}^*}{L_{42}^*} \right) G_{D1}, \end{aligned}$$

$$(77)$$

$$\begin{aligned} G_{B1} &= \frac{L_{42}}{(L_{31}^* L_{42}^* - L_{32}^* L_{41}^*) K_{53} K_{42}^*} \left[ K_{53} (L_{51} K_{42}^* - L_{52} K_{41}^*) - L_{53} (K_{51} K_{42}^* - K_{52} K_{41}^*) \right] \\ G_{B2} &= -\frac{L_{32}^*}{L_{42}^*} G_{B1}, \\ G_{B3} &= -\frac{1}{L_{53}} \left( L_{33} - L_{43} \frac{L_{32}^*}{L_{42}^*} \right) G_{B1}. \end{aligned}$$

$$(78)$$

It can be observed clearly from Eqs. (75)-(78) that the field intensity factors depend only on the material properties and the applied

loads, and are independent of size of the yielding zone. This feature is similar to that based on the 3D EMPS model as presented above and to that using the corresponding 2D EMBD model (Zhao and Fan, 2008).

We also point out that even though the expression for the stress intensity factor given by Eq. (39) based on the EMPS model looks different than that given by Eq. (75a) based on the EMBD model, we would suspect they should be very close to each other based on the existing 2D analyses (e.g. Zhang et al., 2005; Fan et al., 2009). To numerically verify this point, we take the MEE medium as an example, which is made of the composite BaTiO<sub>3</sub>–CoFe<sub>2</sub>O<sub>4</sub> with CoFe<sub>2</sub>O<sub>4</sub> as matrix and BaTiO<sub>3</sub> as inhomogeneity, with the volume fraction of the inhomogeneity being  $V_i = 0.5$  (Zhao et al., 2007). For this MEE material, we have

$$\begin{split} L_{F2} &= 8.71989 \times 10^8, \quad L_{F3} = 9.22607 \times 10^5, \quad G_{F1} = 0.981534, \\ G_{F2} &= 8.55887 \times 10^8, \quad G_{F3} = 9.05570 \times 10^5. \end{split}$$

Substituting these parameters into Eq. (39) and Eq. (75a) gives the stress intensity factor

As we can see obviously that they are very close to each other. Also for this MEE composite, we find that the other constants in Eqs. (77) and (78) are

$$\begin{aligned} G_{D1} &= 1.92514 \times 10^{-11}, \quad G_{D2} = 0.016787, \quad G_{D3} \\ &= 0.0000177615, \quad G_{B1} = 1.82033 \times 10^{-9}, \quad G_{B2} \\ &= 1.58731, \quad G_{B3} = 0.00167945. \end{aligned} \tag{81}$$

Substituting the obtained extended intensity factors into the local *J*-integral (Zhao and Fan, 2008), it is found that the local *J*-integral is dominated by the stress intensity factor  $K_I^F$  and the contribution of the electric and magnetic intensity factors to the local *J*-integral can be neglected. In this sense, the EMPS model and the EMBD model are equivalent to each other in the analysis of fracture behaviors in MEE materials.

3.2. Case 2: c ≥ b

In this case, the dimensionless size of the electric yielding zone is given by

$$x_2 = \frac{1 + y_7}{\sqrt{1 + 2y_7}},\tag{82}$$

where

$$x_2 = \frac{c}{a}, \quad y_7 = \frac{E_0}{E_b}.$$
 (83)

The dimensionless size of the magnetic yielding zone is given by

$$x_1 = \frac{1 + y_8}{\sqrt{1 + 2y_8}},\tag{84}$$

in which

$$x_1 = b/a, \quad y_8 = (E_0 - (K_{42}/K_{52})H_0)/(E_b - (K_{42}/K_{52})H_b).$$
 (85)

#### 4. Conclusions

Analytical solutions are derived for the fracture problem of a penny shaped crack in 3D MEE media based on the nonlinear EMPS and EMBD models and using the extended displacement disconti-

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nuity method. Based on our solutions, the following conclusions can be drawn.

- (1) The sizes of the electric and magnetic yielding zones are different based on the two nonlinear models. They depend on the applied mechanical-electric-magnetic loadings, as well as on the polarization saturations and breakdown strengths.
- (2) The local extended intensity factors along the crack front based on EMPS or EMBD model are equal whether if the electric yielding zone is larger than the magnetic yielding one, and are independent of the electric and magnetic yielding zones and the yielding properties of the MEE materials. Furthermore, the stress intensity factor based on the EMBD model is very close to that based on the EMPS model.
- (3) Since the local J-integral can be calculated from the field intensity factors, the *J*-integral criterion  $J^{(l)} = J_c$  can be used to predict the fracture behavior of MEE materials under applied mechanical-electric-magnetic loadings. Based on the two nonlinear models, our analytical solutions show that the local J-integral is dominated by the stress intensity factor  $K_{I}^{F}$ . Thus, the K-criterion is equivalent to the J-criterion. Furthermore, since the stress intensity factor based on the EMPS model is very close to the one based on the EMBD model, the two nonlinear models are equivalent to each other in the analysis of fracture behaviors in MEE materials.

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 $\sum_{i=1}^{n}$ 

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#### **Appendix A. Material related constants**

The material-related constants in Eqs. (4)–(6) are given by

$$L_{31} = -\sum_{i=1}^{4} \vartheta_{i1} [-c_{13}D_i + (c_{33}A_i - e_{33}B_i - f_{33}C_i)s_i],$$

$$L_{32} = -\sum_{i=1}^{4} \vartheta_{i1} [-e_{31}D_i + (e_{33}A_i + \varepsilon_{33}B_i + g_{33}C_i)s_i],$$

$$L_{33} = -\sum_{i=1}^{4} \vartheta_{i1} [-f_{31}D_i + (f_{33}A_i + g_{33}B_i + \mu_{33}C_i)s_i];$$
(A1)

$$L_{41} = -\sum_{i=1}^{4} \vartheta_{i2} [-c_{13}D_i + (c_{33}A_i - e_{33}B_i - f_{33}C_i)s_i],$$

$$L_{42} = -\sum_{i=1}^{4} \vartheta_{i2} [-e_{31}D_i + (e_{33}A_i + \varepsilon_{33}B_i + g_{33}C_i)s_i],$$

$$L_{43} = -\sum_{i=1}^{4} \vartheta_{i2} [-f_{31}D_i + (f_{33}A_i + g_{33}B_i + \mu_{33}C_i)s_i];$$
(A2)

$$\begin{split} L_{51} &= -\sum_{i=1}^{4} \vartheta_{i3} [-c_{13}D_i + (c_{33}A_i - e_{33}B_i - f_{33}C_i)s_i], \\ L_{52} &= -\sum_{i=1}^{4} \vartheta_{i3} [-e_{31}D_i + (e_{33}A_i + \varepsilon_{33}B_i + g_{33}C_i)s_i], \\ L_{53} &= -\sum_{i=1}^{4} \vartheta_{i3} [-f_{31}D_i + (f_{33}A_i + g_{33}B_i + \mu_{33}C_i)s_i]. \end{split}$$
(A3)

where  $s_i$  are the roots of the material characteristic equation,  $\vartheta_{ii}$ ,  $A_i$ ,  $B_i$ ,  $C_i$  and  $D_i$  are all material-related constants given in Zhao et al. (2007).

#### References

- Beom, H.G., Kim, Y.H., Cho, C., Kim, C.B., 2006a. A crack with an electric displacement saturation zone in an electrostrictive material. Arch. Appl. Mech. 76, 19-31
- Beom, H.G., Kim, Y.H., Cho, C., Kim, C.B., 2006b. Asymptotic analysis of an impermeable crack in an electrostrictive material subjected to electric loading. Int. J. Solids Struct. 43, 6869–6886. Bhargava, R.R., Gupta, S., 2012. Mathematical model for crack arrest of a
- transversely cracked piezoelectromagnetic strip Part I. Appl. Math. Modell. 36, 3502-3512.
- Budiansky, B., Hutchinson, J.W., 1978. Analysis of closure in fatigue crack growth. J. Appl. Mech. 45, 267-275.
- Chaiyat, S., Jin, X., Keer, L.M., Kiattikomol, K., 2008. Analytical and numerical evaluation of crack-tip plasticity of an axisymmetrically loaded penny-shaped crack. C. R. Mecanique 336, 54–68.
- Chang, D.H., Kotousov, A., 2012. A strip yield model for two collinear cracks. Eng. Fract. Mech. 90, 121-128.
- Chen, M.C., Tang, R.J., 1997. A closed type solution of hyper-singular integral equation for a penny crack subjected to uniform pressure in an infinite homogeneous elastic solid. Shanghai J. Mech. (in Chinese) 18, 248-251.
- Collins, R.A., Cartwright, D.J., 2001. An analytical solution for two equal-length collinear strip yield cracks. Eng. Fract. Mech. 68, 915-924.
- Crapps, J., Daniewicz, S.R., 2010. Weight function based Dugdale model for mixedmode crackproblems with arbitrary crack surface tractions. Eng. Fract. Mech. 77.793-802.
- Danyluck, H.T., Singh, B.M., Vrbik, J., 1995. A Dugdale-type estimation of the plastic zone for a penny-shaped crack in a thick transversely isotropic layer due to radial shear. Eng. Fract. Mech. 51 (5), 735-740.
- Dugdale, D.S., 1960. Yielding of steel sheets containing slits. J. Mech. Phys. Solids 8, 100-104.
- Fan, C.Y., Zhao, M.H., 2011. Nonlinear fracture of 2D magnetoelectroelastic media:
- analytical and numerical solutions. Int. J. Solids Struct. 48, 2383–2392. Fan, C.Y., Zhao, M.H., Zhou, Y.H., 2009. Numerical solution of polarization saturation/dielectric breakdown model in 2D finite piezoelectric media. J. Mech. Phys. Solids 57, 1527-1544.
- Fan, C.Y., Zhao, Y.F., Zhao, M.H., Pan, E., 2012. Analytical solution of a semipermeable crack in a 2D piezoelectric medium based on the PS model. Mech. Res. Commun. 40, 34-40.
- Gao, C.F., Noda, N., Zhang, T.Y., 2006. Dielectric breakdown model for a conductive crack and electrode in piezoelectric materials. Int. J. Eng. Sci. 44 (3-4), 256-272.
- Gao, H.J., Barnett, D.M., 1996. An invariance property of local energy release rates in a strip saturation model of piezoelectric fracture. Int. J. Fract. 79, R25-R29.
- Gao, H.J., Zhang, T.Y., Tong, P., 1997. Local and global energy release rates for an electrically yielded crack in a piezoelectric ceramic. J. Mech. Phys. Solids 45, 491-510
- Janson, J., 1977. Dugdale-crack in a material with continuous damage formation. Eng. Fract. Mech. 9, 890-899.
- Jeong, K.M., Kim, I.O., Beom, H.G., 2004. Effect of electric displacement saturation on the stress intensity factor for a crack in a ferroelectric ceramic. Mech. Res. Commun. 31, 373-382.
- Lapusta, Y., Loboda, V., 2009. Electro-mechanical yielding for a limited permeable crack in an interlayer between piezoelectric materials. Mech. Res. Commun. 36. 183-192.
- Li, S., 2003. On saturation-strip model of a permeable crack in a piezoelectric ceramic. Acta Mech. 165, 47-71.
- Li, X., Chen, W., Wang, H., Wang, G., 2012. Crack tip plasticity of a penny-shaped Dugdale crack in a power-law graded elastic infinite medium. Eng. Fract. Mech. 88.1-14.
- Loboda, V., Lapusta, Y., Govorukha, V., 2008. Mechanical and electrical yielding for an electrically insulated crack in an interlayer between piezoelectric materials. Int. J. Eng. Sci. 46, 260-272.
- Loboda, V., Lapusta, Y., Sheveleva, A., 2010. Limited permeable crack in an interlayer between piezoelectric materials with different zones of electric saturation and mechanical yielding. Int. J. Solids Struct. 47, 1796–1806. McMeeking, R.M., 2001. Towards a fracture mechanics for brittle piezoelectric and
- dielectric materials. Int. J. Fract. 108, 25-41.
- Mou, Y., Han, R.P., 1994. Damage zones based on Dugdale model for materials. Int. J. Fract. 68, 245-259.
- Ru, C.Q., Mao, X., 1999. Conducting crack in a piezoelectric ceramics of limited electrical polarization. J. Mech. Phys. Solids 47, 2125-2146.
- Ru, C.O., 1999. Effect of electrical polarization saturation on stress intensity factors in a piezoelectric ceramic. Int. J. Solids Struct. 36, 869-883.
- Wang, T.C., 2000. Analysis of strip electric saturation model of crack problem in piezoelectric materials. Int. J. Solids Struct. 37, 6031-6049.
- Wu, S.X., Mai, Y.W., Cotterell, B., 1992. A model of fatigue crack growth based on Dugdale model and damage accumulation. Int. J. Fract. 57, 253-267.

- Xu, W., Wu, X.R., 2012. Weight functions and strip-yield model analysis for three collinear cracks. Eng. Fract. Mech. 85, 73–87. Zhang, N., Gao, C.F., 2012. Effects of electrical breakdown on a conducting crack or
- electrode in electrostrictive solids. Eur. J. Mech. A/Solids 32, 62-68.
- Zhang, T.Y., Gao, C.F., 2004. Fracture behaviors of piezoelectric materials. Theor. Appl. Fract. Mech. 41, 339-379.
- Zhang, T.Y., 2004. Dielectric breakdown model for an electrical impermeable crack in a piezoelectric material. Comput. Mater. Con. 1, 107–115.
   Zhang, T.Y., Zhao, M.H., Gao, C.F., 2005. The strip dielectric breakdown model. Int. J. Fract. 132, 311–327.
- Zhang, T.Y., Zhao, M.H., Tong, P., 2002. Fracture of piezoelectric ceramics. Adv. Appl.
- Mech. 38, 147–289.
   Zhao, M.H., Fan, C.Y., 2008. Strip electric-magnetic breakdown model in a magnetoelectroelastic medium. J. Mech. Phys. Solids 56 (12), 3441–3458.
   Zhao, M.H., Fan, C.Y., Yang, F., Liu, T., 2007. Analysis method of planar cracks of
- arbitrary shape in the isotropic plane of a three-dimensional transversely
- Isotrary shape in the isotropic plane medium. Int. J. Solids Struct. 44, 4505–4523.
   Zhao, M.H., Shen, Y.P., Liu, G.N., Liu, Y.J., 1999. Dugdale model solutions for a penny-shaped crack in three-dimensional transversely isotropic piezoelectric media by boundary-integral equation method. Eng. Anal. Bound. Elem. 23, 573–576.