RESEARCH PAPER

Analytical solutions of uniform extended dislocations and tractions over a circular area in anisotropic magnetoelectroelastic bimaterials

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Abstract In this paper, we derive the analytical solutions in a three-dimensional anisotropic magnetoelectroelastic bimaterial space subject to uniform extended dislocations and tractions within a horizontal circular area. By virtue of the Stroh formalism and Fourier transformation, the final expression of solutions in the physical domain contains only line integrals over $[0, 2\pi]$ rather than infinite integrals. As the reduced cases, the half-space and homogeneous full-space solutions can be directly derived from the present solutions. Also, in terms of material domains, the present solutions can be reduced to the piezoelectric, piezomagnetic, purely elastic materials with different symmetries of material property. To carry out numerical calculations, Gauss quadrature is adopted. In the numerical examples, the effect of different loading locations on the response at the interface is analyzed. It is shown that, when the magnetic traction or electric dislocation is applied, the physical quantities on the interface may not decrease monotonically as the loading area moves away from the interface. The distributions of different in-plane physical quantities on the upper and lower interfaces under various extended horizontal loadings are compared and the differences are discussed. The work presented in this paper can serve as benchmarks for future numerical studies in related research fields.

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X.-C. Shang Dept. of Applied Mechanics, University of Science and Technology, 100083 Beijing, China Keywords Anisotropic bimaterial space \cdot Magnetoelectroelastic \cdot Extended dislocations \cdot Extended tractions \cdot Circular loading \cdot Analytic solutions

1 Introduction

Magnetoelectroelastic (MEE) materials and composites belong to the smart or intelligent material systems. Due to their full coupling properties among magnetic, electric, and elastic fields, MEE materials can convert energies from one form to the other (among the mechanical, electric and magnetic ones), and thus they have potential applications, say, in energy harvest, among others [1, 2]. MEE materials and composites are also good candidates for manufacturing various sensors and actuators with applications in hi-tech areas such as ultrasonic and microwave fields [2, 3]. Besides, MEE composites offer engineers great opportunities to make structures capable of producing the desired internal or environment changes [4]. As such, MEE materials and composites are continuously receiving attentions from researchers in various fields.

MEE materials could contain various defects (such as cracks, inclusions and inhomogeneities) arising from the manufacturing process, even be damaged under service environments, which has motivated some recent studies. Li [4] studied the multiple inclusions and inhomogeneities embedded in an infinite three-dimensional (3D) MEE matrix and developed a numerical method for evaluating the MEE Eshelby tensors in the general material case with ellipsoidal inclusion shape. Liu et al. [5] derived the Green's function for an infinite two-dimensional (2D) anisotropic MEE medium containing an elliptical cavity using the extended Stroh formalism combined with the conformal mapping and the Laurent series expansion. The 2D polygonal and generalshaped inclusion problems in anisotropic MEE full-plane, half-plane, and bimaterial-plane were also solved [6-8]. Gao [8] investigated the fracture problem for an elliptical cavity in a MEE solid and obtained the closed-form solution for a mode III crack. Qin [9] considered various defects embedded in an infinite MEE matrix induced by a line temperature discontinuity and a line heat source and obtained the Green's functions using the Stroh formalism, conformal mapping and perturbation technique. Zhao et al. [10] derived the extended displacement discontinuity Green's functions for 3D transversely isotropic MEE media using the integral equation method and studied two coplanar and parallel rectangular cracks. Wang and Pan [11] derive the solutions for 2D and (quasi-static) time-dependent Green's functions in anisotropic MEE multiferroic bimaterials with a viscous interface subject to extended line force and dislocation by virtue of the unified Stroh formalism. To the authors' knowledge, however, no solution is available to the case where the extended dislocations and tractions are applied over a circular area within a general anisotropic MEE bimaterial space. This motivates the present study.

In this paper, we solve analytically the MEE fields in a 3D general anisotropic MEE bimaterial space subject to uniform extended dislocations and tractions applied to a horizontal circular area. We first derive the solutions in the Fourier-transformed domain using the Stroh formalism. The solutions are comprised of two parts: An infinite-space or Kelvin-type solution, and the complementary term or Mindlin-type solution. Making use of the integral properties of the Bessel functions, we derive the final solutions in the physical domain in an explicit and elegant form, which contains only the line integral over $[0, 2\pi]$. It is worth to note that our solutions are very general and contain the solutions of the piezoelectric, piezomagnetic or magnetoelectric material cases and the half-space or full-space domains as the special cases. In numerical examples, the responses on the interface between two material half spaces under different loadings are studied and some interesting features are observed. The effect of the circular loading distance to the interface on the response is also analyzed. It is shown that the displacement on the interface does not always decrease monotonically as the loading circle moves away from the interface.

2 Mathematical model

2.1 The governing equations

For MEE materials, the extended equilibrium equations in terms of the extended stresses σ_{iJ} can be expressed as

$$\sigma_{iJ,i} + f_J = 0,\tag{1}$$

where f_J is the extended body force, and repeated lowercase (uppercase) indices take the summation from 1 to 3 (1 to 5). An index following the subprime "*i*" indicates the derivative with respect to the coordinate x_i .

The generalized and fully coupled constitutive equations in terms of the extended material coefficients c_{iJKl} have the following form

$$\sigma_{iJ} = c_{iJKl} u_{K,l}.$$
 (2)

Substituting Eq. (2) into Eq. (1), we obtain the governing equations in terms of the extended displacements u_K as

$$c_{iJKl}u_{K,li} + f_J = 0.$$
 (3)

It should be noted that the governing equations for the fully coupled MEE systems (3) are exactly the same in the mathematical form as its purely elastic counterpart, except for the difference in the dimension of the involved quantities. This implies that the solution method developed in anisotropic elasticity can be directly applied to the anisotropic MEE case. Furthermore, once we find the general solution to the 3D fully coupled MEE system (3), we can reduce our solution to the 3D piezoelectric, piezomagnetic, and purely elastic cases by setting the corresponding material constants to be zero. For example, reducing the uppercase index from 5 to 4 (as its upper limit) will give the solutions to either the piezoelectric or piezomagnetic case. For the piezomagnetic case, the piezoelectric quantities associated with index 4 need to be replaced by the piezomagnetic quantities associated with index 5. For the anisotropic elastic case, all the indices are limited to 3.

2.2 MEE bimaterial system

We now consider the bimaterial system of general linear anisotropic MEE materials, as shown in Fig. 1. The upper $(x_3 > 0 \text{ or } z > 0)$ and lower $(x_3 < 0 \text{ or } z < 0)$ half spaces are assigned as Materials 1 and 2, respectively. Within a horizontal circular area of radius *R* in Material 1 at $x_3 = h$, either the extended dislocation vector or extended traction vector is applied. The goal of this article is to derive the solutions of the elastic, electric, and magnetic fields in response to the uniform "extended" dislocation and traction given in the circular area.



Fig. 1 Sketch of an anisotropic MEE bimaterial space subject to a uniform traction or dislocation within a circular area of radius *R* which is centered at (x, y, z) = (0, 0, h)

In order to find the solution to this problem, the bimaterial full-space is divided into three subdomains: $x_3 < 0$, $0 < x_3 < h$, and $x_3 > h$. Since there is no body source in these subdomains, we have $f_J = 0$ in Eq. (3). However, the solutions should satisfy the following conditions: the solution in subdomains $x_3 < 0$ and $0 < x_3 < h$ should satisfy the continuity conditions on the extended displacement and traction across the interface $x_3 = 0$; the solution in subdomains $0 < x_3 < h$ and $x_3 > h$ should satisfy the suitable discontinuity conditions at $x_3 = h$; the solutions in the upper and lower half spaces should approach zero when the field variable $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ approaches infinity. These conditions are addressed below in terms of their corresponding equations.

The interface of the two half spaces is assumed to be perfect; namely the extended displacement vector $\boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5)^{\mathrm{T}} \equiv (u_1, u_2, u_3, \varphi, \psi)^{\mathrm{T}}$ and the extended traction vector $\boldsymbol{t} = (t_1, t_2, t_3, t_4, t_5)^{\mathrm{T}} = (t_1, t_2, t_3, D_3, B_3)^{\mathrm{T}}$ are required to satisfy the following continuity conditions across the interface $x_3 = 0$,

$$u(x_1, x_2, +0) - u(x_1, x_2, -0) = \mathbf{0},$$

$$t(x_1, x_2, +0) - t(x_1, x_2, -0) = \mathbf{0}.$$
(4)

On the horizontal plane $x_3 = h$ between the subdomains $0 < x_3 < h$ and $x_3 > h$ within the same material domain, the extended displacement and traction vectors should be continuous outside the loading circle r > R. However, within the circle $r \leq R$, the following discontinuity conditions should be satisfied.

For the case of applied extended dislocations, the continuity conditions at $x_3 = h$ are

$$u_{J}(x_{1}, x_{2}, h + 0) - u_{J}(x_{1}, x_{2}, h - 0)$$

$$= \begin{cases} d_{J} = \begin{cases} b_{j}, & J = j = 1, 2, 3, \\ \Delta \varphi, & J = 4, \\ \Delta \psi, & J = 5, \end{cases} \quad 0 \leq r \leq R, \\ 0, & r > R, \end{cases}$$
(5)

$$t_J(x_1, x_2, h + 0) - t_J(x_1, x_2, h - 0) = 0, \quad 0 < r < \infty,$$

where the extended dislocations d_J are the given constants which include the elastic, electric and magnetic dislocations applied over the circular area $r \leq R$.

For the case of applied extended tractions, we have

$$u_J(x_1, x_2, h+0) - u_J(x_1, x_2, h-0) = 0, \quad 0 < r < \infty,$$

$$t_J(x_1, x_2, h+0) - t_J(x_1, x_2, h-0)$$

$$= \begin{cases} T_J = \begin{cases} T_j, & J = j = 1, 2, 3, \\ D_3, & J = 4, \\ B_3, & J = 5, \\ 0, & r > R, \end{cases}$$
(6)

where the extended tractions T_J are the given constants which include the elastic, electric and magnetic tractions ap-

plied over the circular area $r \leq R$.

In addition, since the extended displacements and stresses at infinity should be zero, we have

$$\lim_{|x| \to \infty} |u_j| = 0,$$

$$\lim_{|x| \to \infty} |\sigma_{ij}| = 0.$$
(7)

Thus, the boundary value problem is to solve the governing equation (3) (with $f_J = 0$) in the three subdomains subject to conditions (5)–(7).

3 General solution

3.1 General solution in the Fourier-transformed domain

The two-dimensional Fourier transform of a function $F(x_1, x_2, x_3)$ is introduced as

$$\tilde{F}(k_1, k_2, x_3) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(x_1, x_2, x_3) e^{i(k_1 x_1 + k_2 x_2)} dx_1 dx_2,$$
(8)

where $i = \sqrt{-1}$ is the unit of the imaginary number. The transformed variables can be expressed in terms of polar co-ordinates in the Fourier-transformed domain as

$$k_1 = \eta \cos \theta, \qquad k_2 = \eta \sin \theta,$$

where η and θ are, respectively, the radial and circumferential coordinates.

The governing equation (1) for each subdomain in the Fourier-transformed domain becomes (with $f_J = 0$)

$$C_{3JK3}\tilde{u}_{K,33} - i(C_{\alpha JK3} + C_{3JK\alpha})k_{\alpha}\tilde{u}_{K,3}$$
$$- C_{\alpha JK\beta}k_{\alpha}k_{\beta}\tilde{u}_{K} = 0, \quad \alpha, \beta = 1, 2,$$
(9)

where the repeated Greek indices α and β take the summation from 1 to 2. In terms of the Stroh formalism, the general solution to Eq. (9) can be written as

$$\tilde{\boldsymbol{u}}(k_1, k_2, x_3) = \mathrm{i}\eta^{-1} \bar{\boldsymbol{A}} \langle \mathrm{e}^{-\mathrm{i}\bar{p}_* \eta x_3} \rangle \bar{\boldsymbol{q}} + \mathrm{i}\eta^{-1} \boldsymbol{A} \langle \mathrm{e}^{-\mathrm{i}p_* \eta x_3} \rangle \boldsymbol{q}', \qquad (10)$$

where the overbar denotes the complex conjugate, \overline{q} and q' are two unknown complex vectors, and the matrices $\langle e^{-ip_*\eta x_3} \rangle$ and A are

$$\langle e^{-ip_*\eta x_3} \rangle = \text{diag}[e^{-ip_1\eta x_3}, e^{-ip_2\eta x_3}, e^{-ip_3\eta x_3}, e^{-ip_4\eta x_3}, e^{-ip_5\eta x_3}],$$
(11)

$$A = [a_1, a_2, a_3, a_4, a_5].$$

The eigenvalue p_m and associated eigenvector \boldsymbol{a}_m are calculated from the following eigenequation

$$[\boldsymbol{Q} + p_m(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p_m^2 \boldsymbol{T}] \boldsymbol{a}_m = \boldsymbol{0}, \quad m = 1, 2, \cdots, 10, \quad (12)$$

where the involved real matrices are defined as

 $(\boldsymbol{Q})_{IK} = C_{1IK1} \cos^2 \theta + C_{2IK2} \sin^2 \theta$ $+ (C_{1IK2} + C_{2IK1}) \cos \theta \sin \theta,$ $(\boldsymbol{R})_{IK} = C_{1IK3} \cos \theta + C_{2IK3} \sin \theta,$

 $(\boldsymbol{T})_{IK} = C_{3IK3}.$

There are five pairs of complex conjugate eigenvalues and associated eigenvectors from Eq. (12). We order the first five with Im $(p_J) > 0$ and a_J , and the remaining five as $p_{J+5} = \overline{p}_J, a_{J+5} = \overline{a}_J (J = 1-5).$

The extended stresses on the x_3 = constant plane are divided into two parts: the extended traction vector

$$\boldsymbol{t} = (\sigma_{31}, \sigma_{32}, \sigma_{33}, \sigma_{34}, \sigma_{35})^{\mathrm{T}} = (\sigma_{31}, \sigma_{32}, \sigma_{33}, D_3, B_3)^{\mathrm{T}},$$

and the extended in-plane stress vector

$$\mathbf{s} = (\sigma_{11}, \sigma_{12}, \sigma_{22}, \sigma_{14}, \sigma_{24}, \sigma_{15}, \sigma_{25})^{\mathrm{T}}$$

$$\equiv (\sigma_{11}, \sigma_{12}, \sigma_{22}, D_1, D_2, B_1, B_2)^{\mathrm{T}},$$

whose general solutions in the Fourier-transformed domain can be written as

$$\widetilde{\boldsymbol{t}}(k_1, k_2, x_3) = \widetilde{\boldsymbol{B}} \langle e^{-i\widetilde{p}_*\eta x_3} \rangle \widetilde{\boldsymbol{q}} + \boldsymbol{B} \langle e^{-ip_*\eta x_3} \rangle \boldsymbol{q}',
\widetilde{\boldsymbol{s}}(k_1, k_2, x_3) = \widetilde{\boldsymbol{C}} \langle e^{-i\widetilde{p}_*\eta x_3} \rangle \widetilde{\boldsymbol{q}} + \boldsymbol{C} \langle e^{-ip_*\eta x_3} \rangle \boldsymbol{q}'.$$
(13)

For convenience, we introduce the following real matrices

$$\boldsymbol{U} = \boldsymbol{H}_1 \cos \theta + \boldsymbol{H}_2 \sin \theta,$$

$$\boldsymbol{V} = \boldsymbol{H}_3$$

where the real matrices H_i of dimensions 7×5 are defined as

$$\boldsymbol{H}_{i} = (\boldsymbol{H}_{i})_{sK} = \begin{cases} C_{s1Ki}, & s = 1, \\ C_{(s-1)2Ki}, & s = 2, 3, \\ C_{(s-3)4Ki}, & s = 4, 5, \\ C_{(s-5)5Ki}, & s = 6, 7, \end{cases} \quad i = 1, 2, 3.$$

As a result, the matrices B and C in Eq. (13) can be expressed as

$$B = R^{\mathrm{T}}A + TAP,$$

$$C = UA + VAP,$$
(14)

$$P = diag[p_1, p_2, p_3, p_4, p_5].$$

It should be noted that the matrices A, B and C, the vectors \overline{q} and q', and the eigenvalues p_J in solutions (10) and (13) are all functions of the circumferential polar coordinate θ .

3.2 The general Fourier-domain solution in the three subdomains

In the Fourier transformed domain, the continuity conditions (4) on the interface $x_3 = 0$ becomes

$$\tilde{\boldsymbol{u}}|_{x_{3}=+0} - \tilde{\boldsymbol{u}}|_{x_{3}=-0} = \boldsymbol{0},$$

$$\tilde{\boldsymbol{t}}|_{x_{3}=+0} - \tilde{\boldsymbol{t}}|_{x_{3}=-0} = \boldsymbol{0}.$$
(15)

On the loading level $x_3 = h$, the discontinuity conditions (5) and (6) in the Fourier domain are

$$\tilde{\boldsymbol{u}}|_{x_3=h+0} - \tilde{\boldsymbol{u}}|_{x_3=h-0} = \begin{cases} \tilde{\boldsymbol{d}}(\eta,\theta), & \text{for dislocation,} \\ \boldsymbol{0}, & \text{for traction,} \end{cases}$$

$$\tilde{\boldsymbol{t}}|_{x_3=h+0} - \tilde{\boldsymbol{t}}|_{x_3=h-0} = \begin{cases} \boldsymbol{0}, & \text{for dislocation,} \\ \tilde{\boldsymbol{T}}(\eta,\theta), & \text{for traction,} \end{cases}$$
(16)

where $\tilde{d}(\eta, \theta)$ and $\tilde{T}(\eta, \theta)$ are, respectively, the Fourier transformation of the given uniform dislocation and traction over the circle. Namely

$$\begin{bmatrix} \tilde{\boldsymbol{d}}(\eta,\theta) \\ \tilde{\boldsymbol{T}}(\eta,\theta) \end{bmatrix} = \int_{0}^{2\pi} \int_{0}^{\infty} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{T} \end{bmatrix} e^{i\eta r \cos(\varphi-\theta)} r dr d\phi.$$
(17)

In addition, the condition (7) at infinity in the Fouriertransformed domain becomes

$$\lim_{|x| \to \infty} |\tilde{\boldsymbol{\iota}}| = 0,$$

$$\lim_{|x| \to \infty} |\tilde{\boldsymbol{\iota}}| = 0.$$
(18)

As in Ref. [12], we now assume that the solution in the upper half space contains two parts—the full-space solution and the complimentary part, whilst the solution in the lower half space contains only the complimentary part. In other words, the solutions in the three Fourier-transformed subdomains can be expressed as follows.

For
$$x_{3} > h$$
 (in Material 1)
 $\tilde{\boldsymbol{u}}^{(1)}(k_{1}, k_{2}, x_{3}) = \pm i\eta^{-1} \overline{\boldsymbol{A}}^{(1)} \langle e^{-i\overline{p}^{(1)}_{*}\eta(x_{3}-h)} \rangle \overline{\boldsymbol{q}}^{\infty}$
 $-i\eta^{-1} \overline{\boldsymbol{A}}^{(1)} \langle e^{-i\overline{p}^{(1)}_{*}\etax_{3}} \rangle \overline{\boldsymbol{q}}^{(1)},$
 $\tilde{\boldsymbol{t}}^{(1)}(k_{1}, k_{2}, x_{3}) = \pm \overline{\boldsymbol{B}}^{(1)} \langle e^{-i\overline{p}^{(1)}_{*}\eta(x_{3}-h)} \rangle \overline{\boldsymbol{q}}^{\infty}$
 $-\overline{\boldsymbol{B}}^{(1)} \langle e^{-i\overline{p}^{(1)}_{*}\etax_{3}} \rangle \overline{\boldsymbol{q}}^{(1)},$
(19)
 $\tilde{\boldsymbol{s}}^{(1)}(k_{1}, k_{2}, x_{3}) = \pm \overline{\boldsymbol{C}}^{(1)} \langle e^{-i\overline{p}^{(1)}_{*}\eta(x_{3}-h)} \rangle \overline{\boldsymbol{q}}^{\infty}$
 $-\overline{\boldsymbol{C}}^{(1)} \langle e^{-i\overline{p}^{(1)}_{*}\etax_{3}} \rangle \overline{\boldsymbol{q}}^{(1)}.$

For
$$0 \le x_3 < h$$
 (in Material 1)

$$\tilde{\boldsymbol{u}}^{(1)}(k_1, k_2, x_3) = i\eta^{-1} \boldsymbol{A}^{(1)} \langle e^{-ip_*^{(1)}\eta(x_3-h)} \rangle \boldsymbol{q}^{\infty} -i\eta^{-1} \overline{\boldsymbol{A}}^{(1)} \langle e^{-i\overline{p}_*^{(1)}\eta x_3} \rangle \overline{\boldsymbol{q}}^{(1)},
\tilde{\boldsymbol{t}}^{(1)}(k_1, k_2, x_3) = \boldsymbol{B}^{(1)} \langle e^{-ip_*^{(1)}\eta(x_3-h)} \rangle \boldsymbol{q}^{\infty} -\overline{\boldsymbol{B}}^{(1)} \langle e^{-i\overline{p}_*^{(1)}\eta x_3} \rangle \overline{\boldsymbol{q}}^{(1)},$$
(20)

$$\tilde{\boldsymbol{s}}^{(1)}(k_1, k_2, x_3) = \boldsymbol{C}^{(1)} \langle \mathrm{e}^{-\mathrm{i} p_*^{(1)} \eta(x_3 - h)} \rangle \boldsymbol{q}^{\infty}$$
$$- \boldsymbol{\overline{C}}^{(1)} \langle \mathrm{e}^{-\mathrm{i} \boldsymbol{\overline{p}}_*^{(1)} \eta x_3} \rangle \boldsymbol{\overline{q}}^{(1)}.$$

For $x_3 < 0$ (in Material 2)

$$\tilde{\boldsymbol{u}}^{(2)}(k_1, k_2, x_3) = i\eta^{-1} \boldsymbol{A}^{(2)} \langle e^{-ip_*^{(-)} \eta x_3} \rangle \boldsymbol{q}^{(2)},$$

$$\tilde{\boldsymbol{t}}^{(2)}(k_1, k_2, x_3) = \boldsymbol{B}^{(2)} \langle e^{-ip_*^{(2)} \eta x_3} \rangle \boldsymbol{q}^{(2)},$$

$$\tilde{\boldsymbol{s}}^{(2)}(k_1, k_2, x_3) = \boldsymbol{C}^{(2)} \langle e^{-ip_*^{(2)} \eta x_3} \rangle \boldsymbol{q}^{(2)}.$$
(21)

The superscripts "1" and "2" denote, respectively, the quantities in Materials 1 and 2, and $\bar{q}^{(1)}$, $q^{(2)}$ and q^{∞} are the three unknown vectors to be determined by conditions (15)–(18). It should be noted that the positive "+" and negative "–" signs in Eq. (19) are for the dislocation and traction cases, respectively.

3.3 Determination of unknown vectors in the solutions

First, the complex vector \mathbf{q}^{∞} can be determined by the discontinuity condition (16) at $x_3 = h$. By means of the orthogonal normalization identity [13] we can verify that the discontinuity condition (16) is satisfied if the unknown vector \mathbf{q}^{∞} in solutions (19) and (20) takes the following expression

$$\boldsymbol{q}^{\infty} = \begin{cases} -\mathrm{i}\eta(\boldsymbol{B}^{(1)})^{\mathrm{T}}\tilde{\boldsymbol{d}}(\eta,\theta), & \text{for dislocation,} \\ (\boldsymbol{A}^{(1)})^{\mathrm{T}}\tilde{\boldsymbol{T}}(\eta,\theta), & \text{for traction.} \end{cases}$$
(22)

Then, substituting solutions (20) and (21) into continuity conditions (15) on the interface $x_3 = 0$ yields

$$\boldsymbol{A}^{(2)}\boldsymbol{q}^{(2)} + \boldsymbol{\bar{A}}^{(1)}\boldsymbol{\bar{q}}^{(1)} = \boldsymbol{A}^{(1)}\langle e^{ip_*^{(1)}\eta h} \rangle \boldsymbol{q}^{\infty},$$

$$\boldsymbol{B}^{(2)}\boldsymbol{q}^{(2)} + \boldsymbol{\bar{B}}^{(1)}\boldsymbol{\bar{q}}^{(1)} = \boldsymbol{B}^{(1)}\langle e^{ip_*^{(1)}\eta h} \rangle \boldsymbol{q}^{\infty}.$$
(23)

To solve the unknown vectors $\overline{q}^{(1)}$ and $q^{(2)}$ from Eq. (23), we introduce the matrices

$$\boldsymbol{M}^{(\alpha)} = -\mathrm{i}\boldsymbol{B}^{(\alpha)}(\boldsymbol{A}^{(\alpha)})^{-1}, \quad \alpha = 1, 2.$$
(24)

Substituting $\mathbf{B}^{(\alpha)} = i\mathbf{M}^{(\alpha)}\mathbf{A}^{(\alpha)}$ ($\alpha = 1, 2$) into Eq. (23) results in a simple expression of the unknown vectors

$$\vec{q}^{(1)} = \boldsymbol{G}_1 \langle \mathrm{e}^{\mathrm{i} p_*^{(1)} \eta h} \rangle \boldsymbol{q}^{\infty},
\boldsymbol{q}^{(2)} = \boldsymbol{G}_2 \langle \mathrm{e}^{\mathrm{i} p_*^{(1)} \eta h} \rangle \boldsymbol{q}^{\infty},$$
(25)

where the matrices G_1 and G_2 are given by

$$G_{1} = -(\bar{A}^{(1)})^{-1}(\bar{M}^{(1)} + M^{(2)})^{-1}(M^{(1)} - M^{(2)})A^{(1)},$$

$$G_{2} = (A^{(2)})^{-1}(\bar{M}^{(1)} + M^{(2)})^{-1}(M^{(1)} + \bar{M}^{(1)})A^{(1)}.$$
(26)

3.4 General solution in the physical domain

The Fourier inverse transform of the function $\tilde{F}(k_1, k_2, x_3)$ is

$$F(x_1, x_2, x_3) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{F}(k_1, k_2, x_3) e^{-i(k_1 x_1 + k_2 x_2)} dk_1 dk_2.$$
(27)

We introduce the cylindrical polar coordinates in the physical domain

 $x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z.$

Then, Eq. (27) becomes

$$F(r,\varphi,z) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{+\infty} \tilde{F}(\eta,\theta,z) e^{-i\eta r \cos(\varphi-\theta)} \eta d\eta d\theta.$$
(28)

Applying the Fourier inverse transform to Eqs. (19)–(21), we obtain the extended displacements and stresses in the three physical subdomains as follows.

For z > h (in Material 1)

$$\begin{bmatrix} \boldsymbol{u}^{(1)}(r,\varphi,z) \\ \boldsymbol{t}^{(1)}(r,\varphi,z) \\ \boldsymbol{s}^{(1)}(r,\varphi,z) \end{bmatrix} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} \left(\pm \begin{bmatrix} i\eta^{-1} \bar{\boldsymbol{A}}^{(1)} \\ \bar{\boldsymbol{B}}^{(1)} \\ \bar{\boldsymbol{C}}^{(1)} \end{bmatrix} \boldsymbol{K}_+^{\infty} \bar{\boldsymbol{q}}^{\infty} - \begin{bmatrix} i\eta^{-1} \bar{\boldsymbol{A}}^{(1)} \\ \bar{\boldsymbol{B}}^{(1)} \\ \bar{\boldsymbol{C}}^{(1)} \end{bmatrix} \boldsymbol{K}_1 \boldsymbol{q}^{\infty} \right) \eta d\eta d\theta.$$
(29)

For
$$0 \le z < h$$
 (in Material 1)

$$\begin{bmatrix} \boldsymbol{u}^{(1)}(r,\varphi,z) \\ \boldsymbol{t}^{(1)}(r,\varphi,z) \\ \boldsymbol{s}^{(1)}(r,\varphi,z) \end{bmatrix} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} \left(\begin{bmatrix} i\eta^{-1}\boldsymbol{A}^{(1)} \\ \boldsymbol{B}^{(1)} \\ \boldsymbol{C}^{(1)} \end{bmatrix} \boldsymbol{K}_-^{\infty} \boldsymbol{q}^{\infty} - \begin{bmatrix} i\eta^{-1}\boldsymbol{\overline{A}}^{(1)} \\ \boldsymbol{\overline{B}}^{(1)} \\ \boldsymbol{\overline{C}}^{(1)} \end{bmatrix} \boldsymbol{K}_1 \boldsymbol{q}^{\infty} \right) \eta d\eta d\theta.$$
(30)

For z < 0 (in Material 2)

$$\begin{bmatrix} \boldsymbol{u}^{(2)}(r,\varphi,z) \\ \boldsymbol{t}^{(2)}(r,\varphi,z) \\ \boldsymbol{s}^{(2)}(r,\varphi,z) \end{bmatrix} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} \begin{bmatrix} i\eta^{-1}\boldsymbol{A}^{(2)} \\ \boldsymbol{B}^{(2)} \\ \boldsymbol{C}^{(2)} \end{bmatrix} \boldsymbol{K}_2 \boldsymbol{q}^{\infty} \eta \mathrm{d}\eta \mathrm{d}\theta. \quad (31)$$

The sign "+" and "-" in solution (29) again is, respectively, for the dislocation and traction case, and the involved matrices are defined as

$$\begin{aligned} (\boldsymbol{K}_{+}^{\infty})_{IJ} &= e^{-i\eta[r\cos(\varphi-\theta)+\vec{p}_{I}^{(1)}(z-h)]} \delta_{IJ}, \\ (\boldsymbol{K}_{-}^{\infty})_{IJ} &= e^{-i\eta[r\cos(\varphi-\theta)+\vec{p}_{I}^{(1)}(z-h)]} \delta_{IJ}, \\ (\boldsymbol{K}_{1})_{IJ} &= e^{-i\eta[r\cos(\varphi-\theta)+\vec{p}_{I}^{(1)}z-p_{J}^{(1)}h]} (\boldsymbol{G}_{1})_{IJ}, \\ (\boldsymbol{K}_{2})_{IJ} &= e^{-i\eta[r\cos(\varphi-\theta)+p_{I}^{(2)}z-p_{J}^{(1)}h]} (\boldsymbol{G}_{2})_{IJ}. \end{aligned}$$

$$(32)$$

Up to now, in the three physical domains, we have found the general solutions (29)–(31) which are expressed by double integrals of the known functions of two variables θ and η . In those solutions, for each material domain ($\alpha = 1, 2$), the eigenvalues $p_I^{(\alpha)}(I = 1, 2, 3, 4, 5)$ and eigenmatrix $\mathbf{A}^{(\alpha)}$ can be solved from the eigenequation (12) and the definition (11), the matrices $B^{(\alpha)}$ and $C^{(\alpha)}$ are from relation (14), and the matrices A_{α} can be calculated from expressions (24) and (26). They are all functions of the variable θ only. The vector q^{∞} is given by expressions (22) and (17), and thus, in general, is associated with both θ and η ; however, for the uniform traction and dislocation loading case, the involved infinite integral with respect to η can be carried out. This is discussed below.

4 Solutions for uniform dislocation and traction cases

For the case of applied uniform dislocation or traction within a circle of radius r = R, we have

$$\begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{T} \end{bmatrix} = H(R-r) \begin{bmatrix} \boldsymbol{d}_0 \\ \boldsymbol{T}_0 \end{bmatrix},$$
(33)

where \boldsymbol{d}_0 and \boldsymbol{T}_0 are constant vectors.

$$H(R - r) = \begin{cases} 1, & r \le R \\ 0, & r > R \end{cases}$$
 is the Heaviside function. Then
the integration with respect to infinity in Eq. (17) can be

the integration with respect to infinity in Eq. (17) can be changed to the integration with respect to R.

Making use of the following expressions [14]

$$\int_{0}^{2\pi} e^{i\eta r \cos(\varphi - \theta)} d\theta = 2\pi J_0(\eta r),$$
$$\int_{0}^{R} J_0(\eta r) r dr = \frac{R J_1(\eta R)}{\eta},$$

and from Eq. (17), we find that the Fourier transformation of the uniform load is

$$\begin{bmatrix} \tilde{\boldsymbol{d}} \\ \tilde{\boldsymbol{T}} \end{bmatrix} = 2\pi R \frac{J_1(\eta R)}{\eta} \begin{bmatrix} \boldsymbol{d}_0 \\ \boldsymbol{T}_0 \end{bmatrix}.$$
(34)

Thus, using the infinite integral formulae [14]

$$\int_{0}^{\infty} e^{-at} J_{1}(bt) \begin{bmatrix} t^{-1} \\ 1 \\ t \end{bmatrix} dt = \begin{bmatrix} \frac{1}{b} \left(\sqrt{a^{2} + b^{2}} - a \right) \\ \frac{1}{b} \left(1 - \frac{a}{\sqrt{a^{2} + b^{2}}} \right) \\ \frac{b}{(a^{2} + b^{2})^{3/2}} \end{bmatrix},$$

the infinite integral with respect to η in solutions (29)–(31) can be further carried out, as discussed below.

4.1 Solutions for uniform extended dislocation case

(1) 7

Under a uniform extended dislocation d_0 within the circle, we find, from Eqs. (22) and (33)

$$\boldsymbol{q}^{\infty} = -2\pi \mathrm{i}RJ_1(\eta R)(\boldsymbol{B}^{(1)})^{\mathrm{T}}\boldsymbol{d}_0.$$

Thus, we have the following analytical solutions in the three subdomains.

For z > h (in Material 1)

$$\boldsymbol{u}^{(1)}(r,\varphi,z) \\ \boldsymbol{t}^{(1)}(r,\varphi,z) \\ \boldsymbol{s}^{(1)}(r,\varphi,z) \end{bmatrix} = \frac{1}{2\pi} \int_{0}^{2\pi} \left(- \left[\frac{\bar{\boldsymbol{A}}^{(1)} \boldsymbol{G}_{u}^{h+}}{\bar{\boldsymbol{B}}^{(1)} \boldsymbol{G}_{\sigma}^{h+}} \right] (\bar{\boldsymbol{B}}^{(1)})^{\mathrm{T}} \\ - \left[\frac{\bar{\boldsymbol{A}}^{(1)} \boldsymbol{G}_{u}^{(1)}}{\bar{\boldsymbol{B}}^{(1)} \boldsymbol{G}_{\sigma}^{(1)}} \right] (\boldsymbol{B}^{(1)})^{\mathrm{T}} \right) \mathrm{d}\boldsymbol{\theta} \cdot \boldsymbol{d}_{0},$$
(35)

For $0 \le z < h$ (in Material 1)

$$\begin{bmatrix} \boldsymbol{u}^{(1)}(r,\varphi,z) \\ \boldsymbol{t}^{(1)}(r,\varphi,z) \\ \boldsymbol{s}^{(1)}(r,\varphi,z) \end{bmatrix}$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \begin{bmatrix} \boldsymbol{A}^{(1)} \boldsymbol{G}_{u}^{h-} - \bar{\boldsymbol{A}}^{(1)} \boldsymbol{G}_{u}^{(1)} \\ \boldsymbol{B}^{(1)} \boldsymbol{G}_{\sigma}^{h-} - \bar{\boldsymbol{B}}^{(1)} \boldsymbol{G}_{\sigma}^{(1)} \\ \boldsymbol{C}^{(1)} \boldsymbol{G}_{\sigma}^{h-} - \bar{\boldsymbol{C}}^{(1)} \boldsymbol{G}_{\sigma}^{(1)} \end{bmatrix} (\boldsymbol{B}^{(1)})^{\mathrm{T}} \mathrm{d}\boldsymbol{\theta} \cdot \boldsymbol{d}_{0}, \quad (36)$$

For z < 0 (in Material 2)

$$\begin{bmatrix} \boldsymbol{u}^{(2)}(r,\varphi,z) \\ \boldsymbol{t}^{(2)}(r,\varphi,z) \\ \boldsymbol{s}^{(2)}(r,\varphi,z) \end{bmatrix} = \frac{1}{2\pi} \int_{0}^{2\pi} \begin{bmatrix} \boldsymbol{A}^{(2)} \boldsymbol{G}_{u}^{(2)} \\ \boldsymbol{B}^{(2)} \boldsymbol{G}_{\sigma}^{(2)} \\ \boldsymbol{C}^{(2)} \boldsymbol{G}_{\sigma}^{(2)} \end{bmatrix} \left(\boldsymbol{B}^{(1)} \right)^{\mathrm{T}} \mathrm{d}\boldsymbol{\theta} \cdot \boldsymbol{d}_{0}, \qquad (37)$$

where

$$\begin{aligned} \left(\boldsymbol{G}_{u}^{h\pm}\right)_{IJ} &= \left(1 - \frac{\mathrm{i}[r\cos(\varphi - \theta) + p_{I}^{\pm}(z - h)]}{\sqrt{R^{2} - [r\cos(\varphi - \theta) + p_{I}^{\pm}(z - h)]^{2}}}\right) \boldsymbol{\delta}_{IJ}, \\ \left(\boldsymbol{G}_{u}^{(\alpha)}\right)_{IJ} &= \left(1 - \frac{\mathrm{i}[r\cos(\varphi - \theta) + \hat{p}_{I}^{(\alpha)}z - p_{J}^{(1)}h]}{\sqrt{R^{2} - [r\cos(\varphi - \theta) + \hat{p}_{I}^{(\alpha)}z - p_{J}^{(1)}h]^{2}}}\right) (\boldsymbol{G}_{\alpha})_{IJ}, \end{aligned}$$
(38)

$$\left(\boldsymbol{G}_{\sigma}^{h\pm}\right)_{IJ} = \frac{-\mathrm{i}R^2}{(R^2 - [r\cos(\varphi - \theta) + p_I^{\pm}(z - h)]^2)^{3/2}}\boldsymbol{\delta}_{IJ},$$
$$\left(\boldsymbol{G}_{\sigma}^{(\alpha)}\right)$$

$$\left(\boldsymbol{G}_{\sigma}^{(\sigma)}\right)_{IJ}$$

$$= \frac{-\mathrm{i}R^2}{(R^2 - [r\cos(\varphi - \theta) + \hat{p}_I^{(\alpha)}z - p_J^{(1)}h]^2)^{3/2}} (\mathbf{G}_\alpha)_{IJ}.$$

In Eq. (38) and below we introduce $\hat{p}_I^{(1)} = \bar{p}_I^{(1)}$, $\hat{p}_I^{(2)} = p_I^{(2)}$, $p_I^+ = \bar{p}_I^{(1)}$, $p_I^- = p_I^{(1)}$, $\alpha = 1, 2$. It is noted that when the loading circle plane and the

It is noted that when the loading circle plane and the field point are both located on the interface (h = z = 0), the displacement kernel (with subscript "u" in Eq. (38)) is weakly singular (order of $1/\sqrt{r}$) and it is integrable in the

Cauchy principal sense; however, the stress kernel (with subscript " σ " in Eq. (38)) has a high order of singularity $(1/r^{3/2})$, which is not integrable even in the Cauchy principal sense. This stress kernel due to the circular dislocation on the interface would result in the Dirac delta function singularity in the extended stress field expression [15–18].

4.2 Solutions for extended uniform traction case

Under a uniform extended traction T_0 within the circle, Eqs. (22) and (33) give us

$$\boldsymbol{q}^{\infty} = 2\pi R \eta^{-1} J_1(\eta R) (\boldsymbol{A}^{(1)})^{\mathrm{T}} \boldsymbol{T}_0.$$

Therefore, for this case, we have the following analytical solutions in the three physical domains.

For
$$z > h$$
 (in Material 1)

$$\begin{bmatrix} \boldsymbol{u}^{(1)}(r,\varphi,z) \\ \boldsymbol{t}^{(1)}(r,\varphi,z) \\ \boldsymbol{s}^{(1)}(r,\varphi,z) \end{bmatrix} = \frac{1}{2\pi} \int_{0}^{2\pi} \left(-\begin{bmatrix} \bar{\boldsymbol{A}}^{(1)} \boldsymbol{G}_{u}^{h+} \\ \bar{\boldsymbol{B}}^{(1)} \boldsymbol{G}_{\sigma}^{h+} \\ \bar{\boldsymbol{C}}^{(1)} \boldsymbol{G}_{\sigma}^{h+} \end{bmatrix} (\bar{\boldsymbol{A}}^{(1)})^{\mathrm{T}} -\begin{bmatrix} \bar{\boldsymbol{A}}^{(1)} \boldsymbol{G}_{u}^{(1)} \\ \bar{\boldsymbol{B}}^{(1)} \boldsymbol{G}_{\sigma}^{(1)} \\ \bar{\boldsymbol{C}}^{(1)} \boldsymbol{G}_{\sigma}^{(1)} \end{bmatrix} (\boldsymbol{A}^{(1)})^{\mathrm{T}} \right) \mathrm{d}\boldsymbol{\theta} \cdot \boldsymbol{T}_{0}.$$
(39)

For $0 \le z < h$ (in Material 1)

$$\begin{bmatrix} \boldsymbol{u}^{(1)}(r,\varphi,z) \\ \boldsymbol{t}^{(1)}(r,\varphi,z) \\ \boldsymbol{s}^{(1)}(r,\varphi,z) \end{bmatrix}$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \begin{bmatrix} \boldsymbol{A}^{(1)} \boldsymbol{G}_{u}^{h-} - \bar{\boldsymbol{A}}^{(1)} \boldsymbol{G}_{u}^{(1)} \\ \boldsymbol{B}^{(1)} \boldsymbol{G}_{\sigma}^{h-} - \bar{\boldsymbol{B}}^{(1)} \boldsymbol{G}_{\sigma}^{(1)} \\ \boldsymbol{C}^{(1)} \boldsymbol{G}_{\sigma}^{h-} - \bar{\boldsymbol{C}}^{(1)} \boldsymbol{G}_{\sigma}^{(1)} \end{bmatrix} (\boldsymbol{A}^{(1)})^{\mathrm{T}} \mathrm{d}\boldsymbol{\theta} \cdot \boldsymbol{T}_{0}. \quad (40)$$

For z < 0 (in Material 2)

$$\begin{bmatrix} \boldsymbol{u}^{(2)}(r,\varphi,z) \\ \boldsymbol{t}^{(2)}(r,\varphi,z) \\ \boldsymbol{s}^{(2)}(r,\varphi,z) \end{bmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} \boldsymbol{A}^{(2)} \boldsymbol{G}_u^{(2)} \\ \boldsymbol{B}^{(2)} \boldsymbol{G}_\sigma^{(2)} \\ \boldsymbol{C}^{(2)} \boldsymbol{G}_\sigma^{(2)} \end{bmatrix} (\boldsymbol{A}^{(1)})^{\mathrm{T}} \mathrm{d}\boldsymbol{\theta} \cdot \boldsymbol{T}_0, \qquad (41)$$

where

$$\begin{aligned} \left(\boldsymbol{G}_{u}^{h\pm}\right)_{IJ} &= \left(\left[r\cos(\varphi-\theta)+p_{I}^{\pm}(z-h)\right]\right. \\ &+\mathrm{i}\,\sqrt{R^{2}-\left[r\cos(\varphi-\theta)+p_{I}^{\pm}(z-h)\right]^{2}}\right) \boldsymbol{\delta}_{IJ} \\ \left(\boldsymbol{G}_{u}^{(\alpha)}\right)_{IJ} &= \left(\left[r\cos(\varphi-\theta)+\hat{p}_{I}^{(\alpha)}z-p_{J}^{(1)}h\right]\right. \\ &+\mathrm{i}\,\sqrt{R^{2}-\left[r\cos(\varphi-\theta)+\hat{p}_{I}^{(\alpha)}z-p_{J}^{(1)}h\right]^{2}}\right) \left(\boldsymbol{G}_{\alpha}\right)_{IJ}, \\ \left(\boldsymbol{G}_{\sigma}^{h\pm}\right)_{IJ} \end{aligned}$$

$$\end{aligned}$$

$$= \left(1 - \frac{\mathrm{i}[r\cos(\varphi - \theta) + p_I^{\pm}(z - h)]}{\sqrt{R^2 - [r\cos(\varphi - \theta) + p_I^{\pm}(z - h)]^2}}\right) \boldsymbol{\delta}_{IJ},$$
$$\left(\boldsymbol{G}_{\sigma}^{(\alpha)}\right)_{IJ}$$
$$= \left(1 - \frac{\mathrm{i}[r\cos(\varphi - \theta) + \hat{p}_I^{(\alpha)}z - p_J^{(1)}h]}{\sqrt{R^2 - [r\cos(\varphi - \theta) + \hat{p}_I^{(\alpha)}z - p_J^{(1)}h]^2}}\right) (\boldsymbol{G}_{\alpha})_{IJ}$$

Similarly, it is observed that when the loading circle plane and the field point are both located on the interface (h = z = 0), the displacement kernel (with subscript "u" in Eq. (42)) is integrable; however, the stress kernel (with subscript " σ " in Eq. (42)) is weakly singular (order of $1/\sqrt{r}$) and thus it is integrable in the Cauchy principal sense. The latter behavior is just the same as in the extended displacement expression due to the interfacial dislocation (Eq. (38) for the displacement kernel). Furthermore, the singular behavior for both extended displacements and stresses on the interface is the same as that on the surface of a half space [19].

5 Numerical examples and discussion

Before applying our analytical solutions to numerical examples, we have first checked various reduced cases for verification. For instance, we checked that, for the reduced purely elastic isotropic half space under a uniform surface loading over a circle, the induced elastic displacements and stresses from our solutions were the same as the well-known solutions [20].

In our numerical studies, the MEE bimaterial space is made of two transversely isotropic materials: The upper half space (Material 1) is made of the pseudo BaTiO₃ [21] and the lower half space (Material 2) made of 50% BaTiO₃ and 50% CoFe₂O₄ MEE composite [22], with their properties given in Appendix. We consider both the extended traction and extended dislocation loadings with the circular loading area being assumed to be in the upper material half space.

5.1 Effect of different loading locations

Figure 2 shows the variation of the dimensionless displacement component u_z (normalized with respect to $u_{z \max}$) on the interface at point (x, y, z) = (0, 0, 0) with respect to the normalized center distance h/R (varying from 1 to 5) of the circular loading area (for fixed radius R = 1 m), induced by the uniform extended traction/dislocation. For the traction loading case, the vertical traction, electric displacement and magnetic induction (with densities of 1 Pa, 1 C/m² and 1T, respectively) are, respectively, applied, and the results are shown in Fig. 2a. For the dislocation loading case, the vertical Burgers component, electrical potential jump and magnetic potential jump (with densities of 10 nm, 1 V and 1 A, respectively) are, respectively, applied, and the results are shown in Fig. 2b. The six maximum values $u_{z\max}$ are

 $5.65 \text{ pm}, 6.13 \text{ mm}, 56.4 \mu\text{m}, 5.01 \text{ nm}, 18.5 \text{ pm}, 2.41 \text{ nm}, \text{ corresponding to the vertical traction, electric displacement, magnetic induction, vertical Burgers component, electrical potential jump, and magnetic potential jump, respectively.$

It is interesting to observe from Fig. 2a that under both mechanical and electric traction loadings, the displacement component u_z at the interface point (0, 0, 0) decreases monotonically as the loading area moves away from the interface. However, under the magnetic induction, the elastic displacement u_z at (0, 0, 0) first increases sharply to a maximum value when h/R increases from 0 to a point near 1. It then decreases with increasing h/R. Under the uniform extended

dislocation, the results are quite different. It shows clearly in Fig. 2b that under the electric potential jump, u_z at (0, 0, 0) first increases to a maximum value when h/R increases from 0 to a point near 1, and then decreases with increasing h/R. Under the vertical Burgers component, u_z at the interface point (0, 0, 0) decreases monotonically with increasing h/R. Under the magnetic potential jump, however, the value of u_z continuously decrease from its maximum at h/R = 0 to a negative value when h/R equals to 2.5. Similar to other two loading cases shown in Fig. 2b, the elastic displacement u_z due to the magnetic potential jump eventually approaches zero when h/R is far away from the interface.



Fig. 2 Variation of the normalized elastic displacement u_z on the interface point (x, y, z) = (0, 0, 0) vs. the normalized vertical distance h/R of the circle, with fixed radius R = 1 m, induced **a** by uniform extended traction (elastic traction $t_z = 1$ Pa, electric displacement $D_3 = 1$ C/m², and magnetic induction $B_3 = 1$ T), and **b** by uniform extended dislocations (elastic dislocation $b_z = 10$ nm, electric potential discontinuity $\Delta \varphi = 1$ V, and magnetic potential discontinuity $\Delta \psi = 1$ A)

5.2 Response on the two sides of the interface, $z = 0^+$ and $z = 0^-$

Figures 3-5 show, respectively, contours of the dimension-

less stress component σ_{xx} (normalized by its maximum 0.231 Pa), electric displacement D_x (normalized by its maximum 65.9 pC/m²) and magnetic induction B_x (normalized by its maximum 2.79 nN/Am) on the interface $z = 0^+$ (a)



Fig. 3 Dimensionless stress component σ_{xx} on the interface induced by a uniform mechanical traction in *x*-direction ($t_x = 1$ Pa) within the circle of radius R = 1 m centered at (x, y, z)/R = (0, 0, 0.5). **a** $z = 0^+$ in Material 1; **b** $z = 0^-$ in Material 2



Fig. 4 Dimensionless electric displacement component D_x on the interface induced by a uniform mechanical traction in x-direction $(t_x = 1 \text{ Pa})$ within the circle of radius R = 1 m centered at (x, y, z)/R = (0, 0, 0.5). **a** $z = 0^+$ in Material 1; **b** $z = 0^-$ in Material 2



Fig. 5 Dimensionless magnetic induction component B_x on the interface induced by a uniform mechanical traction in x-direction ($t_x = 1$ Pa) within the circle of radius R = 1 m centered at (x, y, z)/R = (0, 0, 0.5). **a** $z = 0^+$ in Material 1; **b** $z = 0^-$ in Material 2

and $z = 0^{-}$ (b) due to a uniform horizontal mechanical traction t_x (with a density of 1 Pa) within the circular area of radius R=1 centered at (x, y, z)/R = (0, 0, 0.5). For σ_{xx} in Fig. 3, it can be observed that its magnitudes on the upper interface $z = 0^+$ are different to the ones on the lower interface $z = 0^{-}$, although the contours are both anti-symmetric with respect to the y-axis, and symmetric with respect to the x-axis. The dimensionless maximum of σ_{xx} are 0.82 and 1, respectively, on the side $z = 0^+$ and $z = 0^-$, with the concentration being located at point (x, y)/R = (-1.3, 0) (where σ_{xx} is positive) and (x, y)/R = (1.3, 0) (where σ_{xx} is negative). The contour shapes for D_x are also similar on both sides of the interface, whilst their dimensionless maximum magnitudes, located at the center (x, y)/R = (0, 0), are 1 and 0.49 on the side of $z = 0^+$ and $z = 0^-$, respectively (Fig. 4). Figure 5 shows the contours of the corresponding dimensionless magnetic induction B_x , which are strikingly different to each other on both sides of the interface. For example, B_x has only one concentration at the center on the upper interface $z = 0^+$

with the maximum value of 1, whilst on the lower interface $z = 0^-$, it has four concentrations located symmetrically with respect to the *x*- and *y*-axes: $(x, y)/R = (\pm 1.5, 0)$ where B_x has a maximum value of 0.45, and $(x, y)/R = (0, \pm 1.4)$ where B_x has a minimum value of -0.14.

Figures 6–8 show, respectively, contours of the dimensionless stress component σ_{xx} (normalized by its maximum 241 Pa), electric displacement D_x (normalized by its maximum 32.3 nC/m²) and magnetic induction B_x (normalized by its maximum 1.82 μ N/Am) on the upper interface $z = 0^+$ (a) and lower $z = 0^-$ (b) due to a uniform mechanical dislocation b_x (with a density of 10 nm) applied to the circle of radius R = 1 centered at (x, y, z)/R = (0, 0, 0.5). As in the uniform traction case, the material property mismatches in the two half spaces cause the difference of the field quantities on both sides of the interface. For instance, while the contour shapes of the stress component σ_{xx} and electric displacement D_x are, respectively, similar on both sides of the interface (Figs. 6 and 7), their magnitudes are different. For



Fig. 6 Dimensionless stress component σ_{xx} on the interface induced by a uniform mechanical dislocation in *x*-direction ($b_x = 10$ nm) within the circle of radius R = 1 m centered at (x, y, z)/R = (0, 0, 0.5). **a** $z = 0^+$ in Material 1; **b** $z = 0^-$ in Material 2



Fig. 7 Dimensionless electric displacement component D_x on the interface induced by a uniform mechanical dislocation in x-direction $(b_x = 10 \text{ nm})$ within the circle of radius R = 1 m centered at (x, y, z)/R = (0, 0, 0.5). **a** $z = 0^+$ in Material 1; **b** $z = 0^-$ in Material 2



Fig. 8 Dimensionless magnetic induction component B_x on the interface induced by a uniform mechanical dislocation in *x*-direction $(b_x = 10 \text{ nm})$ within the circle of radius R = 1 m centered at (x, y, z)/R = (0, 0, 0.5). **a** $z = 0^+$ in Material 1; **b** $z = 0^-$ in Material 2

e

q

e

ε

 σ_{xx} (Fig. 6), the field concentrations on both upper and lower interfaces are located anti-symmetrically at $(x, y)/R = (\pm 1.3, 0)$ with a dimensionless maximum magnitude of 0.75 (on the side $z = 0^+$) and 1 (on the side $z = 0^-$). From Figs. 7 and 8, we observe that the field distribution of D_x and B_x are complicated with more concentrations, although they are all symmetric with respect to both the *x*- and *y*axes. The dimensionless maximum values of D_x are 1 and 0.488, respectively, on the upper and lower interfaces (near $(x, y)/R = (\pm 1.3, 0)$). The dimensionless maximum values of B_x are 0.85 and 1, respectively, on the upper and lower interface (also near $(x, y)/R = (\pm 1.3, 0)$).

6 Conclusions

In this paper, we derived the analytical solutions in a 3D anisotropic MEE bimaterial space subject to uniform extended dislocations and tractions within a horizontal circular area. In the numerical examples, the effect of different loading locations on the response at the interface was analyzed. It is interesting to observe that the physical quantities on the interface do not decrease monotonically as the loading area moves away from the interface when the magnetic traction or electric dislocation is applied. The distributions of different in-plane physical quantities on the upper interface $z = 0^+$ and the lower interface $z = 0^-$ under different horizontal extended loadings were compared and the differences were discussed. This work could not only serve as a benchmark for future numerical studies in related research fields, but also be employed as the special Green's functions in the boundary integral equations based on the tractions or dislocations kernel functions.

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Appendix

The material coefficients of pseudo $BaTiO_3$ for upper half space (Material 1) (z > 0).

(1) Elastic constants

<i>c</i> =	166	77	78	0	0	0	GPa
	77	166	124	0	0	0	
	78	124	162	0	0	0	
	0	0	0	43	0	0	
	0	0	0	0	43	0	
	0	0	0	0	0	44.5	

(2) Piezoelectric constants

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 11.6 & 0 \\ 0 & 0 & 0 & 11.6 & 0 & 0 \\ -4.4 & -4.4 & 9.3 & 18.6 & 0 & 0 \end{bmatrix} C/m^2$$

(3) Dielectric permeability coefficients

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 11.2 & 0 & 0 \\ 0 & 11.2 & 0 \\ 0 & 0 & 12.6 \end{bmatrix} \text{nC/Vm}$$

(4) Piezomagnetic constants

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 550 & 0 \\ 0 & 0 & 550 & 0 & 0 \\ 580.3 & 580.3 & 699.7 & 0 & 0 & 0 \end{bmatrix}$$
N/Am

(5) Magnetoelectric coefficients $\alpha_{ij} = 0$ (*i*, *j* = 1, 3) (in Ns/VC)

(6) Magnetic permeability coefficients

$$\boldsymbol{\mu} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix} \mu \mathrm{N}\mathrm{s}^{2}/\mathrm{C}^{2}$$

The material coefficients of the MEE BaTiO₃-CoFe₂O₄ composite (50% BaTiO₃ and 50% CoFe₂O₄) for lower half space (Material 2) (z < 0).

(1) Elastic constants 225 125 124 0 0 n 125 225 124 0 0 0 124 124 216 0 0 0 *c* = GPa 0 0 0 44 0 0 0 0 0 0 44 0 0 0 0 0 50 0

(2) Piezoelectric constants

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 5.8 & 0 \\ 0 & 0 & 0 & 5.8 & 0 & 0 \\ -2.2 & -2.2 & 9.3 & 0 & 0 & 0 \end{bmatrix} C/m^2$$

(3) Dielectric permeability coefficients

$$= \begin{bmatrix} 5.64 & 0 & 0\\ 0 & 5.64 & 0\\ 0 & 0 & 6.35 \end{bmatrix} nC/Vm$$

(4) Piezomagnetic constants

	0	0	0	0	275	0	
i =	0	0	0	275	0	0	N/Am
	290.2	290.2	350	0	0	0	

(5) Magnetoelectric coefficients $\alpha_{ij} = 0$ (for *i*, *j* = 1, 3) (in Ns/VC).

(6) Magnetic permeability coefficients

$$\boldsymbol{\mu} = \begin{bmatrix} 297 & 0 & 0 \\ 0 & 297 & 0 \\ 0 & 0 & 83.5 \end{bmatrix} \mu \mathrm{Ns}^2/\mathrm{C}^2$$

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