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Bending analyses of 1D orthorhombic quasicrystal plates

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ABSTRACT

The meshless Petrov–Galerkin method (MLPG) is applied to plate bending analysis in 1D orthorhombic quasicrystals (QCs) under static and transient dynamic loads. The Bak and elasto-hydrodynamic models are applied for phason governing equation in the elastodynamic case. The phason displacement for the orthorhombic QC in the first-order shear deformation plate theory depends only on the in-plane coordinates on the mean plate surface. Nodal points are randomly distributed over the mean surface of the considered plate. Each node is the center of a circle surrounding this node. The coupled governing partial differential equations are satisfied in a weak-form on small fictitious subdomains. The spatial variations of the phonon and phason displacements are approximated by the moving least-squares (MLS) scheme. After performing the spatial MLS approximation, a system of ordinary differential equations (ODEs) for nodal unknowns is obtained. The system of the ODEs of the second order is solved by the Houbolt finite-difference scheme. Our numerical examples demonstrate clearly the effect of the coupling parameter on both static and dynamic phonon/phason deflections.

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1. Introduction

Quasicrystals (QCs) have a solid structure with a long-range quasiperiodic translational order and a long-range orientational order. The icosahedral QC was first discovered in 1984 by Shechtman et al. (1984). Depending upon their crystallography, the orthorhombic QCs may belong to the class of one-dimensional (1D) QCs (Fan, 2011). A 1D QC is defined as a three-dimensional (3D) body which is periodic, for example, in the x-y plane and quasiperiodic in the third direction. These materials have potential engineering applications. The electronic structure and the optic, magnetic, thermal and mechanical properties of QCs were intensively investigated both experimentally and theoretically (Wollgarten et al., 1993; Park et al., 2005; Wang and Pan, 2008; Altay and Dokmeci, 2012). Elasticity is one of the important properties of QCs. The elastic behavior of QCs is different from that of usual crystals. The Landau density wave theory (Bak, 1985; Levine et al., 1985; Lubensky, 1988) can be considered as a base of the elastic theory of QCs. In terms of the elastic theory of crystals the displacement field (phonon displacement field) represents the phenomenological field corresponding to the translational motion of atoms in crystals. Due to quasi-periodic lattice structure in QCs, additional degrees of freedom corresponding to atomic rearrangements are introduced in the phenomenological theory via phason displacements. The generalized theory of elasticity of QCs was developed by Ding et al. (1993, 1994). Chen et al. (2004) developed a general theory of 3D elastic problems of 1D hexagonal QCs. Some solutions for qusicrystal problems with possible practical applications can be found in the book by Fan (2011). Fan (2013) reviewed the most recent progress on the mathematical theory and methods of mechanics of quasicrystals, including elasticity, plasticity, defects, dynamics, and fracture among many other topics.

Elastodynamics of QCs brings some additional challenges. A consistent opinion on governing equations for phason fields is missing. According to Bak (1985) the phason describes particular structure disorders in QCs, and it can be formulated in a six-dimensional space. Since there are six continuous symmetries, there exit six hydrodynamic vibration modes. Then, phonons and phasons play similar roles in the dynamics and both fields should be described by similar governing equations, namely the balance of momentum. Lubensky et al. (1985) thought that the phason field should be described by a diffusion equation with a very large diffusion time. According to them, phasons are insensitive to spatial translations and phason modes represent the relative motion of the constituent density waves.

Incorporating QC materials into plate or beam structures seems to be very interesting. Recently, Gao (2010) has developed an exact theory for 1D QC beams. The reciprocal theorem is applied for plate bending of 1D hexagonal QCs to obtain the appropriate stress and mixed boundary conditions for plates of general edge geometry and loading (Gao et al., 2007). Furthermore, for 1D QC a refined theory of thick plates was established by Gao and Ricoeur (2011) from the general solution of QCs and the Lure method without

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employing ad hoc stress or deformation assumptions. The purpose of this paper is to develop a reliable computational method for general bending problems of QC plates. In recent years, meshless formulations are becoming popular due to their high adaptability and low costs in preparing input and output data in numerical analysis. The term "meshless" or "meshfree" stems from the ability of an approximation or interpolation scheme to be constructed entirely from a set of nodes without connecting them to elements. Meshless methods for solving partial differential equations (PDEs) in physics and engineering sciences are a powerful new alternative to the traditional mesh-based techniques. The elimination of shear locking in thin walled structures by FEM is difficult and the associated techniques are less accurate. However, the moving leastsquare (MLS) approximation ensures C^1 continuity which satisfies the Kirchhoff hypotheses. The continuity of the MLS approximation is given by the minimum between the continuity of the basis functions and that of the weight function. As such, continuity can be tuned to a desired degree. Previous results showed excellent convergence for linear problems (Krysl and Belytschko 1996; Liew et al., 2004; Atluri et al. 2000); however, up to now the formulation has not been applied to the deflection analysis of QC plates. One of the most rapidly developed meshfree methods is the meshless local Petrov-Galerkin (MLPG) method (Atluri, 2004). The MLPG method has attracted much attention in the past decade and it has been successfully applied also to plate problems (Long and Atluri, 2002; Sladek et al., 2006, 2007; Soric et al., 2004).

In the present paper, the MLPG is applied to plate bending analysis in orthorhombic QCs under both static and transient dynamic loads. The quasiperiodicity of 1D QCs is considered along the plate thickness. The MLPG formulation is developed based on the Bak model (1985). The Reissner-Mindlin theory reduces the original 3D thick plate problem to a 2D problem. It is shown that the phason displacement for the orthorhombic QC in the first-order shear deformation plate theory depends only on the in-plane coordinates over the mean plate surface. Then, nodal points are randomly distributed over the mean surface of the considered plate. Each node is the center of a circle surrounding this node. The coupled governing PDEs are satisfied in a weak-form on the small fictitious subdomains. The spatial variations of the phonon and phason displacements are approximated by the MLS scheme (Zhu et al., 1998). After performing the spatial MLS approximation, a system of ODEs for the unknowns at nodes is obtained. Then, the system of the second-order ODEs resulting from the equations of motion is solved by the Houbolt finite-difference scheme (Houbolt, 1950) as a time-stepping method.

Numerical results for simply-supported and clamped square plates under a static and impact loads on the top surface of the plate are presented to illustrate the efficiency of the proposed method and to demonstrate the possible effect of the coupling parameters on the induced phason/phonon deflections.

2. Local integral equations for orthorhombic QC plates

We consider a plate of total thickness *h* made of homogeneous orthorhombic QC material properties with its mean surface occupying the domain Ω in the plane $\mathbf{x} \equiv (x_1, x_2)$. The axis $x_3 \equiv z$ is perpendicular to the mid-plane (Fig. 1).

The rectangular Cartesian coordinate system is introduced such that the bottom and top surfaces of plate is placed in the planes $x_3 = h/2$ and $x_3 = -h/2$, respectively. In the Reissner–Mindlin theory the transverse shear strains are assumed as constant throughout the plate thickness and thus some correction coefficients are required for computation of transverse shear forces in that theory. Then, the spatial phonon displacement field, due to transverse loading and expressed in terms of displacement components u_1 , u_2 , and u_3 , has the following form (Mindlin, 1951; Reddy, 1997)



Fig. 1. Sign convention of bending moments and forces for a plate.

$$u_{1}(\mathbf{x}, x_{3}, \tau) = u_{10}(\mathbf{x}, \tau) + x_{3}\psi_{1}(\mathbf{x}, \tau), u_{2}(\mathbf{x}, x_{3}, \tau) = u_{20}(\mathbf{x}, \tau) + x_{3}\psi_{2}(\mathbf{x}, \tau), u_{3}(\mathbf{x}, \tau) = u_{30}(\mathbf{x}, \tau)$$
(1)

where τ is the time variable, and $\psi_{\alpha}(x_1, x_2, \tau)$ and $u_{30}(x_1, x_2, \tau)$ represent the rotations around the in-plane axes and the out-of-plane deflection, respectively (Fig. 1). It is assumed that u_3 is independent on x_3 . The in-plane displacements in x_1 - and x_2 -directions are denoted by u_{10} and u_{20} .

The linear phonon strains are given by

$$\begin{aligned} \varepsilon_{11}(\mathbf{x}, x_3, \tau) &= u_{10,1}(\mathbf{x}, \tau) + x_3 \psi_{1,1}(\mathbf{x}, \tau), \\ \varepsilon_{22}(\mathbf{x}, x_3, \tau) &= u_{20,2}(\mathbf{x}, \tau) + x_3 \psi_{2,2}(\mathbf{x}, \tau), \\ \varepsilon_{12}(\mathbf{x}, x_3, \tau) &= \frac{1}{2}(u_{10,2} + u_{20,1}) + \frac{1}{2}x_3[\psi_{1,2}(\mathbf{x}, \tau) + \psi_{2,1}(\mathbf{x}, \tau)], \\ \varepsilon_{13}(\mathbf{x}, \tau) &= [\psi_1(\mathbf{x}, \tau) + u_{30,1}(\mathbf{x}, \tau)]/2, \\ \varepsilon_{23}(\mathbf{x}, \tau) &= [\psi_2(\mathbf{x}, \tau) + u_{30,2}(\mathbf{x}, \tau)]/2 \\ \varepsilon_{33}(\mathbf{x}, \tau) &= u_{30,3}(\mathbf{x}, \tau) = 0. \end{aligned}$$

$$(2)$$

The generalized Hooke's law for plane elasticity of orthorhombic QC is given as (Fan, 2011)

$$\sigma_{11} = c_{11}\varepsilon_{11} + c_{12}\varepsilon_{22} + c_{13}\varepsilon_{33} + R_1w_{33},$$

$$\sigma_{22} = c_{12}\varepsilon_{11} + c_{22}\varepsilon_{22} + c_{23}\varepsilon_{33} + R_2w_{33},$$

$$\sigma_{31} = 2c_{55}\varepsilon_{31} + R_6w_{31},$$

$$\sigma_{32} = 2c_{44}\varepsilon_{32} + R_5w_{32},$$

$$\sigma_{12} = \sigma_{21} = 2c_{66}\varepsilon_{12},$$

$$H_{31} = 2R_6\varepsilon_{31} + K_1w_{31},$$

$$H_{32} = 2R_5\varepsilon_{32} + K_2w_{32},$$

$$H_{33} = R_1\varepsilon_{11} + R_2\varepsilon_{22} + R_3\varepsilon_{33} + K_3w_{33},$$
(3)

where $\varepsilon_{ij}(\mathbf{x}, x_3, \tau)$ are the phonon strains, and the phason strains $w_{ij}(\mathbf{x}, x_3, \tau)$ are defined as

$$w_{ij}(\mathbf{x}, x_3, \tau) = w_{i,j}(\mathbf{x}, x_3, \tau) \tag{4}$$

It is noted that the phason strains $w_{ij}(\mathbf{x})$ are not symmetric in contrast to the phonon strains $\varepsilon_{ij}(\mathbf{x})$. The material coefficients c_{ij} , R_i and K_i in Eq. (3) denote the classical phonon elastic coefficients, the phonon-phason coupling parameters and the phason elastic coefficients, respectively.

It is well-known that the quasi-periodicity leads to two different elementary excitations in the material: phonons u_i and phasons w_i . The phonon modes may be understood as vibrations of the QC lattice which lead to elastic wave propagation. A phason field can be described by either wave propagation or diffusion. Both models are equivalent in static case. According to Bak (1985) the phason structure disorders are realized by fluctuations in QCs. The balance of momentum is valid for phonon deformation and similarly for phason oscillations with the same mass of density in relaxation processes. Then, the model is described by the following governing equations:

$$\sigma_{ij,j}(\mathbf{x}, x_3, \tau) + X_i(\mathbf{x}, x_3, \tau) = \rho \ddot{u}_i(\mathbf{x}, x_3, \tau), \tag{6}$$

$$H_{ii,j}(\mathbf{x}, \mathbf{x}_3, \tau) + G_i(\mathbf{x}, \mathbf{x}_3, \tau) = \rho \ddot{w}_i(\mathbf{x}, \mathbf{x}_3, \tau), \tag{7}$$

where \ddot{u}_i , \ddot{w}_i , ρ , X_i and G_i denote the acceleration of the phonon and phason displacements, the mass density, and the body force vectors, respectively. Both governing equations have mathematically similar structures. The dots over a quantity indicate differentiations with respect to time τ .

Lubensky et al. (1985) pointed out that the phonon and phason fields play different roles in the hydrodynamics of quasicrystals, because phason displacements are insensitive to spatial translations. Furthermore, the relaxation of the phason strain is diffusive and is much slower than rapid relaxation of conventional phonon strains. The elasto-hydrodynamic model was introduced by Fan et al. (2009). It is a combination of elastodynamics originating from Bak's arguments and the hydrodynamics of Lubensky et al. (1985). The corresponding governing equations in this case have the following forms

$$\sigma_{ijj}(\mathbf{x}, x_3, \tau) + X_i(\mathbf{x}, x_3, \tau) = \rho \ddot{u}_i(\mathbf{x}, x_3, \tau), \tag{8}$$

$$H_{ii,i}(\mathbf{x}, x_3, \tau) + G_i(\mathbf{x}, x_3, \tau) = D\dot{w}_i(\mathbf{x}, x_3, \tau),$$
(9)

where $D = 1/\Gamma_w$ with Γ_w being the kinematic coefficient of phason field of the material defined by Lubensky et al. (1985).From the kinematical equation (1) it follows directly that $u_{3,3} \equiv 0$. Similarly, one can assume $w_{3,3} \equiv 0$. Then, the last constitutive relationship in Eq. (3) becomes

$$H_{33} = R_1 \varepsilon_{11} + R_2 \varepsilon_{22} \tag{10}$$

One can define the integral quantities such as the bending moments $M_{\alpha\beta}$, normal forces $T_{\alpha\beta}$ and the shear forces Q_{α} (Sladek et al., 2007) as

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} x_3 dx_3,$$

$$T_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} dx_3,$$

$$Q_{\alpha} = \kappa \int_{-h/2}^{h/2} \sigma_{\alpha3} dx_3,$$

$$P_{\alpha} = \int_{-h/2}^{h/2} H_{3\alpha} dx_3,$$

(11)

where the Greek indices take values 1, 2 and $\kappa = 5/6$ according to the Reissner plate theory. It should be noted that the phason forces P_{α} are generalized forces induced by generalized stresses, which have finite values due to coupling parameters at a pure phonon load. However, since a physical interpretation on these phason forces is still missing, we are unable, in this paper, to prescribe finite values of them as boundary conditions.

Substituting constitutive equations (3) and kinematic equations (2) and (4) into the moment and force resultants (11) allows the expression of the bending moments $M_{\alpha\beta}$ and shear forces Q_{α} in terms of the rotations, deflection and phason displacements as

$$\begin{split} M_{11}(\mathbf{x},\tau) &= \int_{-h/2}^{h/2} \{ c_{11}[u_{10,1} + x_3\psi_{1,1}] + c_{12}[u_{20,2} + x_3\psi_{2,2}] \} x_3 \\ dx_3 &= c_{11} \frac{h^3}{12} \psi_{1,1} + c_{12} \frac{h^3}{12} \psi_{2,2}, \\ M_{22}(\mathbf{x},\tau) &= \int_{-h/2}^{h/2} \{ c_{12}[u_{10,1} + x_3\psi_{1,1}] + c_{22}[u_{20,2} + x_3\psi_{2,2}] \} x_3 \\ dx_3 &= c_{12} \frac{h^3}{12} \psi_{1,1} + c_{22} \frac{h^3}{12} \psi_{2,2}, \end{split}$$

$$M_{12}(\mathbf{x},\tau) = \int_{-h/2}^{h/2} c_{66}[u_{10,2} + x_3\psi_{1,2} + u_{20,1} + x_3\psi_{2,1}]x_3$$

$$dx_3 = c_{66}\frac{h^3}{12}(\psi_{1,2} + \psi_{2,1}),$$
(12)

$$\begin{split} T_{11}(\mathbf{x},\tau) &= \int_{-h/2}^{h/2} \{ c_{11}[u_{10,1} + x_3\psi_{1,1}] + c_{12}[u_{20,2} + x_3\psi_{2,2}] \} \\ dx_3 &= c_{11}hu_{10,1} + c_{12}hu_{20,2}, \\ T_{22}(\mathbf{x},\tau) &= \int_{-h/2}^{h/2} \{ c_{12}[u_{10,1} + x_3\psi_{1,1}] + c_{22}[u_{20,2} + x_3\psi_{2,2}] \} \\ dx_3 &= c_{12}hu_{10,1} + c_{22}hu_{20,2}, \\ T_{12}(\mathbf{x},\tau) &= \int_{-h/2}^{h/2} \{ c_{66}[u_{10,2} + u_{20,1} + x_3(\psi_{1,2} + \psi_{2,1})] \} \\ dx_3 &= c_{66}h(u_{10,2} + u_{20,1}), \end{split}$$
(13)

$$Q_{1}(\mathbf{x},\tau) = \kappa \int_{-h/2}^{h/2} [c_{55}(\psi_{1} + u_{30,1}) + R_{6}w_{3,1}] dx_{3} = \kappa h[c_{55}(\psi_{1} + u_{30,1}) + R_{6}w_{3,1}], Q_{2}(\mathbf{x},\tau) = \kappa \int_{-h/2}^{h/2} [c_{44}(\psi_{2} + u_{30,2}) + R_{5}w_{3,2}] dx_{3} = \kappa h[c_{44}(\psi_{2} + u_{30,2}) + R_{5}w_{3,2}],$$
(14)

$$P_{1}(\mathbf{x},\tau) = \int_{-h/2}^{h/2} [R_{6}(\psi_{1} + u_{30,1}) + K_{1}w_{3,1}] dx_{3} = h\{R_{6}[\psi_{1}(\mathbf{x},\tau) + u_{30,1}(\mathbf{x},\tau)] + K_{1}w_{3,1}(\mathbf{x},\tau)\},$$
(15)
$$P_{2}(\mathbf{x},\tau) = \int_{-h/2}^{h/2} [R_{5}(\psi_{2} + u_{30,2}) + K_{2}w_{3,2}] dx_{3} = h\{R_{5}[\psi_{2}(\mathbf{x},\tau) + u_{30,2}(\mathbf{x},\tau)] + K_{2}w_{3,2}(\mathbf{x},\tau)\}.$$

If the Mindlin plate bending theory is considered to describe the quasicrystal plates, the following governing equations have to be satisfied for both Bak's and elasto-hydrodynamic models (Reddy, 1997):

$$M_{\alpha\beta,\beta}(\mathbf{x},\tau) - Q_{\alpha}(\mathbf{x},\tau) = I_M \ddot{\psi}_{\alpha}(\mathbf{x},\tau), \tag{16}$$

$$Q_{\alpha,\alpha}(\mathbf{x},\tau) + q(\mathbf{x},\tau) = I_Q \ddot{\boldsymbol{u}}_{30}(\mathbf{x},\tau), \tag{17}$$

$$T_{\alpha\beta,\beta}(\mathbf{x},\tau) + q_{\alpha}(\mathbf{x},\tau) = I_{Q}\ddot{u}_{\alpha0}(\mathbf{x},\tau), \tag{18}$$

and the integral form of Eq. (7) in terms of the Bak model is given as

$$P_{\alpha,\alpha}(\mathbf{x},\tau) + g(\mathbf{x},\tau) = I_Q \ddot{w}_3(\mathbf{x},\tau), \tag{19}$$

Or in the elasto-hydrodynamic model (9) the last governing equation has the following form

$$P_{\alpha,\alpha}(\mathbf{x},\tau) + g(\mathbf{x},\tau) = I_D \dot{w}_3(\mathbf{x},\tau), \quad \mathbf{x} \in \Omega.$$
(20)

In Eqs. (16)-(20),

$$\begin{split} I_{M} &= \frac{\rho h^{3}}{12}, \quad I_{Q} = \rho h, \quad g(\mathbf{x}, \tau) \\ &= h[R_{1}\psi_{1,1}(\mathbf{x}, \tau) + R_{2}\psi_{2,2}(\mathbf{x}, \tau)] + \int_{-h/2}^{h/2} G_{3}(\mathbf{x}, x_{3}, \tau) \, dx_{3}, \end{split}$$

 $I_D = Dh.$

A transversal load is denoted by $q(\mathbf{x}, \tau)$, and $q_{\alpha}(\mathbf{x}, \tau)$ represents the in-plane loads.

Time-harmonic load is a special case of the general dynamic analysis. Time variation of physical fields is given by the frequency of excitation ω . Then the governing equations for the amplitudes are given by, in terms of the Bak model,

$$M_{\alpha\beta,\beta}(\mathbf{x},\omega) - Q_{\alpha}(\mathbf{x},\omega) = -I_{M}\omega^{2}\psi_{\alpha}(\mathbf{x},\omega), \qquad (21)$$

$$Q_{\alpha,\alpha}(\mathbf{x},\omega) + q(\mathbf{x},\omega) = -I_Q \omega^2 u_{30}(\mathbf{x},\omega), \qquad (22)$$

$$T_{\alpha\beta,\beta}(\mathbf{x},\omega) + q_{\alpha}(\mathbf{x},\omega) = -I_{Q}\omega^{2}u_{\alpha0}(\mathbf{x},\omega), \qquad (23)$$

$$P_{\alpha,\alpha}(\mathbf{x},\omega) + g(\mathbf{x},\omega) = -I_{Q}\omega^{2}w_{3}(\mathbf{x},\omega), \quad \mathbf{x} \in \Omega,$$
(24)

To solve the corresponding initial-boundary value problem, we apply the local boundary-domain integral equation method with meshless approximations. The MLPG method constructs the weakform over local subdomains such as Ω_s , which is a small region taken for each node inside the global domain (Atluri, 2004). The local subdomains could be of any geometrical shape and size. In the current paper, the local subdomains are taken to be of a circular shape. The local weak-form of the governing equations (16)–(20) for $\mathbf{x}^i \in \Omega_s^i$ can be written as

$$\int_{\Omega_{s}^{l}} [M_{\alpha\beta,\beta}(\mathbf{x},\tau) - Q_{\alpha}(\mathbf{x},\tau) - I_{M}\ddot{\psi}_{\alpha}(\mathbf{x},\tau)] v_{\alpha\gamma}^{*}(\mathbf{x}) d\Omega = 0,$$
(25)

$$\int_{\Omega_{s}^{i}} [\mathcal{Q}_{\alpha,\alpha}(\mathbf{x},\tau) + q(\mathbf{x},\tau) - l_{\mathcal{Q}}\ddot{u}_{30}(\mathbf{x},\tau)] \nu^{*}(\mathbf{x}) d\Omega = 0,$$
(26)

$$\int_{\Omega_{\mathbf{x}}^{l}} [T_{\alpha\beta,\beta}(\mathbf{x},\tau) + q_{\alpha}(\mathbf{x},\tau) - I_{Q}\ddot{u}_{\alpha0}(\mathbf{x},\tau)] v_{\alpha\gamma}^{*}(\mathbf{x}) d\Omega = 0,$$
(27)

$$\int_{\Omega_{s}^{l}} [P_{\alpha,\alpha}(\mathbf{x},\tau) + g(\mathbf{x},\tau) - I_{Q}\ddot{w}_{3}(\mathbf{x},\tau)] \upsilon^{*}(\mathbf{x}) d\Omega = 0, \qquad (28)$$

or

$$\int_{\Omega_s^i} [P_{\alpha,\alpha}(\mathbf{x},\tau) + g(\mathbf{x},\tau) - I_D \dot{w}_3(\mathbf{x},\tau)] v^*(\mathbf{x}) d\Omega = 0,$$
(29)

where $v_{\alpha\beta}^*(\mathbf{x})$ and $v^*(\mathbf{x})$ are the weight or test functions.

Applying the Gauss divergence theorem to Eqs. (25)–(29), one obtains

$$\int_{\partial\Omega_{s}^{i}} M_{\alpha}(\mathbf{x},\tau) \upsilon_{\alpha\gamma}^{*}(\mathbf{x}) d\Gamma - \int_{\Omega_{s}^{i}} M_{\alpha\beta}(\mathbf{x},\tau) \upsilon_{\alpha\gamma,\beta}^{*}(\mathbf{x}) d\Omega - \int_{\Omega_{s}^{i}} Q_{\alpha}(\mathbf{x},\tau) \upsilon_{\alpha\gamma}^{*}(\mathbf{x}) d\Omega - \int_{\Omega_{s}^{i}} I_{M} \ddot{\psi}_{\alpha}(\mathbf{x},\tau) \upsilon_{\alpha\gamma}^{*}(\mathbf{x}) d\Omega = \mathbf{0},$$
(30)

$$\begin{aligned} \int_{\partial\Omega_{s}^{i}} Q_{\alpha}(\mathbf{x},\tau) n_{\alpha}(\mathbf{x}) \, \boldsymbol{\nu}^{*}(\mathbf{x}) \, d\Gamma &- \int_{\Omega_{s}^{i}} Q_{\alpha}(\mathbf{x},\tau) \, \boldsymbol{\nu}^{*}_{,\alpha}(\mathbf{x}) \, d\Omega \\ &- \int_{\Omega_{s}^{i}} I_{Q} \ddot{u}_{30}(\mathbf{x},\tau) \, \boldsymbol{\nu}^{*}(\mathbf{x}) \, d\Omega + \int_{\Omega_{s}^{i}} q(\mathbf{x},\tau) \, \boldsymbol{\nu}^{*}(\mathbf{x}) \, d\Omega = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} \int_{\partial\Omega_{s}^{l}} T_{\alpha}(\mathbf{x},\tau) \, v_{\alpha\gamma}^{*}(\mathbf{x}) \, d\Gamma &- \int_{\Omega_{s}^{l}} T_{\alpha\beta}(\mathbf{x},\tau) \, v_{\alpha\gamma,\beta}^{*}(\mathbf{x}) \, d\Omega \\ &+ \int_{\Omega_{s}^{l}} q_{\alpha}(\mathbf{x},\tau) \, v_{\alpha\gamma}^{*}(\mathbf{x}) \, d\Omega - \int_{\Omega_{s}^{l}} I_{Q} \ddot{u}_{\alpha0}(\mathbf{x},\tau) \, v_{\alpha\gamma}^{*}(\mathbf{x}) \, d\Omega = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} \int_{\partial\Omega_{s}^{i}}P_{\alpha}(\mathbf{x},\tau)n_{\alpha}(\mathbf{x})\upsilon^{*}(\mathbf{x})\,d\Gamma &-\int_{\Omega_{s}^{i}}P_{\alpha}(\mathbf{x},\tau)\upsilon_{,\alpha}^{*}(\mathbf{x})\,d\Omega \\ &-\int_{\Omega_{s}^{i}}I_{Q}\ddot{w}_{3}(\mathbf{x},\tau)\upsilon^{*}(\mathbf{x})\,d\Omega + \int_{\Omega_{s}^{i}}g(\mathbf{x},\tau)\upsilon^{*}(\mathbf{x})\,d\Omega = 0. \end{aligned}$$
(33)

An alternative to Eq. (33) is

$$\int_{\partial\Omega_{s}^{i}} P_{\alpha}(\mathbf{x},\tau) n_{\alpha}(\mathbf{x}) \upsilon^{*}(\mathbf{x}) d\Gamma - \int_{\Omega_{s}^{i}} P_{\alpha}(\mathbf{x},\tau) \upsilon^{*}_{,\alpha}(\mathbf{x}) d\Omega - \int_{\Omega_{s}^{i}} I_{D} \dot{w}_{3}(\mathbf{x},\tau) \upsilon^{*}(\mathbf{x}) d\Omega + \int_{\Omega_{s}^{i}} g(\mathbf{x},\tau) \upsilon^{*}(\mathbf{x}) d\Omega = 0,$$
(34)

where $\partial \Omega_s^l$ is the boundary of the local subdomain and

$$M_{\alpha}(\mathbf{x},\tau) = M_{\alpha\beta}(\mathbf{x},\tau)n_{\beta}(\mathbf{x}), \quad T_{\alpha}(\mathbf{x},\tau) = T_{\alpha\beta}(\mathbf{x},\tau)n_{\beta}(\mathbf{x})$$

are the normal bending moment and the traction vector, respectively, and n_{α} is the unit outward normal vector to boundary $\partial \Omega_s^i$. The local weak-forms (30)–(34) are the starting point for deriving local integral equations on the basis of appropriate test functions. Unit step functions are chosen for the test functions $w_{\alpha\beta}^*(\mathbf{x})$ and $w^*(\mathbf{x})$ in each subdomain

$$\nu_{x\gamma}^{*}(\mathbf{x}) = \begin{cases} \delta_{x\gamma} & \text{at } \mathbf{x} \in (\Omega_{s} \cup \partial \Omega_{s}) \\ \mathbf{0} & \text{at } \mathbf{x} \notin (\Omega_{s} \cup \partial \Omega_{s}) \end{cases}, \quad \nu^{*}(\mathbf{x}) = \begin{cases} 1 & \text{at } \mathbf{x} \in (\Omega_{s} \cup \partial \Omega_{s}) \\ \mathbf{0} & \text{at } \mathbf{x} \notin (\Omega_{s} \cup \partial \Omega_{s}) \end{cases}.$$
(35)

Then, the local weak-forms (30)–(34) are transformed into the following local integral equations (LIEs)

$$\int_{\partial\Omega_{s}^{i}}M_{\alpha}(\mathbf{x},\tau)\,d\Gamma - \int_{\Omega_{s}^{i}}Q_{\alpha}(\mathbf{x},\tau)\,d\Omega - \int_{\Omega_{s}^{i}}I_{M}\ddot{\psi}_{\alpha}(\mathbf{x},\tau)\,d\Omega = 0,\tag{36}$$

$$\int_{\partial\Omega_{s}^{i}} Q_{\alpha}(\mathbf{x},\tau) n_{\alpha}(\mathbf{x}) d\Gamma - \int_{\Omega_{s}^{i}} I_{Q} \ddot{u}_{30}(\mathbf{x},\tau) d\Omega + \int_{\Omega_{s}^{i}} q(\mathbf{x},\tau) d\Omega = \mathbf{0}, \quad (37)$$

$$\int_{\partial\Omega_{s}^{i}}T_{\alpha}(\mathbf{x},\tau)\,d\Gamma + \int_{\Omega_{s}^{i}}q_{\alpha}(\mathbf{x},\tau)\,d\Omega - \int_{\Omega_{s}^{i}}I_{Q}\ddot{u}_{\alpha0}(\mathbf{x},\tau)\,d\Omega = \mathbf{0},\tag{38}$$

$$\int_{\partial\Omega_{s}^{l}}P_{\alpha}(\mathbf{x},\tau)n_{\alpha}(\mathbf{x})\,d\Gamma + \int_{\Omega_{s}^{l}}g(\mathbf{x},\tau)\,d\Omega - \int_{\Omega_{s}^{l}}I_{\varrho}\ddot{w}_{3}(\mathbf{x},\tau)\,d\Omega = 0.$$
 (39)

Similarly, an alternative to Eq. (39) is

$$\int_{\partial\Omega_{s}^{i}} P_{\alpha}(\mathbf{x},\tau) n_{\alpha}(\mathbf{x}) d\Gamma + \int_{\Omega_{s}^{i}} g(\mathbf{x},\tau) d\Omega - \int_{\Omega_{s}^{i}} I_{D} \dot{w}_{3}(\mathbf{x},\tau) d\Omega = 0.$$
(40)

In the above local integral equations, the trial functions for rotations $\psi_{\alpha}(\mathbf{x}, \tau)$, transversal displacements $u_{30}(\mathbf{x}, \tau)$, in-plane displacements $u_{\alpha0}(\mathbf{x}, \tau)$ and phason displacement $w_3(\mathbf{x}, \tau)$, are chosen as the MLS approximations over a number of nodes randomly spreading within the domain of influence. The above six equations can be then solved for the six unknown quantities.

3. Numerical solution

In general, a meshless method uses a local interpolation to represent the trial function with the values (or the fictitious values) of the unknown variable at some randomly located nodes. The MLS approximation (Lancaster and Salkauskas, 1981; Nayroles et al., 1992) used in the present analysis may be considered as one of such schemes. According to the MLS method (Atluri, 2004), the approximation of the field variable $u \in \{u_{10}, u_{20}, \psi_1, \psi_2, u_{30}, w_3\}$ can be given as

$$u^{h}(\mathbf{x}) = \sum_{i=1}^{m} p_{i}(\mathbf{x}) a_{i}(\mathbf{x}) = \mathbf{p}^{T}(\mathbf{x}) \mathbf{a}(\mathbf{x}), \qquad (41)$$

where $\mathbf{p}^{T}(\mathbf{x}) = \{p_1(\mathbf{x}), p_2(\mathbf{x}), \dots, p_m(\mathbf{x})\}\$ is a vector of complete basis functions of order *m* and $\mathbf{a}(\mathbf{x}) = \{a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, a_m(\mathbf{x})\}\$ is a vector of unknown parameters that depends on **x**. For example, in 2D problems

$$\mathbf{p}^{T}(\mathbf{x}) = \{1, x_1, x_2\}$$
 for $m = 3$

and

$$\mathbf{p}^{T}(\mathbf{x}) = \{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2\}$$
 for $m = 6$

are linear and quadratic basis functions, respectively.

The approximation functions for the generalized variables can be written as (Atluri, 2004)

$$u^{h}(\mathbf{x},\tau) = \mathbf{N}^{T}(\mathbf{x}) \cdot \hat{u} = \sum_{a=1}^{n} N^{a}(\mathbf{x}) \hat{u}^{a}(\tau),$$
(42)

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where the nodal values $\hat{u}^a(\tau)$ are fictitious parameters for the approximated field variable and $N^a(\mathbf{x})$ is the shape function associated with node *a*. The number of nodes *n* used in the approximation is determined by the weight function. A 4th order spline-type weight function is applied in the present work.

The directional derivatives of the approximated field $u(\mathbf{x}, \tau)$ are expressed in terms of the same nodal values as

$$u_{,\alpha}(\mathbf{x},\tau) = \sum_{\alpha=1}^{n} \hat{u}^{\alpha}(\tau) N^{\alpha}_{,\alpha}(\mathbf{x}).$$
(43)

According to (43), one obtains the approximation for the bending moments (12) as well as for $M_{\alpha}(\mathbf{x}, \tau) = M_{\alpha\beta}(\mathbf{x}, \tau)n_{\beta}(\mathbf{x})$ or $\mathbf{M}(\mathbf{x}, \tau) = [M_1(\mathbf{x}, \tau)M_2(\mathbf{x}, \tau)]^T$:

$$\mathbf{M}(\mathbf{x},\tau) = \mathbf{N}_1 \sum_{a=1}^{n} \mathbf{B}_1^a(\mathbf{x}) \boldsymbol{\psi}^a(\tau), \tag{44}$$

where the vector $\boldsymbol{\psi}^{a}(\tau)$ is defined as a column vector $\boldsymbol{\psi}^{a}(\tau) = \left[\hat{\psi}_{1}^{a}(\tau), \hat{\psi}_{2}^{a}(\tau)\right]^{T}$, the matrices $\mathbf{N}_{1}(\mathbf{x})$ are related to the normal vector $\mathbf{n}(\mathbf{x})$ on $\partial\Omega_{s}$ by

$$\mathbf{N}_1(\mathbf{x}) = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}.$$

Other matrices and vectors in Eq. (44) are represented in terms of the shape functions and their gradients as

$$\mathbf{B}_{1}^{a}(\mathbf{x}) = \frac{h^{3}}{12} \begin{bmatrix} c_{11}N_{,1}^{a} & c_{12}N_{,2}^{a} \\ c_{12}N_{,1}^{a} & c_{22}N_{,2}^{a} \\ c_{66}N_{,2}^{a} & c_{66}N_{,1}^{a} \end{bmatrix}.$$
(45)

Similarly, one can obtain the approximation for the shear forces

$$\mathbf{Q}(\mathbf{x},\tau) = \kappa \sum_{a=1}^{n} [\mathbf{C} \mathbf{N}^{a}(\mathbf{x}) \boldsymbol{\psi}^{a}(\tau) + \mathbf{C} \mathbf{L}^{a}(\mathbf{x}) \hat{u}_{30}^{a}(\tau) + \mathbf{R} \mathbf{L}^{a}(\mathbf{x}) \hat{w}_{3}^{a}(\tau)], \quad (46)$$

where $\boldsymbol{Q}(\boldsymbol{x},\tau) = [Q_1(\boldsymbol{x},\tau),Q_2(\boldsymbol{x},\tau)]^T$ and

$$\mathbf{C} = h \begin{bmatrix} c_{55} & 0 \\ 0 & c_{44} \end{bmatrix}, \quad \mathbf{L}^{a}(\mathbf{x}) = \begin{bmatrix} N_{,1}^{a} \\ N_{,2}^{a} \end{bmatrix}, \quad \mathbf{R} = h \begin{bmatrix} R_{6} & 0 \\ 0 & R_{5} \end{bmatrix}.$$

The traction vector $\bm{T}(\bm{x},\tau) = \left[T_1(\bm{x},\tau),T_2(\bm{x},\tau)\right]^T$ is approximated by

$$\mathbf{T}(\mathbf{x},\tau) = \mathbf{N}_1 \sum_{a=1}^{n} \mathbf{B}^a(\mathbf{x}) \mathbf{u}_0^a(\tau), \tag{47}$$

where

$$\mathbf{B}^{a}(\mathbf{x}) = h \begin{bmatrix} c_{11}N_{,1}^{a} & c_{12}N_{,2}^{a} \\ c_{12}N_{,1}^{a} & c_{22}N_{,2}^{a} \\ c_{66}N_{,2}^{a} & c_{66}N_{,1}^{a} \end{bmatrix},$$

and the vector $\mathbf{u}_0^a(\tau)$ is defined as a column vector $\mathbf{u}_0^a(\tau) = [\hat{u}_{10}^a(\tau), \hat{u}_{20}^a(\tau)]^T$.

Finally, for the approximation of $\mathbf{P}(\mathbf{x}, \tau) = [P_1(\mathbf{x}, \tau), P_2(\mathbf{x}, \tau)]^T$, we have

$$\mathbf{P}(\mathbf{x},\tau) = \sum_{a=1}^{n} [\mathbf{R} \boldsymbol{\psi}^{a}(\tau) + \mathbf{R} \mathbf{L}^{a}(\mathbf{x}) \hat{u}_{30}^{a}(\tau) + \mathbf{K} \mathbf{L}^{a}(\mathbf{x}) \hat{w}_{3}^{a}(\tau)], \qquad (48)$$

with $\mathbf{K} = h \begin{bmatrix} K_{1} & 0\\ 0 & K_{2} \end{bmatrix}.$

Then, insertion of the MLS-discretized moment, traction and shear force fields (44)–(47), and (48) into the local integral equations (36)–(40) yields the discretized local integral equations

$$\sum_{a=1}^{n} \left[\int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{N}_{1}(\mathbf{x}) \mathbf{B}_{1}^{a}(\mathbf{x}) d\Gamma - \kappa \int_{\Omega_{s}^{i}} \mathbf{C} \mathbf{N}^{a}(\mathbf{x}) d\Omega \right] \boldsymbol{\psi}^{a}(\tau) - \sum_{a=1}^{n} \left(\mathbf{E} I_{M} \int_{\Omega_{s}^{i}} \mathbf{N}^{a}(\mathbf{x}) d\Omega \right) \ddot{\boldsymbol{\psi}}^{a}(\tau) - \sum_{a=1}^{n} \tilde{u}_{30}^{a}(\tau) \left(\kappa \int_{\Omega_{s}^{i}} \mathbf{C} \mathbf{L}^{a}(\mathbf{x}) d\Omega \right) - \sum_{a=1}^{n} \hat{w}_{3}^{a}(\tau) \left(\kappa \int_{\Omega_{s}^{i}} \mathbf{R} \mathbf{L}^{a}(\mathbf{x}) d\Omega \right) = - \int_{\Gamma_{sM}^{i}} \tilde{\mathbf{M}}(\mathbf{x}, \tau) d\Gamma,$$
(49)

$$\begin{split} \sum_{a=1}^{n} \left(\kappa \int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{C}_{n}(\mathbf{x}) N^{a}(\mathbf{x}) d\Gamma \right) \hat{\psi}^{a}(\tau) \\ &+ \sum_{a=1}^{n} \left(\kappa \int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{C}_{n}(\mathbf{x}) \mathbf{L}^{a}(\mathbf{x}) d\Gamma \right) \hat{u}_{30}^{a}(\tau) \\ &+ \sum_{a=1}^{n} \left(\kappa \int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{R}_{n}(\mathbf{x}) \mathbf{L}^{a}(\mathbf{x}) d\Gamma \right) \hat{w}_{3}^{a}(\tau) \\ &- \sum_{a=1}^{n} \left(I_{Q} \int_{\Omega_{s}^{i}} N^{a}(\mathbf{x}) d\Omega \right) \\ \ddot{u}_{30}^{a}(\tau) =- \int_{\Omega_{s}^{i}} q(\mathbf{x}, \tau) d\Omega, \end{split}$$
(50)

$$\sum_{a=1}^{n} \left[\int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{N}_{1}(\mathbf{x}) \mathbf{B}^{a}(\mathbf{x}) d\Gamma \right] \mathbf{u}_{0}^{a}(\tau) - \sum_{a=1}^{n} \left[\mathbf{E} I_{Q} \int_{\Omega_{s}^{i}} N^{a}(\mathbf{x}) d\Omega \right] \ddot{u}_{0}^{a}(\tau)$$
$$= -\int_{\tilde{\Gamma}_{sT}^{i}} \tilde{\mathbf{T}}(\mathbf{x},\tau) d\Gamma - \int_{\Omega_{s}^{i}} \mathbf{q}(\mathbf{x},\tau) d\Omega, \tag{51}$$

$$\sum_{a=1}^{n} \left(\int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{R}_{n}(\mathbf{x}) N^{a}(\mathbf{x}) d\Gamma \right) \hat{\psi}^{a}(\tau) \\ + \sum_{a=1}^{n} \left(\int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{R}_{n}(\mathbf{x}) \mathbf{L}^{a}(\mathbf{x}) d\Gamma \right) \hat{u}_{30}^{a}(\tau) \\ + \sum_{a=1}^{n} \left(\int_{\partial \tilde{\Omega}_{s}^{i}} \mathbf{K}_{n}(\mathbf{x}) \mathbf{L}^{a}(\mathbf{x}) d\Gamma \right) \hat{w}_{3}^{a}(\tau) \\ - \sum_{a=1}^{n} \left(I_{Q} \int_{\Omega_{s}^{i}} N^{a}(\mathbf{x}) d\Omega \right) \ddot{w}_{3}^{a}(\tau) \\ = - \int_{\Omega_{s}^{i}} \mathbf{g}(\mathbf{x}, \tau) d\Omega,$$
(52)

or alternative to Eq. (52), we have

$$\sum_{a=1}^{n} \left(\int_{\partial \bar{\Omega}_{s}^{i}} \mathbf{R}_{n}(\mathbf{x}) N^{a}(\mathbf{x}) d\Gamma \right) \hat{\Psi}^{a}(\tau) \\ + \sum_{a=1}^{n} \left(\int_{\partial \bar{\Omega}_{s}^{i}} \mathbf{R}_{n}(\mathbf{x}) \mathbf{L}^{a}(\mathbf{x}) d\Gamma \right) \hat{u}_{30}^{a}(\tau) \\ + \sum_{a=1}^{n} \left(\int_{\partial \bar{\Omega}_{s}^{i}} \mathbf{K}_{n}(\mathbf{x}) \mathbf{L}^{a}(\mathbf{x}) d\Gamma \right) \hat{w}_{3}^{a}(\tau) \\ - \sum_{a=1}^{n} \left(I_{D} \int_{\Omega_{s}^{i}} N^{a}(\mathbf{x}) d\Omega \right) \dot{w}_{3}^{a}(\tau) \\ = - \int_{\Omega_{s}^{i}} g(\mathbf{x}, \tau) d\Omega,$$
(53)

in which

$$\mathbf{E} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix},$$

 $\tilde{\mathbf{M}}(\mathbf{x}, \tau)$ represents the prescribed bending moments on Γ_{sM}^i , $\tilde{\mathbf{T}}(\mathbf{x}, \omega)$ is the prescribed traction vector on Γ_{sT}^i , and

$$\mathbf{C}_{n}(\mathbf{x}) = h(n_{1}, n_{2}) \begin{pmatrix} c_{55} & 0\\ 0 & c_{44} \end{pmatrix} = h(c_{55}n_{1}, c_{44}n_{2}),$$
$$\mathbf{R}_{n}(\mathbf{x}) = h(n_{1}, n_{2}) \begin{pmatrix} R_{6} & 0\\ 0 & R_{5} \end{pmatrix} = h(R_{6}n_{1}, R_{5}n_{2}),$$
$$\mathbf{K}_{n}(\mathbf{x}) = h(n_{1}, n_{2}) \begin{pmatrix} K_{1} & 0\\ 0 & K_{2} \end{pmatrix} = h(K_{1}n_{1}, K_{2}n_{2}).$$

Eqs. (49)–(53) are considered on the subdomains adjacent to the interior nodes \mathbf{x}^i as well as for the source point \mathbf{x}^i located on the global boundary Γ . We point out that

$$\partial \tilde{\Omega}^i_s = \partial \Omega^i_s$$
 and $\tilde{\Gamma}^i_{sM} = \{ \emptyset \}, \quad \tilde{\Gamma}^i_{sT} = \{ \emptyset \}, \quad \text{if } \mathbf{x}^i \in \Omega,$

whilst for the boundary point $\mathbf{x}^i \in \Gamma$ we define

$$\begin{split} \partial \tilde{\Omega}_{s}^{i} &= L_{s}^{i} = \partial \Omega_{s}^{i} \cap \Omega, \quad \tilde{\Gamma}_{sM}^{i} = \Gamma_{sM}^{i} = \partial \Omega_{s}^{i} \cap \Gamma_{M}, \quad \tilde{\Gamma}_{sT}^{i} = \Gamma_{sT}^{i} \\ &= \partial \Omega_{s}^{i} \cap \Gamma_{T} \end{split}$$

with $\partial \Omega_s^i = L_s^i \cup \Gamma_{sM}^i \cup \Gamma_{sT}^i$ and Γ_M or Γ_T being the part of the global boundary with prescribed bending moment or in-plane tractions, respectively.

It should be noted that there are neither Lagrange multipliers nor penalty parameters introduced into the local weak-forms (25)–(29) because the essential boundary conditions on $\Gamma_{s\psi}^i$ and $\Gamma_{su_3}^i$ (part of the global boundary with prescribed rotations and deflection) and $\Gamma_{su_0}^i$ (part of the global boundary with prescribed in-plane displacements) can be imposed directly, using the interpolation approximation (41)

$$\sum_{a=1}^{n} N^{a}(\mathbf{x}^{i})\hat{u}^{a}(\tau) = \tilde{u}(\mathbf{x}^{i},\tau) \quad \text{for } \mathbf{x}^{i} \in \Gamma_{s\psi}^{i} \text{ or } \Gamma_{su}^{i} \text{ or } \Gamma_{sw}^{i},$$
(54)

where $\tilde{u}(\mathbf{x}^i, \tau)$ is the prescribed phonon and phason values on the boundary Γ_{sy}^i , Γ_{su}^i and Γ_{sw}^i . For a clamped plate, the rotations, deflection and phason displacement are vanishing on the fixed edge, and Eq. (54) is used at all the boundary nodes in such a case. However, for a simply-supported plate, the deflection $\tilde{u}_{30}(\mathbf{x}^i, \tau)$, bending moments and phason displacement $\tilde{w}_3(\mathbf{x}^i, \tau)$ are prescribed, while the rotations are unknowns. Then, the discretized LIE (49) is employed at $\mathbf{x}^i \in \Gamma_{sM}^i$.

The local boundary-domain integral equations (49)-(53) together with the collocation equations (54) on the global boundary for essential conditions are recast into a complete system of ODEs in terms of the Bak model

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{F}\mathbf{x} = \mathbf{Y},\tag{55}$$

where the column-vector **x** is formed by the nodal unknowns $\{u_{10}^a, u_{20}^a, \psi_1^a, \psi_2^a, u_{30}^a, w_3^a\}$. The Houbolt method (Houbolt, 1950) is applied for the second order ODE (55).

The discretized equations for the elasto hydrodynamic model can be written in the following form

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{Y}.$$
 (56)

The Houbolt finite-difference scheme is applied to approximate the acceleration

$$\ddot{\mathbf{x}}_{\tau+\Delta\tau} = \frac{2\mathbf{x}_{\tau+\Delta\tau} - 5\mathbf{x}_{\tau} + 4\mathbf{x}_{\tau-\Delta\tau} - \mathbf{x}_{\tau-2\Delta\tau}}{\Delta\tau^2},\tag{57}$$

and the backward difference method to approximate the diffusion term

$$\dot{\mathbf{x}}_{\tau+\Delta\tau} = \frac{\mathbf{x}_{\tau+\Delta\tau} - \mathbf{x}_{\tau}}{\Delta\tau}.$$
(58)

4. Numerical examples

A square plate with a side-length a = 0.254 m is analyzed to verify the proposed computational method. The total thickness of the plate is h = 0.012 m In the review paper by Fan (2013) some relevant measured data were collected for certain important quasicrystal systems, which are necessary for understanding the physics of the materials. The measurement of elastic constants and results were also given in Fan (2013). The material properties of the plate corresponding to Al–Ni–Co QC are

$$c_{11} = c_{22} = 23.43 \cdot 10^{10} \text{ Nm}^{-2}, \quad c_{12} = 5.74 \cdot 10^{10} \text{ Nm}^{-2},$$

 $c_{44} = c_{55} = 7.19 \cdot 10^{10} \text{ Nm}^{-2},$

$$K_1 = 12.2 \cdot 10^{10} \text{ Nm}^{-2}, \quad K_2 = 2.4 \cdot 10^{10} \text{ Nm}^{-2},$$

 $c_{66} = (c_{11} - c_{12})/2,$

$$ho = 4180 \ {
m kg/m^3}, \quad \Gamma_w = 4.8 \cdot 10^{-19} \ {
m m^3 \ s/kg}$$

On the top surface a uniform mechanical load is applied with an intensity $q = 1 \cdot 10^7 \text{ N/m}^2$. Simply-supported boundary conditions are considered here. In our numerical calculations, 441 nodes with a regular distribution were used for the approximation of the rotations, deflection, in-plane displacements and phason displacement in the neutral plane. The origin of the coordinate system is located at the center of the plate. The variation of the deflection in the plate along normalized coordinate x_1/a with fixed $x_2 = 0$ is presented in Fig. 2. We assumed the same values for both coupling parameters $R = R_5 = R_6$, which were further normalized by the stiffness parameter $M = c_{66}$. Numerical results for the phonon displacement u_{30} are given for various coupling parameters, R/M. For the case of vanishing coupling parameter, R/M = 0, the phonon displacements reduce to the ones in conventional elasticity (i.e., the deflection). We point out that a good agreement between the FEM and present MLPG results was obtained for this special case of R/M = 0, which partially validated our computer code based on MLPG. For *R/M* different from zero, our results reveal that the phonon displacement increases with increasing value of the coupling parameter. The variation of the phason displacements along normalized coordinate x_1/a is presented in Fig. 3. It shows that the phason displacements are strongly



Fig. 2. Variation of the phonon deflection along normalized coordinate x_1/a for a simply-supported square plate.



Fig. 3. Variation of the phason deflection along normalized coordinate x_1/a for a simply-supported square plate.

dependent on the coupling parameter. For vanishing value of the coupling parameter, the phason displacement should be identically zero (i.e., the horizontal axis). We should also remark that the bending moments are independent of the coupling parameter, as can be observed directly from Eq. (12), and that the interaction between phonon and phason activities could be complicated (see, e.g., Ricker and Trebin, 2002; Mariano, 2006).

We have analyzed also the same plate with doubled thickness, h = 0.024 m to show the influence of the plate thickness on the phonon and phason deflections. Phonon deflections are presented in Fig. 4. One can observe that the influence of the coupling parameter *R* is stronger for the plate with doubled thickness. However, the absolute value of the plate deflection is smaller for this plate due to the larger flexural rigidity. A similar behavior can be observed also for the phason deflection presented in Fig. 5.

A clamped square plate under a uniform static load is analyzed in the next numerical example. The geometrical and material parameters are the same as for the previous simply-supported plate. Numerical results for the phonon displacement u_{30} are given for various coupling parameters, R/M as shown in Fig. 6. For the case of vanishing coupling parameter, R/M = 0, the phonon displacements are reduced to the conventional elasticity results. The results in Fig. 6 reveal that the phonon displacement increases with increasing value of the coupling parameter. The influence of the coupling parameter is more significant for the clamped plate than for the simply-supported plate, as can be directly observed by comparing Fig. 6 to Fig. 2.



Fig. 4. Variation of the phonon deflection along normalized coordinate x_1/a for a simply-supported square plate with doubled plate thickness.



Fig. 5. Variation of the phason deflection along normalized coordinate x_1/a for a simply-supported square plate with doubled plate thickness.



Fig. 6. Variation of the phonon deflection along normalized coordinate x_1/a for a clamped square plate.

The variation of the bending moment M_{11} is shown in Fig. 7. The bending moment at the center of the plate $M_{11}^{stat}(a/2) = 0.1426 \cdot 10^5$ Nm is used as a normalized parameter. The bending moment value is independent of the coupling parameters $R = R_5 = R_6$. This follows directly from the bending moment expressions (10).



Fig. 7. Variation of the bending moment along normalized coordinate x_1/a for a clamped square plate.

We have analyzed also the same plate with the doubled thickness, h = 0.024 m to show the influence of the plate thickness on the phonon deflections. Phonon deflections are presented in Fig. 8. Similar to the simply-supported case, for the clamped plate the influence of the coupling parameter *R* is stronger for the plate with doubled thickness.

In the next example, we analyze the same clamped square plate under an impact load with Heaviside time variation qH(t - 0). The normalized phonon and phason deflections at the plate center are presented in Figs. 9 and 10 for two different coupling parameters. The central plate deflections are normalized by the corresponding static quantity valid for the crystal, $u_{30}^{stat} = 0.1646 \cdot 10^{-2}$ m. One can observe that the phonon deflection for a finite value of the coupling parameter is only slightly larger than the corresponding deflection in conventional elasticity. The phason deflection is nonzero only for a finite value of the couple parameter. We point out that we have selected the time-step $\Delta \tau = 0.125 \cdot 10^{-4}$ s in our numerical analyses with the numerical results being presented based on the Bak model.

The normalized bending moments at the plate center, $M(A) = M_{11}(0,0)/M_{11}^{stat}$ and at the center of the clamped side, $M(B) = M_{11}(a/2, a/2)/M_{11}^{stat}$, are presented in Figs. 11. The corresponding numerical results are normalized by the static value of the bending moment at the central point valid for the crystal, $M_{11}^{stat}(0,0) = 0.1426 \cdot 10^5$ Nm. The peak values of the bending moments are approximately twice the corresponding static quantities.



Fig. 8. Variation of the phonon deflection along normalized coordinate x_1/a for a clamped square plate with doubled plate thickness.



Fig. 9. Temporal variations of the normalized phonon deflection for the clamped plate under an impact load.



Fig. 10. Temporal variations of the normalized phason deflection for the clamped plate under an impact load.



Fig. 11. Temporal variations of the normalized bending moments for the clamped plate under an impact load.

Numerical results are given for the crystal (vanishing couple parameters) case only since the bending moment is the same for the crystal and QC.

5. Conclusions

The following conclusions can be drawn from the present study:

- 1. The meshless local Petrov–Galerkin method is applied to analyze plate bending problems in orthorhombic QC plate under static and transient dynamic loads. The phason displacement for the orthorhombic QC in the first-order shear deformation plate theory depends only on the in-plane coordinates over the mean plate surface. The Reissner–Mindlin theory reduces the original 3D thick plate problem to a 2D plate problem.
- 2. Nodal points are randomly distributed over the mean plane of the considered plate. Each node is the center of a circle surrounding it. The weak form on small subdomains with the Heaviside step function as the test function is applied to derive the local integral equations. After performing the spatial MLS approximation, a system of ODEs for certain nodal unknowns is obtained. The system of ODEs of the second order is solved by the Houbolt finite-difference scheme.
- 3. The present numerical results are believed to be new since the commercial computer codes based on the FEM or BEM cannot be applied to analyze the boundary value problems in QCs.

- 4. Numerical results showed that coupling material parameters have a small influence on the phonon deflection for real values of material parameters in QCs. A stronger influence of the coupling parameters on the phonon deflection is observed for plates with larger plate thickness.
- 5. The coupling parameter *R* has a vanishing influence on the phonon deflection in terms of the Bak model.

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