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Analysis of an interfacial crack in a piezoelectric bi-material via the extended Green's functions and displacement discontinuity method



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ABSTRACT

Based on the extended Stroh formalism, we first derive the extended Green's functions for an extended dislocation and displacement discontinuity located at the interface of a piezoelectric bi-material. These include Green's functions of the extended dislocation, displacement discontinuities within a finite interval and the concentrated displacement discontinuities, all on the interface. The Green's functions are then applied to obtain the integro-differential equation governing the interfacial crack. To eliminate the oscillating singularities associated with the delta function in the Green's functions, we represent the delta function in terms of the Gaussian distribution function. In so doing, the integro-differential equation is reduced to a standard integral equation for the interfacial crack problem in piezoelectric bi-material with the extended displacement discontinuities being the unknowns. A simple numerical approach is also proposed to solve the integral equation for the Gaussian parameter on the discontinuities near the crack tip. In numerical examples, the effect of the Gaussian parameter on the numerical results is discussed, and the influence of different extended loadings on the interfacial crack behaviors is further investigated.

1. Introduction

Because of their accumulative effects, multi-layered structures are often preferred in the manufacture of composite materials and are widely used in engineering to enhance the efficiency and sensitivity of materials or structures. However, there is a high possibility of failure at the interface between layers in such composite materials. Cracks leading to fracture often occur at the interface between two different constituents, e.g., the fiber and matrix in a composite. Fracture behaviors of the interfacial crack have attracted increasing attention because of the need in understanding the real failure regime of such composites (e.g., Comninou, 1977a,b; Atkinson, 1977; Dundurs and Gautesen, 1988; Qu and Li, 1991; Hutchinson and Suo, 1992). Due to its coupled mechanical and electric properties, piezoelectric material has been widely used in intelligent structures and systems. Research on interface cracks has thus been extended to piezoelectric media (e.g., Suo et al., 1992; Beom and Atluri, 1996; Chen et al., 1998; Oin and Mai, 1999; Zhang et al., 2002).

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Early solutions to the interfacial crack problems in the context of linear elastic fracture mechanics showed that there is an oscillatory singularity near the crack tip (Williams, 1959; England, 1965; Erdogan, 1965). This kind of singularity shows an unsatisfactory behavior, namely rapid oscillations in the stress and displacement fields, implying the physically impossible phenomenon of interpenetration (England, 1965; Comninou, 1990). To overcome this feature, an alternative technique, which has been used frequently in the numerical calculation of fracture parameters for interfacial cracks, is to insert a thin, isotropic, homogeneous layer (interlayer) at the interface with the crack being located within the interlayer. The problem is thus effectively converted to that of a crack in a homogeneous medium. On the other hand, Eshelby (1951) treated the crack as the pile-up of interfacial dislocations and used a Dirac delta function to model the stress oscillatory singularity in the fundamental solutions of interface dislocations. Thus, using the Green's function of dislocations in purely elastic bi-materials, which contains a delta function on the interface (Comninou, 1977a,b), Zhang and Wang (2013) becomes an attractive model for removing the physically unreasonable oscillation singularity associated with the interfacial crack where the delta function in the fundamental solution is represented by a locally-distributed continuous function. Near an interfacial crack tip, besides the classical singularity $r^{-1/2}$ and the well-known oscillatory singularity $r^{-1/2\pm i\epsilon}$, the extended stresses in piezoelectric bi-materials have a

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new type of singularity, namely $r^{-1/2\pm\kappa}$ (Kuo and Barnett, 1991; Suo et al., 1992; Ou and Wu, 2003). This new type of singularity also exists in three-dimensional piezoelectric bi-materials (Zhao et al., 2008). It is also found that in a transversely isotropic bi-material with an impermeable interface crack, the ε -type and κ -type singularities cannot coexist (Ou and Wu, 2003; Zhao et al., 2008).

Motivated by the recent work of Zhang and Wang (2013), we present a new approach to remove the oscillatory singularities at the tip of the interfacial crack in piezoelectric bi-materials. It is based on the powerful displacement discontinuity method (i.e., Zhao et al., 2008) combined with the representation of the delta function by the Gaussian distribution function. This paper is arranged as follows. In Section 2, we introduce the extended Stroh formalism. In Section 3, we derive the Green's functions of extended dislocations, uniform extended displacement discontinuities within a finite interval, and the concentrated extended displacement discontinuities on the interface of the piezoelectric bi-material. In Section 4, the obtained Green's functions are first applied to the interfacial crack problem, resulting in the integrodifferential equations. Then, the involved delta function in the solution is represented by the Gaussian distribution function, which helps reduce the integro-differential equations to the standard integral equations in terms of extended displacement discontinuities for the interfacial crack in the piezoelectric bi-material. The formulations for the extended stress intensity factors and local J-integral of the interfacial crack are presented in Section 5. Section 6 describes the numerical method for solving the unknown displacement discontinuities along the interfacial crack. In Section 7, various numerical results are presented along with a detailed discussion on the effect of the *ɛ*-parameter in the Gaussian distribution function. We finally draw our conclusions in Section 8.

2. Extended Stroh formalism

In the Cartesian coordinate system, x_i (i = 1, 2, 3), the governing equations for a linear piezoelectric solid without body force and free of electric charge can be given by equilibrium Equation (1), kinematic Equation (2), and constitutive Equation (3) below

$$\sigma_{ijj} = 0, \quad D_{i,i} = 0, \tag{1}$$

$$\varepsilon_{ij} = \frac{1}{2} \left(u_{ij} + u_{j,i} \right), \quad E_i = -\varphi_{,i}, \tag{2}$$

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \tag{3a}$$

$$D_k = e_{kij}\varepsilon_{ij} + \kappa_{kl}E_l. \tag{3b}$$

In Eqs. (1)–(3), σ_{ij} and D_i are, respectively, the components of the stress and electric displacement; ε_{ij} and E_i are, respectively, the components of the strain and electric field; u_i and φ are the elastic displacement and electric potential, respectively; c_{ijkl} , e_{kij} and κ_{ij} are, respectively, the elastic constants, piezoelectric constants and dielectric permittivities. A subscript comma denotes the partial differentiation with respect to the coordinate followed.

For two-dimensional deformations in the x_1 - x_2 plane, in which the extended displacement vector $\mathbf{u} = (u_1 \ u_2 \ u_3 \ u_4)^T \equiv (u_1 \ u_2 \ u_3 \ \varphi)^T$ depends only on the x_1 and x_2 coordinates, the general solution takes the form of (Barnett and Lothe, 1975; Ting, 1996)

$$\mathbf{u} = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}} \,\overline{\mathbf{f}(z)},\tag{4}$$

$$\Phi = \mathbf{B}\mathbf{f}(z) + \overline{\mathbf{B}}\,\overline{\mathbf{f}(z)},\tag{5}$$

where an overbar indicates a complex conjugate; matrices $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ and $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$, with \mathbf{a}_j and \mathbf{b}_j being the eigenvectors (j = 1-4); $\mathbf{f}(z) = (f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4))^T$ is

an analytic function vector with $z_{\alpha} = x_1 + p_{\alpha}x_2$ and p_{α} a complex eigenvalue having a positive imaginary part; $\mathbf{\Phi}$ is the extended stress function vector, from which the stress vectors can be obtained as

$$\Sigma_2 = (\sigma_{21} \ \sigma_{22} \ \sigma_{23} \ D_2)^1 = \Phi_{,1}, \tag{6a}$$

$$\Sigma_1 = (\sigma_{11} \ \sigma_{12} \ \sigma_{13} \ D_1)^{\mathrm{T}} = -\Phi_{,2}. \tag{6b}$$

The eigenvalue p_{α} is obtained by solving the following standard eigensystem of equations (Ting, 1996; Zhang et al., 2002)

$$\begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = p \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix},$$
(7)

where

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^{\mathrm{T}}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q} = \mathbf{N}_3^{\mathrm{T}}, \tag{8}$$

$$\mathbf{Q} = \begin{pmatrix} c_{i1k1} & e_{11i} \\ e_{11i}^T & -\kappa_{11} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} c_{i1k2} & e_{21i} \\ e_{12i}^T & -\kappa_{12} \end{pmatrix},
\mathbf{T} = \begin{pmatrix} c_{i2k2} & e_{22i} \\ e_{22i}^T & -\kappa_{22} \end{pmatrix}, \quad i, k = 1, 2, 3.$$
(9)

The eigenvectors are uniquely determined except for an arbitrary multiple constant. Thus, they can be normalized to satisfy (Ting, 1996; Zhang et al., 2002)

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} + \overline{\mathbf{A}\mathbf{A}^{\mathrm{T}}} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \overline{\mathbf{B}\mathbf{B}^{\mathrm{T}}} = \mathbf{0}, \ \mathbf{B}\mathbf{A}^{\mathrm{T}} + \overline{\mathbf{B}\mathbf{A}^{\mathrm{T}}} = \mathbf{A}\mathbf{B}^{\mathrm{T}} + \overline{\mathbf{A}\mathbf{B}^{\mathrm{T}}} = \mathbf{I},$$
 (10)

where ${\bm I}$ is a 4×4 unit matrix. For convenient analysis below, we also define a new matrix ${\bm Y}$ as

$$\mathbf{Y} = \mathbf{i}\mathbf{A}\mathbf{B}^{-1}, \quad \mathbf{i} = \sqrt{-1}. \tag{11}$$

3. Green's functions of extended displacement discontinuities on the interface of a piezoelectric bi-material

3.1. Green's functions of extended dislocations along the interface in the piezoelectric bi-material

For a piezoelectric bi-material, the coordinate system is oriented with the x_1 -axis being along the interface and x_2 -axis perpendicular to the interface. The superscripts "+" and "–" denote, respectively, the quantities in the upper half-plane where $x_2 > 0$ and those in the lower half-plane where $x_2 < 0$. We now assume that an extended dislocation **b** = $(b_1, b_2, b_3, b_4)^T$ is applied on the interface along the whole negative x_1 -axis. We point out that in the extended dislocation expression, b_i (i = 1, 2, 3) denotes the x_i -component of the Burgers vector and $b_4 \equiv \Delta \varphi$ denotes the electric dislocation. Thus, the corresponding boundary conditions on the interface can be expressed as

$$\|\mathbf{u}(x_1)\| \equiv \mathbf{u}^+(x_1, \mathbf{0}^+) - \mathbf{u}^-(x_1, \mathbf{0}^-) = \mathbf{b}H(-x_1), \tag{12}$$

$$\Sigma_2^+(x_1,0^+) - \Sigma_2^-(x_1,0^-) = \mathbf{0}, \tag{13}$$

where $\|\mathbf{u}(x_1)\| = (\|u_1\| \|u_2\| \|u_3\| \|\varphi\|)^T$ denotes the corresponding displacement discontinuity (or opening displacement) and the electric potential discontinuity (or electric potential jump), and H(x) is the Heaviside function defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$
(14)

Following Nakahara and Willis (1973) and Zhang et al. (1998), we assume the analytic function vector $\mathbf{f}(z)$ in Eqs. (4) and (5) for an extended line dislocation located at $(x_1, x_2) = (0, 0)$ as

$$\mathbf{f}(z) = \langle \ln z_{\alpha} \rangle \mathbf{d},\tag{15}$$

where $\langle \ln z_{\alpha} \rangle = diag(\ln z_1 \ln z_2 \ln z_3 \ln z_4)$, and $\mathbf{d} = (d_1 d_2 d_3 d_4)^{\mathrm{T}}$ is an unknown vector to be determined by the boundary conditions.

Following Ting (1996), the general solutions in both half planes, including the extended displacements, the extended stress functions, the extended tractions and the extended in-plane stresses, can be expressed as

$$\begin{aligned} \mathbf{u}^{\pm} &= \mathbf{A}^{\pm} \langle \ln z_{\alpha}^{\pm} \rangle \mathbf{d}^{\pm} + \overline{\mathbf{A}^{\pm}} \langle \overline{\ln z_{\alpha}^{\pm}} \rangle \mathbf{d}^{\pm}, \\ \mathbf{\Phi}^{\pm} &= \mathbf{B}^{\pm} \langle \ln z_{\alpha}^{\pm} \rangle \mathbf{d}^{\pm} + \overline{\mathbf{B}^{\pm}} \langle \overline{\ln z_{\alpha}^{\pm}} \rangle \overline{\mathbf{d}^{\pm}}, \end{aligned}$$
(16a)

$$\begin{split} \boldsymbol{\Sigma}_{2}^{\pm} &= \mathbf{B}^{\pm} \langle 1/\boldsymbol{z}_{\alpha}^{\pm} \rangle \mathbf{d}^{\pm} + \overline{\mathbf{B}^{\pm}} \langle 1/\overline{\boldsymbol{z}_{\alpha}^{\pm}} \rangle \overline{\mathbf{d}^{\pm}}, \\ \boldsymbol{\Sigma}_{1}^{\pm} &= -\mathbf{B}^{\pm} \langle \boldsymbol{p}_{\alpha}^{\pm}/\boldsymbol{z}_{\alpha}^{\pm} \rangle \mathbf{d}^{\pm} - \overline{\mathbf{B}^{\pm}} \langle \overline{\boldsymbol{p}_{\alpha}^{\pm}}/\overline{\boldsymbol{z}_{\alpha}^{\pm}} \rangle \overline{\mathbf{d}^{\pm}}. \end{split}$$
(16b)

where

$$\langle 1/z_{\alpha}^{\pm} \rangle = diag(1/z_{1}^{\pm} \ 1/z_{2}^{\pm} \ 1/z_{3}^{\pm} \ 1/z_{4}^{\pm}),$$

$$\langle p_{\alpha}^{\pm}/z_{\alpha}^{\pm} \rangle = diag(p_{1}^{\pm}/z_{1}^{\pm} \ p_{2}^{\pm}/z_{2}^{\pm} \ p_{3}^{\pm}/z_{3}^{\pm} \ p_{4}^{\pm}/z_{4}^{\pm}).$$

$$(17)$$

It is noted that when x_2 in Eqs. (16) and (17) approaches zero from either side of the interface, we obtain the corresponding solutions on the interface. Following Nakahara and Willis (1973) and Wang and Kuang (2000), we define

$$\ln z_{\alpha} = \ln |x_1| \pm i\pi H(-x_1), \quad \text{as } x_2 = 0^{\pm}.$$
(18)

Thus, the extended displacements at the upper and lower interfaces have the form

$$\mathbf{u}^{+} = \mathbf{A}^{+} \mathbf{d}^{+} (\ln |x_{1}| + i\pi H(-x_{1})) + \overline{\mathbf{A}^{+} \mathbf{d}^{+}} (\ln |x_{1}| - i\pi H(-x_{1})), \quad (19a)$$

$$\mathbf{u}^{-} = \mathbf{A}^{-}\mathbf{d}^{-}(\ln|x_{1}| - i\pi H(-x_{1})) + \overline{\mathbf{A}^{-}\mathbf{d}^{-}}(\ln|x_{1}| + i\pi H(-x_{1})).$$
(19b)

From Eq. (16b), the extended stresses at the upper and lower interfaces can be expressed as

$$\Sigma_{2}^{+}(x_{1},0^{+}) = \mathbf{B}^{+}\mathbf{d}^{+}\frac{1}{x_{1}+0^{+}i} + \overline{\mathbf{B}^{+}\mathbf{d}^{+}}\frac{1}{x_{1}-0^{+}i},$$
(20a)

$$\Sigma_{2}^{-}(x_{1},0^{-}) = \mathbf{B}^{-}\mathbf{d}^{-}\frac{1}{x_{1}-0^{-}i} + \overline{\mathbf{B}^{-}\mathbf{d}^{-}}\frac{1}{x_{1}+0^{-}i}.$$
 (20b)

Using the well-known identity in the theory of generalized functions (Gelfand and Shilov, 1964; Nakahara and Willis, 1973), we can express Eq. (20) alternatively as

$$\boldsymbol{\Sigma}_{2}^{+}(\boldsymbol{x}_{1},\boldsymbol{0}^{+}) = \mathbf{B}^{+}\mathbf{d}^{+}\left(\frac{1}{\boldsymbol{x}_{1}} - i\pi\delta(\boldsymbol{x}_{1})\right) + \overline{\mathbf{B}^{+}\mathbf{d}^{+}}\left(\frac{1}{\boldsymbol{x}_{1}} + i\pi\delta(\boldsymbol{x}_{1})\right), \qquad (21a)$$

$$\boldsymbol{\Sigma}_{2}^{-}(\boldsymbol{x}_{1},\boldsymbol{0}^{-}) = \boldsymbol{B}^{-}\boldsymbol{d}^{-}\left(\frac{1}{\boldsymbol{x}_{1}} + i\pi\delta(\boldsymbol{x}_{1})\right) + \overline{\boldsymbol{B}^{-}\boldsymbol{d}^{-}}\left(\frac{1}{\boldsymbol{x}_{1}} - i\pi\delta(\boldsymbol{x}_{1})\right), \quad (21b)$$

where $\delta(x_1)$ is the Dirac delta function.

It is noted now that by substituting Eq. (19) into the discontinuous displacement boundary conditions on the interface (i.e., Eq. (12)), we obtain

$$\begin{pmatrix} \mathbf{A}^{+}\mathbf{d}^{+} + \mathbf{A}^{+}\mathbf{d}^{+} - \mathbf{A}^{-}\mathbf{d}^{-} - \overline{\mathbf{A}^{-}\mathbf{d}^{-}} \end{pmatrix} \ln |x_{1}| + i\pi \left(\mathbf{A}^{+}\mathbf{d}^{+} - \overline{\mathbf{A}^{+}\mathbf{d}^{+}} + \mathbf{A}^{-}\mathbf{d}^{-} - \overline{\mathbf{A}^{-}\mathbf{d}^{-}} \right) H(-x_{1}) = \mathbf{b} H(-x_{1}).$$
 (22)

Similarly, substituting Eq. (21) into the continuous traction boundary conditions on the interface, (i.e., Eq. (13)), we have

$$\left(\mathbf{B}^{+}\mathbf{d}^{+} + \overline{\mathbf{B}^{+}\mathbf{d}^{+}} - \mathbf{B}^{-}\mathbf{d}^{-} - \overline{\mathbf{B}^{-}\mathbf{d}^{-}} \right) \frac{1}{x_{1}} - i\pi \left(\mathbf{B}^{+}\mathbf{d}^{+} - \overline{\mathbf{B}^{+}\mathbf{d}^{+}} + \mathbf{B}^{-}\mathbf{d}^{-} - \overline{\mathbf{B}^{-}\mathbf{d}^{-}} \right) \delta(x_{1}) = \mathbf{0}.$$
 (23)

Thus, by comparing the coefficients of the same terms on both sides of Eqs. (22) and (23), we obtain

$$\mathbf{A}^{+}\mathbf{d}^{+} + \overline{\mathbf{A}^{+}\mathbf{d}^{+}} - \mathbf{A}^{-}\mathbf{d}^{-} - \overline{\mathbf{A}^{-}\mathbf{d}^{-}} = \mathbf{0},$$
(24)

$$\mathbf{A}^{+}\mathbf{d}^{+} - \overline{\mathbf{A}^{+}\mathbf{d}^{+}} + \mathbf{A}^{-}\mathbf{d}^{-} - \overline{\mathbf{A}^{-}\mathbf{d}^{-}} = \frac{1}{i\pi}\mathbf{I},$$
(25)

$$\mathbf{B}^{+}\mathbf{d}^{+} + \overline{\mathbf{B}^{+}\mathbf{d}^{+}} - \mathbf{B}^{-}\mathbf{d}^{-} - \overline{\mathbf{B}^{-}\mathbf{d}^{-}} = \mathbf{0},$$
(26)

$$B^{+}d^{+} - B^{+}d^{+} + B^{-}d^{-} - \overline{B^{-}d^{-}} = 0.$$
(27)

From Eqs. (24)–(27), we can solve the two unknown vectors ${\bm d}^+$ and ${\bm d}^-$ as

$$\mathbf{d}^{-} = \frac{1}{2\pi} (\mathbf{B}^{-})^{-1} \left(\overline{\mathbf{Y}^{+}} + \mathbf{Y}^{-} \right)^{-1} \mathbf{b},$$
(28a)

$$\mathbf{d}^{+} = \frac{1}{2\pi} (\mathbf{B}^{+})^{-1} (\mathbf{Y}^{+} + \overline{\mathbf{Y}^{-}})^{-1} \mathbf{b}.$$
 (28b)

By substituting Eq. (28) into Eq. (16), we can finally derive the fundamental solutions of the extended dislocations on the interface of a piezoelectric bi-material. For instance, for the extended stresses on the interface, we have

$$\Sigma_{2}^{+}(x_{1},0^{+}) = \frac{1}{\pi} \operatorname{Re}\left(\left(\mathbf{Y}^{+} + \overline{\mathbf{Y}^{-}}\right)^{-1}\right) \mathbf{b} \frac{1}{x_{1}} + \operatorname{Im}\left(\left(\mathbf{Y}^{+} + \overline{\mathbf{Y}^{-}}\right)^{-1}\right) \mathbf{b} \delta(x_{1}).$$
(29)

Eq. (29) is useful in the analysis below.

3.2. Green's functions of uniform extended displacement discontinuities over a finite length on the interface

Without loss of generality, we now assume that there is line element of length 2*a* on the interface ($x_2 = 0$) within, say, $x_1 \in (-a, a)$. Along this element, uniform extended displacement discontinuities are distributed. Namely,

$$\|\mathbf{u}(x_1)\| \equiv \mathbf{u}^+(x_1, \mathbf{0}^+) - \mathbf{u}^-(x_1, \mathbf{0}^-) = \|\mathbf{u}\|, \quad x_1 \in (-a, a).$$
 (30)

Making use of the fundamental solution (29) for the extended dislocations along the interface, we find the extended stresses on the interface due to uniform extended displacement discontinuities over the finite length $x_1 \in (-a, a)$ along the interface, which are

$$\Sigma_{2}^{+}(x_{1},0^{+}) = \frac{1}{\pi} \mathbf{L} \|\mathbf{u}\| \left(\frac{1}{x_{1}-a} - \frac{1}{x_{1}+a}\right) + \mathbf{F} \|\mathbf{u}\| (\delta(x_{1}-a) - \delta(x_{1}+a)),$$
(31)

where

$$\mathbf{L} = \operatorname{Re}\left(\left(\mathbf{Y}^{+} + \overline{\mathbf{Y}^{-}}\right)^{-1}\right), \quad \mathbf{F} = \operatorname{Im}\left(\left(\mathbf{Y}^{+} + \overline{\mathbf{Y}^{-}}\right)^{-1}\right). \tag{32}$$

Eq. (31) indicates that the Green's stresses due to uniform extended displacement discontinuities over a finite interface element are composed of two parts: the first part corresponds to the conventional one which is similar to that in an homogeneous material; the second part is proportional to the delta function which is similar to that associated with an elastic interfacial crack (Comninou, 1990). The second part is troublesome since it leads to the oscillating singularities.

3.3. Green's functions of concentrated interfacial extended displacement discontinuities

The concentrated extended displacement discontinuity located at $(x_1, x_2) = (0, 0)$ is defined as

$$\|\mathbf{u}^p\| = \lim_{n \to \infty} (2a\|\mathbf{u}\|). \tag{33}$$

Substituting this definition into Eq. (31), we obtain the Green's functions for the concentrated extended displacement discontinuities on the interface of piezoelectric bi-materials as:

$$\Sigma_{2}^{+}(x_{1},0^{+}) = \frac{1}{\pi} \mathbf{L} \|\mathbf{u}^{p}\| \frac{1}{x_{1}^{2}} - \mathbf{F} \|\mathbf{u}^{p}\| \delta_{x_{1}}(x_{1}),$$
(34)

where $\delta_{x_1}(x_1)$ is the derivative of the delta function defined as:

$$\delta_{x_1}(x_1) = \frac{\mathrm{d}\delta(x_1)}{\mathrm{d}x_1}.$$
(35)

4. Displacement discontinuity integral equations for an interfacial crack in a piezoelectric bi-material

4.1. Extended displacement discontinuity integro-differential equations

Using the Green's function solutions for the concentrated extended displacement discontinuities given in Eq. (34) and also making use of the Somigliana identity, we derive the following extended displacement discontinuity integro-differential equation for an interfacial crack of length l ($x_1 \in (0, l)$, Fig. 1) in a piezoelectric bi-material

$$\int_{0}^{l} \frac{1}{\pi} \mathbf{L} \| \mathbf{u}(y_{1}) \| \frac{1}{(x_{1} - y_{1})^{2}} dy_{1} - \mathbf{F} \frac{\mathbf{d} \| \mathbf{u}(x_{1}) \|}{\mathbf{d}x_{1}} = -\mathbf{t}(x_{1}),$$
(36)

where $\mathbf{t}(x_1)$ is the prescribed extended traction on the crack surface, which satisfies

$$\mathbf{t}(x_1) = \mathbf{t}(x_1, \mathbf{0}^+) = -\mathbf{t}(x_1, \mathbf{0}^-).$$
(37)

We remark that it is the differential term in the integro-differential Equation (36) which makes the singularity at the crack tip more complex. Besides the classical singularity of order 1/2, the other two singularity indices, namely the κ -type and the oscillating ε -type, appear in the piezoelectric bi-materials (Ou and Wu, 2003; Zhao et al., 2008).

4.2. Treatment of oscillating singularities at the tip of an interfacial crack

The delta function is known to make the stress near the interfacial crack tip oscillatory. However, the delta function can be approximated by various spreading density functions. For example, the Gaussian distribution is a popular one for representing the del-



Fig. 1. A piezoelectric bi-material with an interfacial crack of length l under a uniform traction applied to its surface. A representative crack element of length 2a is also shown.

ta function in physical and engineering analyses. Thus, in the present study, we approximate the delta function by the Gaussian distribution:

$$\delta(\mathbf{x}_1) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left[-\frac{\mathbf{x}_1^2}{2\varepsilon^2}\right].$$
(38)

It is noted that in Eq. (38) parameter ε is a small value and its optimal value could be greatly different for different problems. For instance, in the study of the Euler–Bernoulli and Timoshenko beams with multiple cracks by Palmeri and Cicirello (2011), a large value for ε (say $\varepsilon = 0.1$) in the Gaussian distribution function is preferred since a small ε would introduce a negative bending stiffness. On the other hand, for the interfacial crack in elastic bi-materials, the optimal values of ε would be in the range of (0.002,0.012) (Zhang and Wang, 2013). The selection of and its effect of the ε value for the present problem will be further discussed in the numerical analysis section.

If the delta function is replaced by Eq. (38), its derivative can be expressed correspondingly by

$$\delta_{x_1}(x_1) = -\frac{x_1}{\sqrt{2\pi\varepsilon^3}} \exp\left[-\frac{x_1^2}{2\varepsilon^2}\right].$$
(39)

Thus, the integro-differential Equation (36) can be rewritten as

$$\int_{0}^{t} \frac{1}{\pi} \mathbf{L} \| \mathbf{u}(y_{1}) \| \frac{1}{(x_{1} - y_{1})^{2}} dy_{1} + \int_{0}^{t} \mathbf{F} \| \mathbf{u}(y_{1}) \| \frac{x_{1} - y_{1}}{\sqrt{2\pi\varepsilon^{3}}}$$
$$\times \exp\left[-\frac{(x_{1} - y_{1})^{2}}{2\varepsilon^{2}} \right] dy_{1} = -\mathbf{t}(x_{1}).$$
(40)

It is observed now that we have successfully converted the original integro-differential Equation (36) into a standard integral equation.

5. Extended stress intensity factors and energy release rate for an interfacial crack in a piezoelectric bi-material

The solution derived in Section 4 can now be applied to the analysis of the interfacial crack tip. In Eq. (40), the kernel function in the first integral expression has the singularity of $O(1/r^2)$, whilst the second one has no singularity. Thus, the singular index in this integral equation is dominated by the classical singular index of 1/2.

For convenient analysis, we assume that the origin of the coordinate axis is at the left crack tip (Fig. 1). Then the extended displacement discontinuity near the left crack tip and in front of it can be asymptotically expressed by

$$|\mathbf{u}(x_1)|| = \mathbf{C}(x_1)^{1/2},\tag{41}$$

where **C** is a proportional constant vector. Thus, the stress vector at distance $|x_1| = r \ (r \ll l)$ near the left crack tip and behind it can be calculated as shown in Eq. (42) below.

$$\Sigma_{2}^{+}(r,0^{+}) = \int_{0}^{l} \frac{1}{\pi} \mathbf{L} \|\mathbf{u}(y_{1})\| \frac{1}{(r+y_{1})^{2}} dy_{1} - \int_{0}^{l} \mathbf{F} \|\mathbf{u}(y_{1})\| \frac{r+y_{1}}{\sqrt{2\pi}\varepsilon^{3}}$$

$$\times \exp\left[-\frac{(r+y_{1})^{2}}{2\varepsilon^{2}}\right] dy_{1}$$

$$\approx \frac{1}{\sqrt{r}} \int_{0}^{\infty} \frac{1}{\pi} \mathbf{L} \mathbf{C} \frac{\xi^{1/2}}{(1+\xi)^{2}} d\xi \qquad (42)$$

$$= \frac{1}{2\sqrt{r}} \mathbf{L} \mathbf{C}.$$

It should be pointed out that the scale of $\xi = y_1/r$ is naturally introduced in Eq. (42). Now the extended stress intensity factors are defined as

$$\mathbf{K} = \lim_{r \to 0} \sqrt{2\pi r} \Sigma_2^+(r, 0^+), \tag{43}$$

where $\mathbf{K} = (K_{II} K_{I} 0 K_{D})^{T}$. Substituting Eq. (42) into Eq. (43) leads to

$$\mathbf{K} = \sqrt{\frac{\pi}{2}} \mathbf{L} \mathbf{C}.$$
 (44)

By considering the definition in Eq. (41), Eq. (44) becomes

$$\mathbf{K} = \lim_{x_1 \to 0} \sqrt{\frac{\pi}{2}} \mathbf{L} \frac{\|\mathbf{u}(x_1)\|}{\sqrt{x_1}}.$$
(45)

In other words, the extended displacement discontinuity near the crack tip can also be expressed by the stress intensity factors ${\bf K}$ as

$$\|\mathbf{u}\| = \sqrt{\frac{2x_1}{\pi}} \mathbf{L}^{-1} \mathbf{K}.$$
(46)

With the stress intensity factors **K**, the *J*-integral can be calculated by (Beom and Atluri, 1996; Zhang et al., 2002)

$$J = \frac{1}{4} \mathbf{K}^{\mathrm{T}} \mathbf{H} \mathbf{K}, \tag{47}$$

where

$$\mathbf{M}_{\beta} = -\left[\operatorname{Re}(\mathbf{Y}^{+} + \mathbf{Y}^{-})\right]^{-1}\left[\operatorname{Im}(\mathbf{Y}^{+} - \mathbf{Y}^{-})\right]. \tag{49}$$

Thus, once the extended displacement discontinuities along the interfacial crack are obtained, the asymptotic expressions presented in this section can be used to find the extended stress intensity factors and the local *J*-integral. This is presented in the following section.

6. Numerical formulation for the unknown displacement discontinuities along the interfacial crack

It is well-known that a crack can be represented by continuously distributed displacement discontinuities. A displacement discontinuity boundary element method (Crouch, 1976) is used in this section to derive the formulation. The interfacial crack is discretized using the constant displacement discontinuity element with the unknown magnitudes of the extended displacement discontinuities being uniformly distributed on each element. The unknown extended displacement discontinuities along the whole interfacial crack can be solved using the Green's functions we derived in this paper. This is discussed below.

According to the solutions in Section 3.2, the Green's function at *i*-th field point due to *j*-th influence element is

$$\mathbf{G}(x_{i1}; x_{j1}) = \frac{1}{\pi} \mathbf{L} \left(\frac{1}{x_{i1} - x_{j1} - a} - \frac{1}{x_{i1} - x_{j1} + a} \right) \\ + \mathbf{F} \frac{1}{\sqrt{2\pi\epsilon}} \left(\exp\left[-\frac{(x_{i1} - x_{j1} - a)^2}{2\epsilon^2} \right] - \exp\left[-\frac{(x_{i1} - x_{j1} + a)^2}{2\epsilon^2} \right] \right).$$
(50)

where x_{i1} and x_{j1} denote the x_1 -coordinate of the *i*-th field point and the center of the *j*-th element, respectively, and *a* is the half length of an element (Fig. 1). If the field points are chosen to be the centers of the elements, we obtain, by superposing the results from all the elements, a system of linear equations expressed as

$$\sum_{i=1}^{4n} G_{ij} \|u_j\| = t_i, \quad i = 1, 4n$$
(51a)

or

$$\mathbf{G}\|\mathbf{u}\| = \mathbf{t},\tag{51b}$$

where **G** is the $4n \times 4n$ coefficient matrix whose components are presented in Eq. (50) with *n* being the total element number; $\|\mathbf{u}\|$ is the vector of the unknown extended displacement discontinuities within the constant elements; and **t** is the prescribed traction vector. By solving Eq. (51), the unknown displacement discontinuities along the interfacial crack can be obtained. Then the displacement discontinuity at the left crack tip $x_1 = 0$ can be calculated by the extrapolation method with the corresponding fitting equation being

$$\|\mathbf{u}\|_{tip} = \beta_1 x_1^{1/2} + \beta_2 x_1^{3/2}, \tag{52}$$

where β_i are the fitting coefficient vectors.

With the calculated displacement discontinuities near the crack tip, the extended stress intensity factors \mathbf{K} and then *J*-integral can be also calculated using the corresponding equations presented in the previous section.

7. Numerical analysis

The popular piezoelectric ceramics PZT4, PZT5H, PZT6B, and PZT7A are considered in our numerical analysis. Material properties in their local coordinate system are listed in Table 1. In all numerical examples, the poling directions of the upper and lower half material planes are assumed to be along the global x_2 -axis. A crack of length l at the interface of the piezoelectric bi-material is considered. Before we present our numerical results, we first study the possible influence of the parameter ε in the Gaussian functions on our numerical solution.

7.1. On the choice of parameter ε

As mentioned in Section 4.2, the value of ε can be different for different problems. Here we propose a numerical method on the suitable selection of the ε value. We mainly focus on the effect of parameter ε on *J*-integral since the latter one is a comprehensive parameter for crack studies. Fig. 2(a)-(c) are the curves of normalized *I*-integral as functions of the ratio $\varepsilon/(2a)$ for three material pairs PZT4/PZT5H, PZT4/PZT6B, and PZT4/PZT7A. In this figure, while the *J*-integral is normalized by σ_{22}^2/c_{33} with c_{33} being the elastic constant of PZT4, 2a is the element length in our numerical calculation which is usually very small (2a = l/180 in this study after various numerical validations). Shown in Fig. 2 are also the results of the corresponding solutions without delta functions in Eq. (31). It can be seen from Fig. 2 that when $\varepsilon/(2a)$ approaches zero, the J-integral including the delta function effect approaches the value based on the formulation without the delta function which is shown by the horizontal straight line. This indicates that the solution without the delta function provides an upper bound for J-integral. However, the J-integral based on our new solution including

Table 1

Material properties of PZT4, PZT5H, PZT6B and PZT7A in their local material coordinate system (m_1, m_2, m_3) with m_3 being their poling direction (only the nonzero components are listed).

	PZT4	PZT5H	PZT6B	PZT7A
$c_{11} (\times 10^9 \mathrm{N/m^2})$	139	126	168	148
$c_{33} (\times 10^9 \text{ N/m}^2)$	113	117	163	131
$c_{12} (\times 10^9 \text{ N/m}^2)$	77.8	55	60	76.2
$c_{13} (\times 10^9 \text{ N/m}^2)$	74.3	53	60	74.2
$c_{44} (\times 10^9 \text{ N/m}^2)$	25.6	35.3	27.1	25.4
e_{31} (C/m ²)	-6.98	-6.5	-0.9	-2.1
e_{33} (C/m ²)	13.84	23.3	7.1	9.5
e_{15} (C/m ²)	13.44	17	4.6	9.7
$\kappa_{11} (\times 10^{-9}{ m C/Vm})$	6	15.1	3.6	8.11
$\kappa_{33} (\times 10^{-9} \text{ C/Vm})$	5.47	13	3.4	7.35



Fig. 2. Effect of the ratio $\varepsilon/(2a)$ on the normalized *J*-integral in the bi-material of (a) PZT4/PZT5H, (b) PZT4/PZT6B and (c) PZT4/PZT7A under loadings of $\sigma_{21} = \sigma_{22} = 20$ MPa, $D_2 = 0.002$ C/m². The constant solution without containing the delta function is also shown for comparison.

the delta function of Gaussian distribution reaches a minimum value near $\varepsilon/(2a) = 0.66$, which could serve as a lower bound. The value of *J*-integral with the delta function effect will always fall between the upper and lower bounds. The difference between its upper and lower bounds is only 4% for PZT4/PZT6B and is less than 1% for the other two material pairs. Actually, it is further noticed that when $\varepsilon/(2a)$ is within (0.66, 1.8), the *J*-integral containing the delta function is nearly constant. Thus, in the following numerical analysis, the ratio is fixed at $\varepsilon/(2a) = 1.35$ (again with the element length 2*a* being fixed 2a = l/180).

7.2. Displacement discontinuities along the interface crack

Fig. 3 shows the normalized extended displacement discontinuity $||u_2||$ (normalized by $l \times 10^{-3}$) along the interface crack of

length *l* under the applied traction $\sigma_{22} = 20$ MPa and electric displacement $D_2 = 0.002$ C/m² for the three pairs of bi-material. The maximum normalized opening displacement is at the center of the crack, being 0.395, 0.372, and 0.377, respectively, for PZT4/PZT4, PZT4/PZT5H, and PZT4/PZT6B. Fig. 4 shows the normalized electric potential jump $||\varphi||$ (normalized by ID_2/κ_{33}) along the interface crack under the loadings of $\sigma_{22} = 20$ MPa and $D_2 = 0.002$ C/m², where κ_{33} is the dielectric constant of PZT4. For the material pairs PZT4/PZT4, PZT4/PZT5H, and PZT4/PZT6B, the normalized electric potential jump has, respectively, a maximum of 0.74, 0.59, and 0.18 in the center of the crack. It is concluded that the relative opening displacement and electric potential jump of the interfacial crack in a homogeneous material are larger than those in a true bi-material system.

7.3. Effect of different loadings on stress intensity factors and J-integral

For the same material pairs, Figs. 5-8 show, respectively, the variation of the normalized normal stress intensity factor K_I (normalized by $\sqrt{\pi l(\sigma_{21}^2 + \sigma_{22}^2)/2}$), shear stress intensity factor $K_{\rm II}$ (normalized by $\sqrt{\pi l(\sigma_{21}^2+\sigma_{22}^2)/2}$), electric displacement intensity factor K_D (normalized by $\sqrt{\pi l/2}D_2$) and *J*-integral (normalized by $l\sigma_{22}^2/c_{33}$) with varying σ_{21}/σ_{22} under electric displacement $D_2 = 0.002 \text{ C/m}^2$. Actually, in the calculation, we fixe $\sigma_{22} \equiv 20 \text{ MPa}$ and change the value of σ_{21} . Figs. 5 and 6 show that, with increasing σ_{21}/σ_{22} , while the normalized $K_{\rm I}$ decreases, the normalized $K_{\rm II}$ increases. It can be further observed from both figures that these two mechanical (stress) intensity factors are insensitive to the different material pairs or the material mismatch. On the other hand, the electric (displacement) intensity factor K_D is very sensitive to the material pairs, as shown in Fig. 7. We point out first that, for the material pairs PZT4/PZT4 which is actually a homogeneous material, the normalized K_D has a constant value of 1.01286 under the given loadings as expected. However, for the true bi-material cases, due to the material mismatch on the interface, the normalized $K_{\rm D}$ decreases linearly with increasing σ_{21}/σ_{22} . It is further noticed that when the ratio of σ_{21}/σ_{22} is larger than 1.9, the value of $K_{\rm D}$ for the material pairs PZT4/PZT5H becomes negative.

Fig. 8 shows that the normalized *J*-integral increases with increasing σ_{21}/σ_{22} . Similar to the mechanical (stress) intensity factors in Figs. 5 and 6, the *J*-integral is insensitive to different material pairs or mismatches. Fig. 9 shows the variation of the normalized *J*-integral (normalized by $l\sigma_{22}^2/c_{33}$) with applied electric displacement D_2 (C/m²) (with fixed mechanical load at



Fig. 3. Variation of the normalized displacement discontinuity $||u_2||$ along the interfacial crack in the piezoelectric bi-material induced by the mixed uniform loading of $\sigma_{22} = 20$ MPa and $D_2 = 0.002$ C/m² applied to the crack surface.



Fig. 4. Variation of the normalized electric potential discontinuity $||\phi||$ along the interface crack in the piezoelectric bi-material induced by the mixed uniform loading of $\sigma_{22} = 20$ MPa and $D_2 = 0.002$ C/m² applied to the crack surface.



Fig. 5. Variation of the normalized normal stress intensity factor K_1 with different σ_{21}/σ_{22} when the applied electric displacement is fixed at $D_2 = 0.002 \text{ C/m}^2$.



Fig. 6. Variation of the normalized shear stress intensity factor K_{II} with different σ_{21}/σ_{22} when the applied electric displacement is fixed at $D_2 = 0.002 \text{ C/m}^2$.

 σ_{22} = 20 MPa). It is interesting to observe that from this figure that, for a given bi-material, the *J*-integral will first increase to a maximum value and then will decrease with increasing applied electric displacement *D*₂. In other words, there is a special applied electric displacement to make the *J*-integral maximum, and on either sides of this applied electric displacement, the local *J*-integral of the interfacial crack will decrease. This special electric displacement value is roughly 0.005, 0.0055, and 0.0025 C/m² for the material pairs PZT4/PZT4, PZT4/PZT5H, and PZT4/PZT6B.



Fig. 7. Variation of the normalized electric displacement intensity factor K_D with different σ_{21}/σ_{22} when the applied electric displacement is fixed at $D_2 = 0.002 \text{ C/m}^2$.



Fig. 8. Variation of the normalized *J*-integral with different σ_{21}/σ_{22} when the applied electric displacement is fixed at $D_2 = 0.002 \text{ C/m}^2$.



Fig. 9. Variation of the normalized *J*-integral with the applied electric displacement D_2 (C/m²) when the applied traction is fixed at σ_{22} = 20 MPa.

8. Conclusions

In this paper, we have derived various extended Green's functions for extended dislocations or extended displacement discontinuities located at the interface of a piezoelectric bi-material. The Green's functions are expressed simply in terms of the Stroh formalism and are applied to obtain the integro-differential equation of the interfacial crack employing the Somigliana identity. The well-known oscillating singularities associated with the interfacial crack are removed by representing the delta function in terms of the Gaussian distribution function. This important operation further reduces the integro-differential equation to a standard integral equation for the interfacial crack problem in piezoelectric bimaterial. A simple numerical approach is also proposed to solve the integral equation for the involved unknown displacement discontinuities. Based on the displacement discontinuities, asymptotical expressions of the extended intensity factors and *J*-integral are further derived. As numerical results is first discussed. It is found that the *J*-integral without the delta function effect can be used as the upper bound in conservative analysis. Furthermore, for the selected Gaussian parameter, the influence of different extended loadings on the interfacial crack behaviors is discussed.

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