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Journal of Sound and Vibration 333 (2014) 4017-4029

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Free vibration of three-dimensional multilayered magneto-electro-elastic plates under combined clamped/free boundary conditions

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ARTICLE INFO

Article history: Received 10 December 2013 Received in revised form 25 March 2014 Accepted 26 March 2014 Handling Editor: S. Ilanko Available online 30 April 2014

ABSTRACT

In this paper, we study the free vibration of multilayered magneto-electro-elastic plates under combined clamped/free lateral boundary conditions using a semi-analytical discrete-layer approach. More specifically, we use piecewise continuous approximations for the field variables in the thickness direction and continuous polynomial approximations for those within the plane of the plate. Group theory is further used to isolate the nature of the vibrational modes to reduce the computational cost. As numerical examples, two cases of the lateral boundary conditions combined with the clamped and free edges are considered. The non-dimensional frequencies and mode shapes of elastic displacements, electric and magnetic potentials are presented. Our numerical results clearly illustrate the effect of the stacking sequences and magneto-electric coupling on the frequencies and mode shapes of the anisotropic magneto-electro-elastic plate, and should be useful in future vibration study and design of multilayered magneto-electro-elastic plates.

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1. Introduction

Multilayered composites offer many useful features as structural components and as such, response of such composites under external loads is an important research subject. Besides the common numerical methods, such as the finite-element and boundary-element methods, various analytical/semi-analytical solutions were presented for layered composite plates. For instance, Vel and Batra [1] presented an analytical three-dimensional (3D) solution for the static deformation of multilayered piezoelectric plates under general boundary conditions in terms of series expansion. The corresponding bending vibration was further solved by Vel et al. [2]. The extended Kantorovich method was also applied to the static bending of layered piezoelectric plates by Kapuria and Kumari [3], and to the 3D deformation of layered elastic plates by Kumari et al. [4] where an iterative scheme was employed.

Recent development of smart materials/structures is receiving widespread attention owing to their potential applications in various engineering fields such as sensors, actuators and microwave devices. As an important member of these smart materials, magneto-electro-elastic (MEE) materials which consist of piezoelectric (PE) and piezomagnetic (PM) phases, are able to facilitate the energy conversion between the electric and magnetic fields. Such a phenomenon is called

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http://dx.doi.org/10.1016/j.jsv.2014.03.035 0022-460X/© 2014 Elsevier Ltd. All rights reserved.







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magnetoelectric (ME) effect which cannot be found in the pure piezoelectric or piezomagnetic material. Since the report on the ME effect by van Suchtelen [5], many interesting studies on MEE materials and structures have been carried out. Among them, the static and dynamic behaviors of typical MEE structures, for example plates and beams, are especially investigated.

For a simply supported multilayered MEE plate, the exact closed-form solution of the deformation under a static mechanical load was derived using the pseudo Stroh formalism [6]. The corresponding free vibration was analyzed by Pan and Heyliger [7]. Another method, the state-space formulation, was also widely used in the analysis of the static and dynamic behaviors of MEE multilayered plates [8,9]. Free vibration of a non-homogeneous transversely isotropic MEE plate was carried out by Chen et al. [10]. Besides, the discrete-layer and domain-discretization methods were also proposed to the analysis of free vibration of anisotropic elastic and MEE plates and shells [11–15].

For a simply supported MEE plate, analytical solutions of the field variables can be found that satisfy exactly the lateral boundary conditions. However, under other lateral boundary conditions such as the clamped or free conditions, one cannot find such analytical expressions of the field variables. Furthermore, many commercial finite-element codes cannot handle the multiphase coupling problem. Thus, in this paper, a semi-analytical discrete-layer model of the governing differential equations is developed and applied to typical layered MEE media under combined clamped and free lateral boundary conditions. Our representative numerical results on the natural frequencies and mode shapes clearly show the unique characteristics of these MEE solids which should be of particular interest to the design of layered MEE composites.

2. Formulation

2.1. Governing equations

While our semi-analytical model can be applied to any layered plate, we consider an anisotropic, MEE, and three-layered rectangular plate with horizontal dimensions *a* and *b* and total thickness *H* (in the vertical or thickness direction) as shown in Fig. 1. A Cartesian coordinate system is attached to the plate and its origin is at one of the four corners on the bottom surface, with the plate occupying the region of $z \ge 0$. The interface of each layer is assumed to be bonded perfectly. In other words, the elastic displacements, electric and magnetic potentials, elastic traction, and the *z*-components of the electric displacement and magnetic induction are continuous across the interfaces.

For a linear, anisotropic MEE solid, the coupled constitutive equation can be written in the following form:

$$\sigma_i = c_{ik}\gamma_k - e_{ki}E_k - q_{ki}H_k, \quad D_i = e_{ik}\gamma_k + \varepsilon_{ik}E_k + d_{ik}H_k, \quad B_i = q_{ik}\gamma_k + d_{ik}E_k + \mu_{ik}H_k, \tag{1}$$

where σ_i , D_i and B_i are the stress, electric displacement and magnetic induction, respectively; γ_k , E_k and H_k are the strain, electric field and magnetic field, respectively; c_{ik} , ε_{ik} and μ_{ik} are the elastic, dielectric, and magnetic permeability coefficients, respectively; e_{ik} , q_{ik} and d_{ik} are the piezoelectric, piezomagnetic and magnetoelectric coefficients, respectively. We remark that various uncoupled cases can be reduced by setting the appropriate coupling coefficients to zero.

The relationship between the strain and displacement, electric (magnetic) field and its potential can be expressed as

$$\gamma_{ii} = 0.5(u_{i,j} + u_{j,i}), \ E_i = -\varphi_i, \ H_i = -\psi_{i,i}, \tag{2}$$

where u_i are the elastic displacements, and φ and ψ are the electric and magnetic potentials, respectively. The subscript after the comma, e.g. ",*i*", in the elastic displacement, electric and magnetic potentials denotes partial derivative with respect to the *i*-th coordinate x_i ($x_1=x, x_2=y, x_3=z$).

For the problem to be considered in this paper, we assume that the body forces, electric charge and current densities are zero; thus the governing equations of motion in the dynamic case are given by

$$\sigma_{ij,\,j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \ D_{j,\,j} = 0, \ B_{j,\,j} = 0,$$
(3)

where ρ is the density of the material.

While general conditions may be prescribed to the lateral boundaries (along the whole thickness of the plate; i.e., for any given *z*-coordinate in the problem domain) of the layered plate, we consider the following two typical cases.



Fig. 1. The three-layered magneto-electro-elastic plate.

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Case I: CCCC, which means that the four sides of the plate are "clamped":

$$u_1 = u_2 = u_3 = \varphi = \psi = 0;$$

on $x = 0$ and $x = a$, with $y \in (0, b)$; on $y = 0$ and $y = b$, with $x \in (0, a)$. (4)

Case II: FCFC, which means that two sides of the plate are "free" (say, y=0 and b) while the other two are clamped (x=0 and a):

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = D_2 = B_2 = 0 \text{ on } y = 0 \text{ and } y = b, \text{ with } x \in (0, a)$$

$$u_1 = u_2 = u_3 = \varphi = \psi = 0 \text{ on } x = 0 \text{ and } x = a, \text{ with } y \in (0, b).$$
(5)

We point out that the clamped and free conditions are terminologies for the mechanical quantities only. For the electrical ones, $\varphi = 0$ indicates a conducting or closed circuit condition, whilst $D_2 = 0$ denotes an insulating or open circuit condition. Similar statements can be made for the corresponding magnetic quantities.

2.2. The semi-analytical discrete-layer model

Our semi-analytical discrete-layer models are based on the idea of allowing for discontinuity of the slope of the primary unknowns because of the continuity of extended traction between dissimilar layers. They provide some benefits over classical single-layer theories in that the known breaks in slope are explicitly represented via C^0 continuity through the thickness. The primary steps of this development are presented below.

In our model, we split the approximation of the primary unknowns into the product of functions in the *z*-direction multiplied by functions in (x,y) and then solve the weak form of the governing equations. The latter can be obtained by forming the weighted residuals

$$0 = \int_{V} \delta u_{i} \left(\sigma_{ijj} - \rho \frac{\partial^{2} u_{i}}{\partial t^{2}} \right) dV; \ 0 = \int_{V} \delta \varphi D_{jj} \ dV; \ 0 = \int_{V} \delta \psi B_{jj} \ dV$$
(6)

These expressions are then integrated by parts to yield the final weak form of the original differential equations [12]. We assume that the five primary field variables ($u \equiv u_1$, $v \equiv u_2$, $w \equiv u_3$, φ, ψ) can be approximated using the form, say for u, as

$$u(x, y, z) = \sum_{j=1}^{n} a_j f_j^u(x, y, z)$$
(7)

where a_j are the unknown constants and f_j^u the given functions of the coordinates. The complete functions f are then split into the following form

$$f_{j}^{u}(x, y, z) = g_{j}(z) h_{j}(x, y)$$
 (8)

Hence the three-dimensional approximations are split into the products of the through-thickness approximation multiplied by the in-plane approximation. For homogeneous solids, such a separation is somewhat inconsequential. But for layered materials, it allows for approximations in the *z*-direction that are piecewise continuous rather than fully continuous over the thickness direction. Since in exact solutions the interface conditions are enforced exactly, this class of approximation allows for a closer representation with those constraints.

In this study, the functions $g_j(z)$ in Eq. (8) are taken to be piecewise linear functions over an individual sub-layer of the laminate. These layers need not be the same as the individual physical layers of the laminate. The functions $h_j(x, y)$ in Eq. (8) are taken to be the products of polynomials over the range of x = (0,a) and of y = (0,b) or $h(x,y) = f_x(x)f_y(y)$. For the case of zero values at the endpoints, such as the clamped plate, the functions $f_x(x(\xi))$ can be written in terms of the normalized coordinates ξ and expressed as

$$f_{X}(X(\xi)) = \frac{(\xi - \xi_{1})(\xi - \xi_{2})\dots(\xi - \xi_{n})}{(\xi_{i} - \xi_{1})(\xi_{i} - \xi_{2})\dots(\xi_{i} - \xi_{n})}$$
(9)

where *n* must be greater than or equal to 2, $\xi_1 = -1$, $\xi_n = 1$, and the remaining ξ_i are at equally spaced locations between (-1,1). The functions $f_y(y)$ have a similar form except for that the normalized coordinate η replaces ξ . These functions have the advantage of having pure symmetry (evenness) or asymmetry (oddness) about their normalized coordinate origins, and hence the resulting approximations can be grouped with that in mind. For example, the first two even functions of this nature are $(1 - \xi_i^2)$ and $(1 - \xi_i^2)(1/9 - \xi_i^2)$. Both of these are symmetric about the mid-line $\xi_i = 0$ and are zero at the endpoints (-1,1). For the case of the free-free condition, the approximation functions in the appropriate direction are replaced with Legendre polynomials. In this case, the lowest order used is not quadratic but rather constant (even) over the domain, and the increasing order of the polynomial retains the even/odd structure about the origin. Hence the even polynomials take the sequential forms 1, $(3x^2-1)/2$, $(35x^4-30x^2+3)/8$, and so on, whilst the odd functions appear as *x*, $(5x^3-3x)/2$, $(63x^5-70x^3+15x)/8$ and so on [16].

The eigenvalue problem that results from the discretization of the governing equations can be separated into different combinations of (u,v,w,φ,ψ) according to their various symmetries about the middle lines of the plate. This allows for a significant reduction in the size of the matrix expression that needs to be solved for any combination of approximations. This is in fact one of the primary advantages of Ritz-based discrete-layer models for the analysis of rectangular plates over

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Designation of the evenness (E) or oddness (O) of the approximation functions for the four basic groups in the semi-analytical discrete-layer theory.

| Displacement | Group number | | | | | | | | | |
|--------------|--------------|---|---|---|---|---|---|---|--|--|
| | 1 | | 2 | | 3 | 3 | | 4 | | |
| | x | у | x | у | x | у | x | у | | |
| и | 0 | E | 0 | 0 | E | Е | E | 0 | | |
| ν | E | 0 | E | E | 0 | 0 | 0 | E | | |
| W | E | E | E | 0 | 0 | Е | 0 | 0 | | |
| φ | E | E | E | 0 | 0 | E | 0 | 0 | | |
| Ψ | Е | Е | Е | 0 | 0 | Е | 0 | 0 | | |

alternatives such as finite element methods. Not only are the global approximation methods generally more accurate per degree-of-freedom, but the applications of group theory significantly reduces the size of the resulting eigenvalue problem. The nature and combination of these approximations are given in Table 1 which shows that one can solve 4 separate smaller eigenvalue problems instead of a large problem. For example, if 16 layers are used that include 6 terms in the *x*- and *y*-direction, the full matrix expression would be of the order 5(16+1)(6)(6)=3060. However, splitting this problem into four smaller groups in *x* and *y*, where only the appropriate odd or even terms are kept, reduces this to solving 4 problems with much smaller dimension 5(16+1)(3)(3)=765.

2.3. Finite element model

To compare and check with our semi-analytic discrete-layer model, we have also built a finite element model based on COMSOL [17]. In terms of COMSOL, once Eqs. (1)–(3) are expressed as a general form with the PDE module, the analysis of the three-dimensional vibration of the MEE plate can be carried out. The general governing equation for the dynamics system in COMSOL software [17] is given in the PDE vector form as

$$\boldsymbol{e}_{a}\frac{\partial^{2}\boldsymbol{u}}{\partial t^{2}}+\boldsymbol{d}_{a}\frac{\partial\boldsymbol{u}}{\partial t}+\nabla\boldsymbol{\bullet}(-\boldsymbol{c}\nabla\boldsymbol{u}-\boldsymbol{\alpha}\boldsymbol{u}+\boldsymbol{\gamma})+\boldsymbol{\beta}\boldsymbol{\bullet}\boldsymbol{u}+\boldsymbol{a}\boldsymbol{u}=\boldsymbol{f}$$
(10)

where e_a is the mass coefficient, d_a damping coefficient, c diffusion coefficient, α conservative flux convection coefficient, β convection coefficient, a absorption coefficient, γ conservative flux source term, and f is the source term. The MEE multilayered plate is assumed to be linear and free of any load, without any damping. Thus, only two coefficient matrices, i.e., those related to the diffusion and mass coefficients, are kept. The elements of these two matrices are given in Appendix A.

3. Results and discussions

We first point out that the convergence of the results using our semi-analytical discrete-layer approximations can be also self-assessed both by increasing the number of sub-layers through the thickness of the laminate and by varying the number of terms used in the in-plane polynomial approximations. For the results presented in this study, we varied the layer numbers as 1, 2, 4, 8, 16, and 32 for the homogeneous plate, and then for each of these discretizations also varied the number of in-plane functions from a single term (for example, $(1-\xi)(1+\xi)(1-\eta)(1+\eta)$ for the clamped-clamped plate with n dimensionless coordinates originated at the plate's in-plane center) up to 36 in-plane terms that include polynomial products up to the order of 12. Using a total of 32 layers and 36 in-plane terms resulted in frequencies that did not change in 4 significant figures from the prior lower level of approximation. Thus, for the layered specimens, we used 36 layers for calculating the frequencies presented in this study.

In order to verify our semi-analytical discrete-layer model, a purely elastic homogeneous square plate with CCCC and FCFC boundary conditions is considered [15]. The top and bottom surfaces of the plate are traction free and the plate is made of hexagonal material, with material coefficients being given in Appendix B.

Table 2 lists the first ten non-dimensional natural frequencies of the square plate of H/a=0.2 where the frequencies are normalized as $\overline{\omega} = \omega H \sqrt{\rho/c_{11}}$ based on our semi-analytical discrete-layer (DL) model, as compared to those in [15] and those based on our FEM formulation of COMSOL [17] with a total of $2 \times 10 \times 10$ elements as implemented by the authors. The group number used in our DL model is also listed in Table 1. It is clearly observed that, the natural frequencies calculated using our semi-analytical DL model are almost identical to those based on a FEM formulation of COMSOL [17] and also those in [15] based on a different finite element approach.

3.1. Eigenfrequencies

Having verified our semi-analytical method for the purely elastic plate, we now apply it to the three-layered plate made of piezoelectric $BaTiO_3$ and magnetostrictive $CoFe_2O_4$ with material properties taken from Pan and Heyliger [7] but making use of the positive magnetic permeability coefficients for $CoFe_2O_4$ as listed in Appendix C. It is noted that the density of the

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Table 2

The first ten non-dimensional natural frequencies of an elastic square plate with hexagonal materials calculated by our semi-analytical discrete-layer (DL), as compared to our FEM, and also those in [15]. The value in the parentheses in DL indicates the group number used.

| No. | CCCC | | | FCFC | | | |
|-----|-----------|--------|-----------|-----------|--------|-----------|--|
| | DL | FEM | Ref. [15] | DL | FEM | Ref. [15] | |
| 1 | 0.3332(1) | 0.3332 | 0.3325 | 0.2193(1) | 0.2194 | 0.2191 | |
| 2 | 0.5987(2) | 0.5988 | 0.5960 | 0.2572(2) | 0.2573 | 0.2566 | |
| 3 | 0.5987(3) | 0.5988 | 0.5960 | 0.3798(2) | 0.3798 | 0.3796 | |
| 4 | 0.7459(2) | 0.7460 | 0.7449 | 0.3967(1) | 0.3968 | 0.3941 | |
| 5 | 0.7459(3) | 0.7460 | 0.7449 | 0.5182(3) | 0.5183 | 0.5163 | |
| 6 | 0.8138(4) | 0.8136 | 0.8081 | 0.5656(4) | 0.5657 | 0.5628 | |
| 7 | 0.9354(1) | 0.9358 | 0.9280 | 0.6260(3) | 0.6261 | 0.6254 | |
| 8 | 0.9478(1) | 0.9482 | 0.9405 | 0.6645(2) | 0.6646 | 0.6570 | |
| 9 | 0.9613(4) | 0.9617 | 0.9591 | 0.6744(4) | 0.6744 | 0.6735 | |
| 10 | 1.1091(4) | 1.1158 | 1.1045 | 0.6958(1) | 0.6959 | 0.6949 | |

| Table 3 |
|---|
| The first ten non-dimensional frequencies for the square homogeneous plate of BBB or FFF under lateral boundary condition CCCC. |

| No. | BBB | BBB | | | | FFF | | | |
|-----|--------|--------|-------------|-------------|--------|--------|-------------|-------------|--|
| | PE(DL) | PE(FE) | Elastic(DL) | Elastic(FE) | PM(DL) | PM(FE) | Elastic(DL) | Elastic(FE) | |
| 1 | 1.7817 | 1.7823 | 1.6479 | 1.6494 | 1.3667 | 1.3674 | 1.3621 | 1.3629 | |
| 2 | 2.9486 | 2.9492 | 2.7236 | 2.7242 | 2.2311 | 2.2318 | 2,2234 | 2.2242 | |
| 3 | 2.9486 | 2.9492 | 2.7236 | 2.7242 | 2.2311 | 2.2318 | 2,2234 | 2.2242 | |
| 4 | 3.2195 | 3.2201 | 3.1474 | 3.1479 | 2.7905 | 2.7915 | 2.7904 | 2.7914 | |
| 5 | 3.2195 | 3.2201 | 3.1474 | 3.1479 | 2.7905 | 2.7915 | 2.7904 | 2.7914 | |
| 6 | 3.7120 | 3.7121 | 3.5958 | 3.5964 | 2.9345 | 2.9353 | 2.9237 | 2.9244 | |
| 7 | 3.9106 | 3.9113 | 3.7107 | 3.7108 | 3.1898 | 3.1899 | 3.1898 | 3.1899 | |
| 8 | 4.4042 | 4.4050 | 4.0486 | 4.0493 | 3.3033 | 3.3041 | 3.2910 | 3.2918 | |
| 9 | 4.5056 | 4.5069 | 4.1059 | 4.1065 | 3.3441 | 3.3451 | 3.3306 | 3.3316 | |
| 10 | 4.6486 | 4.6495 | 4.5659 | 4.5668 | 3.8735 | 3.8743 | 3.8580 | 3.8589 | |

Table 4

The first ten non-dimensional frequencies for the square plate with BFB under lateral boundary condition CCCC.

| No. | PE(DL) | PE(FE) | PM(DL) | PM(FE) | MEE(DL) | MEE(FE) |
|-----|--------|--------|---------------------|--------|---------|---------|
| 1 | 1.3428 | 1.3434 | 1.2889 | 1.2894 | 1.3452 | 1.3458 |
| 2 | 2.2191 | 2.2199 | 2.1305 | 2.1312 | 2.2231 | 2.2238 |
| 3 | 2.2191 | 2.2199 | 2.1305 | 2.1312 | 2.2231 | 2.2238 |
| 4 | 2.6177 | 2.6182 | 2.5753 | 2.5759 | 2.6178 | 2.6184 |
| 5 | 2.6177 | 2.6182 | 2.5753 | 2.5759 | 2.6178 | 2.6184 |
| 6 | 2.9348 | 2.9357 | 2.8132 | 2.8139 | 2.9404 | 2.9413 |
| 7 | 2.9939 | 2.9940 | 2.9931 | 2.9931 | 2.9939 | 2.9940 |
| 8 | 3.3061 | 3.3073 | 3.1668 | 3.1677 | 3.3123 | 3.3134 |
| 9 | 3.3689 | 3.3703 | 3.2129 | 3.2139 | 3.3758 | 3.3772 |
| 10 | 3.7728 | 3.7736 | 3.7213 ^a | 3.7298 | 3.7729 | 3.7738 |

^a There is another repeated frequency from other groups (2 and 3) which is 3.7289.

two materials is assumed to be equal with ρ_{max} =5800 kg/m³ being used in the calculation. The three layers are assumed to have equal thickness. Four stacking sequences of the plate are considered. They are BBB, BFB, FBF and FFF with B representing BaTiO₃ and F representing CoFe₂O₄, as in [7]. The plate is considered to be square in the horizontal plane with a side length *a*=1 m, and it has a total thickness of 0.3 m in the vertical direction. While the top and bottom surfaces are assumed extended traction free as in [7], the lateral boundary conditions are either CCCC or FCFC type.

Tables 3–7 list the first ten non-dimensional frequencies for the layered plate with frequencies being normalized by $\overline{\omega} = \omega a \sqrt{\rho_{\text{max}}/c_{\text{max}}}$, and with c_{max} being the maximum value of the stiffness coefficients (i.e., $c_{\text{max}} = 286 \times 10^9 \text{ N/m}^2$). In these tables, we have also included the results corresponding to the different uncoupled cases: PE and PM represent, respectively, the material where the piezomagnetic and piezoelectric coupling coefficients are zero, and Elastic represents the material

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| Table 5 | | | | | | | | |
|-------------------------------|----------------|--------------|------------|-----------|---------|----------|-----------|-------|
| The first ten non-dimensional | frequencies fo | r the square | plate with | FBF under | lateral | boundary | condition | CCCC. |

| No. | PE(DL) | PE(FE) | PM(DL) | PM(FE) | MEE(DL) | MEE(FE) |
|-----|--------|--------|--------|--------|---------|---------|
| 1 | 1.4453 | 1.4463 | 1.3911 | 1.3920 | 1.4477 | 1.4486 |
| 2 | 2.3591 | 2.3602 | 2.2704 | 2.2715 | 2.3631 | 2.3641 |
| 3 | 2.3591 | 2.3602 | 2.2704 | 2.2715 | 2.3641 | 2.3641 |
| 4 | 2.7592 | 2.7603 | 2.7377 | 2.7388 | 2.7596 | 2.7607 |
| 5 | 2.7592 | 2.7603 | 2.7377 | 2.7388 | 2.7596 | 2.7607 |
| 6 | 3.1106 | 3.1117 | 2.9855 | 2.9867 | 3.1162 | 3.1173 |
| 7 | 3.1622 | 3.1623 | 3.1619 | 3.1620 | 3.1622 | 3.1623 |
| 8 | 3.5005 | 3.5017 | 3.3609 | 3.3622 | 3.5069 | 3.5080 |
| 9 | 3.5537 | 3.5552 | 3.4014 | 3.4030 | 3.5609 | 3.5624 |
| 10 | 3.9687 | 3.9702 | 3.9401 | 3.9390 | 3.9691 | 3.9706 |

| Table (| 6 |
|---------|---|
|---------|---|

The first ten non-dimensional frequencies for the square plate with BFB under lateral boundary condition FCFC.

| No. | PE(DL) | PE(FE) | PM(DL) | PM(FE) | MEE(DL) | MEE(FE) |
|-----|--------|--------|--------|--------|---------|---------|
| 1 | 0.9054 | 0.9059 | 0.8726 | 0.8730 | 0.9069 | 0.9047 |
| 2 | 0.9836 | 0.9839 | 0.9646 | 0.9649 | 0.9844 | 0.9848 |
| 3 | 1.2038 | 1.2040 | 1.2020 | 1.2022 | 1.2038 | 1.2040 |
| 4 | 1.5609 | 1.5610 | 1.5146 | 1.5147 | 1.5623 | 1.5623 |
| 5 | 1.8998 | 1.9006 | 1.8348 | 1.8355 | 1.9028 | 1.9036 |
| 6 | 2.0254 | 2.0261 | 1.9704 | 1.9710 | 2.0283 | 2.0289 |
| 7 | 2.2101 | 2.2107 | 2.1856 | 2.1862 | 2.2102 | 2.2108 |
| 8 | 2.2359 | 2.2360 | 2.2202 | 2.2203 | 2.2359 | 2.2361 |
| 9 | 2.4412 | 2.4415 | 2.3945 | 2.3947 | 2.4443 | 2.4416 |
| 10 | 2.5091 | 2.5095 | 2.4363 | 2.4366 | 2.5124 | 2.5127 |

Table 7The first ten non-dimensional frequencies for the square plate with FBF under lateral boundary condition FCFC.

| No. | PE(DL) | PE(FE) | PM(DL) | PM(FE) | MEE(DL) | MEE(FE) |
|-----|--------|--------|--------|--------|---------|---------|
| 1 | 0.9803 | 0.9811 | 0.9459 | 0.9467 | 0.9818 | 0.9826 |
| 2 | 1.0633 | 1.0639 | 1.0425 | 1.0431 | 1.0641 | 1.0647 |
| 3 | 1.2721 | 1.2724 | 1.2714 | 1.2717 | 1.2721 | 1.2724 |
| 4 | 1.7141 | 1.7142 | 1.6803 | 1.6804 | 1.7155 | 1.7156 |
| 5 | 2.0214 | 2.0225 | 1.9529 | 1.9541 | 2.0245 | 2.0256 |
| 6 | 2.1700 | 2.1709 | 2.1039 | 2.1049 | 2.1728 | 2.1738 |
| 7 | 2.3321 | 2.3330 | 2.3206 | 2.3216 | 2.3323 | 2.3333 |
| 8 | 2.3630 | 2.3632 | 2.3550 | 2.3553 | 2.3631 | 2.3634 |
| 9 | 2.5735 | 2.5738 | 2.5475 | 2.5478 | 2.5738 | 2.5741 |
| 10 | 2.7116 | 2.7120 | 2.6324 | 2.6330 | 2.7149 | 2.7154 |

which is purely elastic without any piezoelectric and piezomagnetic coupling. In each of these cases, the results from our semi-analytical DL models are compared with those from our FE formulation based on COMSOL. In every case but one (the lowest mode of the BFB CFCF plate as highlighted in bold italics in Table 6), the DL results are slightly lower, and hence slightly more accurate.

Comparing Tables 3–5, we notice that while under the lateral boundary condition CCCC, there are two repeated modes (modes 2 and 3, and 4 and 5), all the modes under CFCF are well separated. We further observe that there are certain modes, which are independent of the PE/PM coupling in the plate. In other words, the PE/PM coupling coefficients have very little or no effect on the mode frequencies. These are highlighted by bold fonts for modes 4, 5, 7 and 10 in Tables 3–5, and modes 3, and 7–9 in Tables 6 and 7. Again, comparing these tables for the independent modes, we find that their mode numbers are different for different boundary conditions. This interesting phenomenon can also be found in the other stacking sequences under other lateral boundary conditions.

3.2. Mode shapes

Figs. 2 and 3 show the first six mode shapes of the elastic displacement component $u_z = w$ of the BFB plate under boundary conditions CCCC and FCFC. Because of symmetric lateral boundary conditions for the CCCC plate, identical mode

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Fig. 2. The first six mode shapes of displacement component $u_z = w$ for a BFB plate under CCCC boundary conditions: (a) $\overline{\omega} = 1.3458$, (b) $\overline{\omega} = 2.2238$, (c) $\overline{\omega} = 2.2238$, (d) $\overline{\omega} = 2.6184$, (e) $\overline{\omega} = 2.6184$, and (f) $\overline{\omega} = 2.9413$.

shapes corresponding to double frequencies can be found in Fig. 2(b) and (c), (d) and (e). It is clear that the mode shapes are identical to those in the purely elastic plate under the same lateral boundary conditions. For the corresponding FCFC boundary case, however, all these mode shapes are separated (Fig. 3) and further are totally different to those in the CCCC case. Furthermore, the effect of layering on the mode shape is also shown clearly in the third mode shape in Fig. 3(c).

Fig. 4 shows the vertical profile of the first mode shapes of the electric and magnetic potentials of the BFB plate under CCCC (Fig. 4(a) and (b)) and FCFC (Fig. 4(c) and (d)) boundary conditions. These mode shapes are on the vertical central plane parallel to the *xoz* plane. It is observed from Fig. 4(a) and (b) that the electric and magnetic potentials φ and ψ reach their maximum in the center of the plate and are quickly reduced to zero on the lateral boundary due to the clamped conditions (CCCC case). For the corresponding FCFC case (Fig. 4(c) and (d)), however, due to the extended traction-free lateral boundary conditions on both sides, the electric and magnetic potential distributions are totally different. Again, the layering effect can be also clearly seen in these two figures for the FCFC case.

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Fig. 3. The first six mode shapes of displacement component $u_z = w$ for a BFB plate with FCFC boundary conditions: (a) $\overline{\omega} = 0.9047$, (b) $\overline{\omega} = 0.9848$, (c) $\overline{\omega} = 1.2040$, (d) $\overline{\omega} = 1.5623$, (e) $\overline{\omega} = 1.9036$, and (f) $\overline{\omega} = 2.0289$.

Fig. 5 presents the first mode shapes of the electric and magnetic potentials of the FBF plate under CCCC and FCFC conditions. While Fig. 5(a) and (b) are somewhat similar to Fig. 4(a) and (b), the shapes of the electric and magnetic potentials in FBF under FCFC (Fig. 5(c) and (d),) are switched when compared to those in Fig. 4(c) and (b). This is due to the fact that while Fig. 4 corresponds to stacking sequence BFB, Fig. 5 corresponds to FBF, i.e., the F and B layers are switched. We further remark that in both Figs. 4 and 5, all the mode shapes are symmetric about the middle plane.

Figs. 6 and 7 show the first mode shapes of the electric field E_z and magnetic field H_z in the vertical central plane parallel to the *xoz* plane of the plate under CCCC and FCFC conditions. Compared to the results for the corresponding potentials in Figs. 4 and 5, we observe a clear discontinuity of fields across the interface in the three-layered plate, which is caused by the discontinuity of the material properties. Furthermore, all the mode shapes of fields are anti-symmetric about the middle plane.

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▼ 0.065

▼ 4.696×10⁻³

Fig. 4. The first mode shapes of the electric (φ) and magnetic (ψ) potentials in the vertical central plane parallel to the *xoz* plane in the BFB plate: CCCC vs. FCFC boundary conditions: (a) electric potential and (b) magnetic potential in CCCC, and (c) electric potential and (d) magnetic potential in FCFC.



Fig. 5. The first mode shapes of electric (φ) and magnetic (ψ) potentials in vertical central plane parallel to the *xoz* plane in the FBF plate: CCCC vs. FCFC boundary conditions: (a) electric potential and (b) magnetic potential in CCCC, and (c) electric potential and (d) magnetic potential in FCFC.

4. Conclusions

In this paper, three-dimensional free vibration of a magneto-electro-elastic (MEE) multilayered plate is studied by using the semi-analytical discrete layer model. Typical lateral boundary conditions such as clamped and free are considered where





Fig. 6. The first mode shapes of the electric field E_z in the vertical central plane parallel to the *xoz* plane in the BFB (a) and FBF (b) plates with CCCC boundary conditions, and in the BFB (c) and FBF (d) plates with FCFC boundary conditions.



Fig. 7. The first mode shapes of the magnetic field H_z in the vertical central plane parallel to the *xoz* plane in the BFB (a) and FBF (b) plates with CCCC boundary conditions, and in the BFB (c) and FBF (d) plates with FCFC boundary conditions.

analytical solution of the corresponding free vibration cannot be found. Our discrete-layer formulation is validated to be accurate and efficient, and it is further applied to the sandwiched MEE plate made of piezoelectric $BaTiO_3$ and magnetostrictive $CoFe_2O_4$. The first ten frequencies are presented and the coupling effect of the material

properties is discussed. For the geometries and boundary conditions considered in this study, we highlight the following results:

- 1. For purely elastic hexagonal materials, the addition of lateral clamped boundary conditions increases the frequencies by 50–100% as compared to the free-boundary case.
- 2. The level of piezoelectric stiffening significantly exceeds that of magnetostrictive stiffening. For instance, under the CCCC lateral boundary condition, the former gives an increase in frequency of up to 10% as compared to under 2% by the latter one.
- 3. Positioning the magnetostrictive layers as the outer layers in the sandwich MEE laminate yields an increase of an average 6–8% in frequency as compared to the laminate with the piezoelectric layers on the outside.

Some typical mode shapes are also presented, which should be of benefit for future research in this direction. It is expected that while the proposed semi-analytical discrete-layer theory can be extended to analyze free vibration of plates with different shapes and functional graded interface, the results presented in the article could be used as benchmarks to future numerical investigation in multilayered MEE composites.

Acknowledgments

The authors would like to thank Prof. Romesh Batra of Virginia Tech. for reading the very first draft and for providing some instructive comments. The authors further thank the Bairen Program in Henan Province and the National Natural Science Foundation of China (No. 11172273) for supporting this research. The contribution of PRH was supported by a grant from the Mountains-Plains Consortium and a grant from the NSF ISTEC Cray Program. We would also like to thank the reviewers for their constructive comments and suggestions on the original version of this paper.

Appendix A

The mass and diffusion coefficients in Eq. (10) are given by

$$\boldsymbol{e}_{a} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 \\ sym & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} \boldsymbol{\Delta}_{11} & \boldsymbol{\Delta}_{12} & \boldsymbol{\Delta}_{13} & \boldsymbol{\Delta}_{14} & \boldsymbol{\Delta}_{15} \\ \boldsymbol{\Delta}_{22} & \boldsymbol{\Delta}_{23} & \boldsymbol{\Delta}_{24} & \boldsymbol{\Delta}_{25} \\ \boldsymbol{\Delta}_{33} & \boldsymbol{\Delta}_{34} & \boldsymbol{\Delta}_{35} \\ sym & \boldsymbol{\Delta}_{44} & \boldsymbol{\Delta}_{45} \\ & & & \boldsymbol{\Delta}_{55} \end{bmatrix}$$
(A.1)

where the sub-matrices in (A.1) for our transversely isotropic MEE materials can be written as

$$\boldsymbol{\Delta}_{11} = \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & 0 \\ 0 & 0 & c_{55} \end{bmatrix}, \, \boldsymbol{\Delta}_{12} = \begin{bmatrix} 0 & c_{12} & 0 \\ c_{66} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \, \boldsymbol{\Delta}_{13} = \begin{bmatrix} 0 & 0 & c_{13} \\ 0 & 0 & 0 \\ c_{55} & 0 & 0 \end{bmatrix}$$
$$\boldsymbol{\Delta}_{22} = \begin{bmatrix} c_{66} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{44} \end{bmatrix}, \, \boldsymbol{\Delta}_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23} \\ 0 & c_{44} & 0 \end{bmatrix}, \, \boldsymbol{\Delta}_{33} = \begin{bmatrix} c_{55} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}$$
$$\boldsymbol{\Delta}_{44} = \begin{bmatrix} -\varepsilon_{11} & 0 & 0 \\ 0 & -\varepsilon_{22} & 0 \\ 0 & 0 & -\varepsilon_{33} \end{bmatrix}, \, \boldsymbol{\Delta}_{55} = \begin{bmatrix} -\mu_{11} & 0 & 0 \\ 0 & -\mu_{22} & 0 \\ 0 & 0 & -\mu_{33} \end{bmatrix}, \, \boldsymbol{\Delta}_{45} = \boldsymbol{0}_{3\times 3}$$
(A.2)

For the piezoelectric material BaTiO₃, the remaining sub-matrices in (A.1) can be written as

$$\boldsymbol{\Delta}_{14} = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & 0 \\ e_{15} & 0 & 0 \end{bmatrix}, \ \boldsymbol{\Delta}_{24} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{32} \\ 0 & e_{24} & 0 \end{bmatrix}, \ \boldsymbol{\Delta}_{34} = \begin{bmatrix} e_{15} & 0 & 0 \\ 0 & e_{24} & 0 \\ 0 & 0 & e_{33} \end{bmatrix}$$
$$\boldsymbol{\Delta}_{15} = \boldsymbol{\Delta}_{25} = \boldsymbol{\Delta}_{35} = \boldsymbol{0}_{3\times3}$$
(A.3)

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For the piezomagnetic material $CoFe_2O_4$ the remaining sub-matrices in (A.1) can be written as

$$\boldsymbol{\Delta}_{15} = \begin{bmatrix} 0 & 0 & q_{31} \\ 0 & 0 & 0 \\ q_{15} & 0 & 0 \end{bmatrix}, \ \boldsymbol{\Delta}_{25} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{32} \\ 0 & q_{24} & 0 \end{bmatrix}, \ \boldsymbol{\Delta}_{35} = \begin{bmatrix} q_{15} & 0 & 0 \\ 0 & q_{24} & 0 \\ 0 & 0 & q_{33} \end{bmatrix}$$
$$\boldsymbol{\Delta}_{14} = \boldsymbol{\Delta}_{24} = \boldsymbol{\Delta}_{34} = \boldsymbol{0}_{3\times3}$$
(A.4)

Appendix B

Material properties of the elastic hexagonal square plate are given by

$$[\mathbf{c}] = \begin{bmatrix} 298.2 & 27.7 & 11.0 & 0 & 0 & 0 \\ & 298.2 & 11.0 & 0 & 0 & 0 \\ & 340.9 & 0 & 0 & 0 \\ & & 165.5 & 0 & 0 \\ & sym. & 165.5 & 0 \\ & & & 135.3 \end{bmatrix}$$
GPa, $\rho = 1850 \text{ kg/m}^3$.

Appendix C

Piezoelectric and piezomagnetic material properties (elastic constants c_{ij} in 10⁹ N/m², piezoelectric constants e_{ij} in C/m², piezomagnetic constants q_{ij} in N/Am, dielectric constants ε_{ij} in 10⁻⁹ C²/Nm², magnetic constants μ_{ij} in 10⁻⁶ Ns²/C², density ρ in kg/m³).

Material coefficients of piezoelectric BaTiO₃

| $c_{11} = c_{22}$ 166 | c ₁₂ 77 | $c_{13} = c_{23}$ 78 | c ₃₃ 162 | <i>c</i> ₄₄ = <i>c</i> ₅₅ 43 | $c_{66} = 0.5(c_{11} - c_{12})$ 44.5 |
|---|-------------------------|---------------------------|-------------------------|---|--------------------------------------|
| $e_{31} = e_{32}$ -4.4 | e ₃₃ 18.6 | $e_{24} = e_{15}$ 11.6 | | | ρ 5800 |
| $\varepsilon_{11} = \varepsilon_{22}$ 11.2 | ε ₃₃ 12.6 | | $\mu_{11} = \mu_{22}$ 5 | μ ₃₃ 10 | |

Material coefficients of magnetostrictive CoFe₂O₄

| $c_{11} = c_{22}$ 286 | c ₁₂ 173 | $c_{13} = c_{23}$ 170.5 | c ₃₃ 269.5 | <i>c</i> ₄₄ = <i>c</i> ₅₅ 45.3 | $c_{66} = 0.5(c_{11} - c_{12})$ 56.5 |
|---|--------------------------|----------------------------|------------------------------|---|---|
| $q_{31} = q_{32}$ 580.3 | q ₃₃ 699.7 | $q_{24} = q_{15}$ 550 | | | ρ 5800 |
| $\varepsilon_{11} = \varepsilon_{22}$ 0.08 | ε ₃₃ 0.093 | | $\mu_{11} = \mu_{22}$ 590 | μ ₃₃ 157 | |

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